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THE COMPLETION OF AN ABELIAN l-GROUP

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Introduction and statement of the main result. A directed partially ordered abelian group $(G, \leqq)$ is a tight Riesz group if for $a_1, a_2, b_1, b_2 \in G$ with $a_i < b_j$, $i, j = 1, 2$, there is an $x \in G$ with $a_i < x < b_j$, $i, j = 1, 2$. The open interval topology on G is the topology having as a base the set of all open intervals $(a, b) = \{x \in G | a < x < b\}$. For any $x \in G$, a neighborhood base at x is the set of all open intervals $(x - a, x + a) = x + (-a, a)$ for $a > 0$.

Let $(G, \leqq)$ be an abelian l-group. A compatible tight Riesz order (CTRO) on G is a directed partial order $\prec$ such that $(G, \prec)$ is a tight Riesz group and the closure of the positive cone of $(G, \prec)$ in the open interval topology on $(G, \prec)$ is the positive cone for the l-group $(G, \leqq)$.

Wirth [**8**, Theorem 2, p. 106] has shown that a proper subset T of the positive cone of $(G, \leqq)$ is the strict cone of a CTRO on G if and only if

(1) T is a dual ideal of $(G, \leqq)$; i.e., $x \in T$, $y \geqq x$ implies $y \in T$ and $x, y \in T$ implies $x \wedge y \in T$;

(2) $T + T = T$;

(3) $\wedge_G T = 0$.

If T is a strict cone of a CTRO on an l-group G, let $\leqq_T$ be the tight Riesz order given by T and let $\mathscr{U}_T$ be the open interval topology on $(G, \leqq_T)$. Loy and Miller [**6**, Theorem 5, p. 228] have shown that $(G, \mathscr{U}_T)$ is a Hausdorff topological group and $(G, \vee, \wedge, \mathscr{U})$ is a topological lattice [**6**, Theorem 1, p. 235].

A subset E of the positive cone of G is said to be a set of topological units if E satisfies

(1′) E is lower directed;

(2′) for each $e \in E$, there is a $d \in E$, with $2d \leqq e$;

(3′) $\wedge_G E = 0$.

Banaschewski [**1**, p. 55] has shown that the set of all $U_e = \{x \in G | -e \leqq x \leqq e\}$ for $e \in E$ defines a closed neighborhood filter at 0 which generates a topology on G under which G is a Hausdorff topological group. The uniform space completion of G with respect to this topology is an l-semigroup [**1**, Theorem 11, p. 62].

If T is a CTRO on G and E is a cofinal subset of T, then E is a set of topological units on G and conversely, if E is a set of topological units in G, $T = \{x \in G | x \geqq e, \text{ some } e \in E\}$ is the strict cone of a CTRO in G.

In this paper all l-groups will be abelian with lattice order $\leqq$. $\leqq_T$ will denote a CTRO with strict positive cone T and $\mathscr{U}_T$, or simply $\mathscr{U}$, will denote

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the open interval topology associated with T. The main result of this paper (Theorem 2.3) is that for an abelian l-group G, there exists a unique minimal l-group H such that H is complete with respect to all the topologies associated with all the CTRO's on H and G is a large l-subgroup of H.

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1. Preliminaries.

LEMMA 1.1 (Loy and Miller [**6**, p. 230]). *Let G be an l-group and let T be the strict cone of a* CTRO *on G. Then* $\mathrm{cl}(a, b) = \{x \in G | a \leqq x \leqq b\}$ *where the closure is with respect to* $\mathscr{U}_T$.

Since $(G, \mathscr{U}_T)$ is a Hausdorff topological group, it is a regular topological space and so for $x \in G$, x has a neighborhood base consisting of closed sets, namely $\{x + U_t | t \in T\}$ where $U_t = \mathrm{cl}(-t, t)$. Thus the topology considered by Banaschewski in [**1**] and the open interval topology of Loy and Miller agree.

PROPOSITION 1.2. *Let G be an l-group and let T be the strict cone of a* CTRO *on G. Let $C_T(G)$ be the completion of G with respect to $\mathscr{U}_T$. Then there exists a unique lattice order on $C_T(G)$ such that*

(1) *for $x, y \in C_T(G)$, $(x, y) \rightarrow x \wedge y$ is uniformly continuous,*

(2) *$C_T(G)$ is an l-group, and*

(3) *G is an l-subgroup of $C_T(G)$.*

Moreover, if $x \in C_T(G)$,

$$x = \vee \{y \in G | y \leqq x\} = \wedge \{z \in G | z \geqq x\}.$$

Proof. Banaschewski [**1**, Theorem 11, p. 62] extends the lattice order on G to $C_T(G)$ by the continuity of the lattice operations on G and shows that $C_T(G)$ is an l-semigroup. That $C_T(G)$ is an l-group then follows by continuity arguments and the fact that G is abelian. The uniqueness follows since G is a dense subspace of the Hausdorff space $C_T(G)$ and the last statement follows from Banaschewski's result [**1**, Theorem 4, p. 58] which states that $C_T(G) \subseteq G^*$, the group of units of the Dedekind completion of G.

For the remainder of this paper, $C_T(G)$ will have the lattice order given above.

COROLLARY 1.3. *If G is archimedean, then $C_T(G)$ is archimedean.*

A convex l-subgroup K of G is said to be closed if whenever $\{x_\nu\} \subseteq K$ and $g = \vee x_\nu$ exists in G, then $g \in K$. G is said to be an a^*-extension of L if L is an l-subgroup of G and the map $K \rightarrow L \cap K$ is a one-to-one map of the closed convex l-subgroups of G onto the closed convex l-subgroups of L. For the properties of a^*-extensions and the existence of a^*-closures, see Bleier and Conrad [**2**].

COROLLARY 1.4. *$C_T(G)$ is an a^*-extension of G.*

2. Statement and proof of main result.

Definition. An l-subgroup H of G is said to be *large* in G if for every non-zero convex l-subgroup K of G, $K \cap H \neq \{0\}$.

LEMMA 2.1. *Let G be a large l-subgroup of H and let T be the strict cone of a* CTRO *on G. Then $T' = \{x \in H | x \geqq t$ for some $t \in T\}$ is the strict cone of a* CTRO *on H and (an isomorphic copy of) $C_T(G)$ is a large l-subgroup of $C_{T'}(H)$.*

Proof. Clearly T' is a dual ideal of H. Let $x, y \in T'$. There are $u, v \in T$ such that $x \geqq u$, $y \geqq v$ and so $x + y \geqq u + v$. Let $x \in T'$. There exist $u, v \in T$ such that $u + v \leqq x$. Thus, $u \leqq x - v$ and $x - v \in T'$. Since $x = (x - v) + v$, $T' + T' = T'$.

Suppose $0 < x < s$ for every $s \in T'$. Since $T' + T' = T'$, $0 < nx < s$ for every $s \in T'$. Since G is large in H, there is a $g \in G$ such that $0 < g < nx$ for some n. Thus $0 < g < s$ for every $s \in T' \supseteq T$. But this contradicts the fact $\wedge_G T = 0$ and so T' is the strict cone of a CTRO on H.

From the definition of T', it is easy to show that $\{x_\alpha\} \subseteq G$ is T-Cauchy if and only if it is T'-Cauchy and that if $\{x_\alpha\}$, $\{y_\alpha\}$ are Cauchy nets in G, then they are T-equivalent if and only if they are T'-equivalent. Thus as topological spaces, $C_T(G) \subseteq C_{T'}(H)$. Continuity arguments show that $C_T(G)$ is an l-subgroup of $C_{T'}(H)$.

Definition. An abelian l-group H is a *$\mathscr{C}$-group* if it is complete with respect to all the topologies associated with all the CTRO's on H.

THEOREM 2.2. *If G is a large l-subgroup of a $\mathscr{C}$-group H, then the intersection U of all l-subgroups of H that contain G and are $\mathscr{C}$-groups is a $\mathscr{C}$-group.*

Proof. Let K be an l-subgroup of H which contains G and is a $\mathscr{C}$-group. Since $G \subseteq U$ and G is large in H, U is large in K. Let T be the strict cone of a CTRO on U and let $T' = \{x \in K | x \geqq t$ for some $t \in T\}$,

$$S = \{y \in H | y \geqq t \text{ for some } t \in T\}.$$

By 2.1, T' and S are CTRO's on K and H respectively. Let $\{x_\alpha\}_\Lambda$ be a T-Cauchy net of elements of U. Since K and H are $\mathscr{C}$-groups, there is a $y_1 \in K$, $y_2 \in H$ such that $y_1 = T' - \lim x_\alpha$, and $y_2 = S - \lim x_\alpha$. By the proof of 2.1, $y_1 = S - \lim x_\alpha \in K$ and U is a $\mathscr{C}$-group. Thus, U is a minimal $\mathscr{C}$-group in which G is large. We will call U a *$\mathscr{C}$-hull* of G.

THEOREM 2.3. *Each l-group admits a unique $\mathscr{C}$-hull.*

Proof. In order to show existence, it suffices to show that G is a large l-subgroup of a $\mathscr{C}$-group H and apply Theorem 2.2. Everett [**3**, Theorem 8, p. 116] has shown that G^*, the group of units of the Dedekind completion, δG, of G is an l-group. Since $\delta(G^*) = \delta G$, $C_T(G^*) \subseteq G^*$ so G^* is a $\mathscr{C}$-group. Let U

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be the $\mathscr{C}$-hull of G contained in G^*. Since $U \subseteq G^*$, each element of U is the supremum of its lower bounds in G. This property characterizes U, for if N is another $\mathscr{C}$-hull of G with this property, then the identity map on G extends naturally to an l-isomorphism $\tau : N \rightarrow \delta G$ and hence into G^*. $N\tau$ is a $\mathscr{C}$-group and thus contains U. Minimality of N forces minimality of $N\tau$, so $N\tau = U$.

To complete the proof, we need only show that any $\mathscr{C}$-hull N of G has the property that each element of N is the supremum of its lower bounds in G. Let K be an l-subgroup of N which contains G and is maximal with respect to the property that each element of K is the supremum of its lower bounds in G. If $K \neq N$, then there is a CTRO T on K with $C_T(K) \neq K \subseteq N$. Since each element of K is the supremum of its lower bounds in G, $K^* = G^*$ so $C_T(K) \subseteq K^* = G^*$. But then each element of $C_T(K)$ is the supremum of its lower bounds in G and this contradicts the maximality of K. Thus $K = N$ and $N \cong U$.

Definition. Let $\bar{G}$ be the $\mathscr{C}$-hull of G.

By 2.1, if T is a CTRO on G, then (an isomorphic copy of) $C_T(G) \subseteq \bar{G}$. By the proof of 2.3, we may assume $\bar{G} \subseteq G^*$. Thus we have the following

COROLLARY 2.4. (1) *If* $g \in \bar{G}$, *then*

$$g = \vee \{x \in G | x \leqq g\} = \wedge \{y \in G | y \in G | y \in G | y \geqq g\}.$$

(2) *If G is archimedean, so is* $\bar{G}$.

If G is totally ordered, then $\bar{G} = G^*$ [**1**, p. 59] but, in general, this is not true (see Example 5).

PROPOSITION 2.5. *If G is divisible and archimedean, then $\bar{G}$ is a vector lattice.*

Proof. Let $g \in \bar{G}$ and let $\bar{G}(g)$ be the convex l-subgroup of $\bar{G}$ generated by g. Let T be a CTRO on $\bar{G}(g)$. Reilly [**7**, Theorem 4.2] has shown that $T' = \{x \in \bar{G} | x \geqq t,$ for some $t \in T\}$ is a CTRO on $\bar{G}$. Let $\{x_\alpha\}_\Lambda$ be T'-Cauchy. Since $\bar{G}$ is a $\mathscr{C}$-group, there is a $y \in G$ such that $y = T' -\lim x_\alpha$. Let $t \in T$. Then there is a $\beta \in \Lambda$ such that $y - t \leqq x_\beta \leqq y + t$. Thus, $x_\beta - t \leqq y \leqq x_\beta + t$. Since $x_\beta - t$, $x_\beta + t \in \bar{G}(g)$ and $\bar{G}(g)$ is convex, $y \in \bar{G}(g)$. Thus, $\bar{G}(g)$ is a $\mathscr{C}$-group, and it suffices to show $\bar{G}(g)$ is a vector lattice. Let K be a divisible archimedean $\mathscr{C}$-group with a strong unit ($\bar{G}(g)$ has these properties). From Wirth's characterization, $T = \{$strong units of $K\}$ is a CTRO on K. Let $a \in T$ and let $B = \{(1/n)a | n \in \mathbf{N}\}$ where $\mathbf{N}$ is the set of natural numbers. Then, $B \subseteq T$ and $\mathscr{B} = \{U_b | b \in B\}$ is a countable base for the open neighborhoods of 0. (If $t \in T$, then there exists an $n \in \mathbf{N}$ such that $nt \geqq a$. Thus, $t \geqq (1/n)a$ and $U_t \supseteq U_{((1/n)a)}$.) Thus $\mathscr{U}_T$ is first countable. Let $r \in \mathbf{R}$, the real numbers, and let $\{p_m\}_\mathbf{N}$ be a sequence of rational numbers converging to r. Let $x \in K$. Since a is a strong unit in K, there is a $k \in \mathbf{N}$ such that $ka \geqq |x|$. Let $b = (1/n)a \in B$. Since $\{p_m\}$ is a Cauchy sequence in $\mathbf{R}$, there is an N such that $s, t \geqq N$ implies $|p_s - p_t| \leqq 1/nk$. Thus,

$$|p_s x - p_t x| = |p_s - p_t||x| \leqq b$$

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and $\{p_m x\}$ is T-Cauchy. Since K is a $\mathscr{C}$-group, there is a $y \in K$ such that $y = T - \lim p_m x$. Define $rx = y = T - \lim p_m x$. Using standard arguments, it can be shown that $(r, x) \to rx$ defines a scalar multiplication on K so that K is a vector lattice.

Examples. (1) A dense archimedean l-group G such that the $\mathscr{C}$-hull $\bar{G}$ of G is not a vector lattice: (G is a dense l-group if for each $0 < x \in G$, there exists a $y \in G$ such that $0 < y < x$.)

Reilly [**7**, Example 2, p. 31] has shown that the l-group G of all periodic sequences of integers under the usual pointwise order and operations is dense and has no CTRO. Thus $G = \bar{G}$ and is not a vector lattice.

Remark. It is easy to see that for any l-group with a cyclic cardinal summand, the $\mathscr{C}$-hull is not a vector lattice.

(2) A divisible l-group G such that $\bar{G}$ is not a vector lattice: Let $G = Q \oplus Q$ (Q = rational numbers) with order $(a, b) \geqq 0$ if $a > 0$ or $a = 0$ and $b \geqq 0$. Then $\bar{G} = Q \oplus \mathbf{R}$ with the "same" lexicographic ordering.

Remark. Holland [**5**, Lemma T4, p. 73] has shown that if Γ is an inversely well-ordered set with no minimal element and if $G = \Pi_\Gamma Q_\gamma$ where $Q_\gamma = Q$ and with the order given by $(g_\gamma) \geqq 0$ if for the largest γ such that $g_\gamma \neq 0$, we have $g_\gamma > 0$ (a Hahn Group), then G is a $\mathscr{C}$-group.

(3) A vector lattice G such that G is not a $\mathscr{C}$-group and $\bar{G}$ is not an a-extension of G (H is an a-extension of G if for $0 < a \in H$, there are $b \in G$, $m, n \in \mathbf{N}$ such that $mb \geqq a$ and $na \geqq b$).

Let G be the l-subgroup of $C[0, 1]$ generated by the polynomials. Then $f \in G$ if and only if f is piecewise a polynomial. Thus if $f \in G$, then $f^{-1}(0)$ has finitely many connected components. Since $g(x) = x \sin 1/x$ for $x \neq 0$ and $g(0) = 0$ is continuous with $g^{-1}(0)$ having countably many components $G \neq C[0, 1]$. The constant function 1 is a strong unit in G and G is an archimedean vector lattice. Thus $T = \{\text{strong units of } G\}$ is the strict cone of a CTRO on G. $\mathscr{U}_T$ is the sup-norm topology on G so $C_T(G) = C[0, 1]$ and G is not complete. If $f \in G$, $g \in C[0, 1]$ are a-equivalent, then $f^{-1}(0) = g^{-1}(0)$ and thus $C[0, 1]$ is not an a-extension of G.

Remark. If G is a divisible archimedean l-group with a strong unit, and $T = \{\text{strong units}\}$ then $C_T(G)$ is the $\mathscr{C}$-hull of G.

(4) Let $G = \oplus_I Q$ be a cardinal sum of rationals. The CTRO's on G are in 1-1 correspondence with finite subsets of I, namely if $A \subseteq I$ is finite, then $T_A = \{f \in G \mid f(i) > 0 \text{ for all } i \in A\}$ is a CTRO on G, and conversely.

$$C_{T_A}(G) = (\oplus_A \mathbf{R}) \oplus (\oplus_{I \setminus A} Q)$$

and the $\mathscr{C}$-hull of G is $\oplus_I \mathbf{R}$.

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(5) An l-group G so that $\bar{G} \neq G^*$: Let

$$G = \{f : \mathbf{N} \to \mathbf{R} | \text{ there is an } m \in N \text{ so that } f(n) = f(m) \text{ for all } n \geqq m\}$$

with the usual pointwise order and addition (the "eventually constant" sequences.). Then $\bar{G}$ is the l-group of convergent sequences while G^* is the l-group of all bounded sequences.

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