Canad. J. Math. Vol. **67** (1), 2015 pp. 55–89
http://dx.doi.org/10.4153/CJM-2014-008-x

On Varieties of Lie Algebras of Maximal Class

Tatyana Barron, Dmitry Kerner, and Marina Tvalavadze

Abstract. We study complex projective varieties that parametrize (finite-dimensional) filiform Lie algebras over $\mathbb{C}$, using equations derived by Millionshchikov. In the infinite-dimensional case we concentrate our attention on $\mathbb{N}$-graded Lie algebras of maximal class. As shown by A. Fialowski there are only three isomorphism types of $\mathbb{N}$-graded Lie algebras $L = \bigoplus_{i=1}^{\infty} L_i$ of maximal class generated by L_1 and L_2, $L = \langle L_1, L_2 \rangle$. Vergne described the structure of these algebras with the property $L = \langle L_1 \rangle$. In this paper we study those generated by the first and q-th components where $q > 2$, $L = \langle L_1, L_q \rangle$. Under some technical condition, there can only be one isomorphism type of such algebras. For $q = 3$ we fully classify them. This gives a partial answer to a question posed by Millionshchikov.

1 Introduction

Much effort has been made to understand the algebraic varieties that parametrize Lie algebras, in particular nilpotent Lie algebras. The long list of literature on this subject includes, in particular, [V, KN, Ha, ENR].

A *filiform* Lie algebra is a nilpotent Lie algebra of maximal class of nilpotency. A generalization of this concept, called *a Lie algebra of maximal class*, and varieties of such algebras were studied in [M1].

A Lie algebra $\mathfrak{g}$ is called *residually nilpotent* if $\bigcap_{i=1}^{\infty} \mathfrak{g}^i = \{0\}$, where $\mathfrak{g}^1 = \mathfrak{g}$ and $\{\mathfrak{g}^i\}$ is the lower central series of $\mathfrak{g}$. A residually nilpotent Lie algebra $\mathfrak{g}$ is called *a Lie algebra of maximal class* if $\Sigma_i(\dim \mathfrak{g}^i/\mathfrak{g}^{i+1} - 1) = 1$. In the finite-dimensional case these are exactly filiform Lie algebras.

An explicit system of quadratic equations that describes a variety of filiform Lie algebras or of Lie algebras of maximal class is provided in [M1]; this is one of the main results of the paper. We use this system to study the topology of the varieties of n-dimensional filiform complex Lie algebras (Section 2).

In Section 3 and in Appendix B we study central extensions of $\mathbb{N}$-graded filiform Lie algebras with lacunas in grading. Although neither finite-dimensional filiform Lie algebras nor infinite-dimensional Lie algebras of maximal class have been classified up to an isomorphism, they have been extensively studied for the last few decades. In [M1] Millionshchikov conjectured the existence of only three isomorphism types of Lie algebras of maximal class generated by the first and the q-th graded components

Received by the editors June 28, 2013; revised March 5, 2014.
Published electronically April 28, 2014.
D.K. was partially supported by FP7-People-MCA-CIG grant 334347.
Research of the first author is supported in part by NSERC.
The first author previously published papers under the name Tatyana Foth.
AMS subject classification: **17B70**, 14F45.
Keywords: filiform Lie algebras, graded Lie algebras, projective varieties, topology, classification.

(*i.e.*, with lacunas in grading from 2 to $q-1$). Using the results on central extensions from Subsections 3.1 and 3.2 we prove Theorem 3.9 and also classify $\mathbb{N}$-graded Lie algebras of maximal class generated by the first and third graded components. Our results coincide with the computer calculations of Vaughan-Lee, who investigated the cases $q = 3, 4$.

Our main results are Theorems 2.2, 2.4, 3.9, and 3.27.

2 Topology of Parameter Spaces

In this section all Lie algebras are finite-dimensional over $\mathbb{C}$.

Equations of the affine variety of n-dimensional filiform complex Lie algebras are given in [M1, equations (19), (28)–(30)]. We are going to consider instead complex projective varieties M_n that are obtained from the varieties in [M1, Theorem 4.3] by removing the Abelian Lie algebra, projectivizing, and "dropping" the variables that do not appear in the equation; see a precise explanation below.

2.1 General Properties of M_n

The parameter space for $n \geq 9$ is defined by the following system of quadratic equations on the variables $x_{j,s}$, $s = 0, \ldots, n-5$, $j = 2, \ldots, \frac{[n-1]}{2}$, (Theorem 4.3 [M1]):

- for n odd:

$$\mathsf{M}_n = \left\{ F_{j,q,r} = 0 \text{ for } 2 \leq j < q,\ j + 2q + r + 1 \leq n,\ r \geq 0 \right\}$$

- for n even:

$$\begin{aligned}\mathsf{M}_n = &\left\{ F_{j,q,r} = 0 \text{ for } 2 \leq j < q,\ j + 2q + r + 1 < n,\ r \geq 0 \right\} \\ &\cap \left\{ F_{j,q,r} + (-1)^{\frac{n}{2}-j-q} x_{-1} G_{j,q,r} = 0 \text{ for } 2 \leq j < q,\ j + 2q + 1 + r = n,\ r \geq 0 \right\} \\ &\cap \left\{ x_{-1} G_{j,q,-1} = 0 \text{ for } 2 \leq j < q,\ j + 2q = n \right\},\end{aligned}$$

where polynomials $F_{j,q,r}$ and $G_{j,q,r}$ are as follows:

$$\begin{aligned}F_{j,q,r} = &\sum_{t=0}^{r} \sum_{l=j}^{[\frac{j+q-1}{2}]} \sum_{m=q+1}^{q+[\frac{j+t}{2}]} (-1)^{l-j+m-q} \binom{q-l-1}{l-j} \binom{j+q-m+t-1}{m-q-1} x_{l,t} x_{m,r-t} \\ &+ \sum_{t=0}^{r} \sum_{l=j}^{[\frac{j+q}{2}]} \sum_{m=q}^{q+[\frac{j+t}{2}]} (-1)^{l-j+m-q} \binom{q-l}{l-j} \binom{j+q-m+t}{m-q} x_{l,t} x_{m,r-t} \\ &+ \sum_{t=0}^{r} \sum_{m=j}^{q+[\frac{j+t}{2}]} (-1)^{m-j+1} \binom{2q-m+t}{m-j} x_{q,t} x_{m,r-t}, \\ G_{j,q,r} = &\sum_{l=j}^{[\frac{j+q-1}{2}]} (-1)^{l} \binom{q-l-1}{l-j} x_{l,r+1} + \sum_{l=j}^{[\frac{j+q}{2}]} (-1)^{l} \binom{q-l}{l-j} x_{l,r+1} - (-1)^{q} x_{q,r+1}.\end{aligned}$$

The variable x_{-1} is denoted by x in [M1]. Our notation emphasizes the weight. The quadratic polynomials $F_{j,q,r}$ are weighted homogeneous; they contain only monomials of the form $x_{a_1,t}x_{a_2,r-t}$. The linear forms $G_{j,q,r}$ contain only monomials of the form $x_{a,r+1}$. Thus, for n odd, or for n even and $x_{-1} = 0$, the variables $x_{a,t}$ with $t > n-9$ do not participate in the defining equations. Moreover, even for $t \le n-9$, many of $x_{a,t}$ do not appear in the defining equations. We shall always consider the "minimal" version of M_n, inside the projective space spanned by those $x_{a,t}$ that do appear. Adding more variables that do not participate in the equations means taking the cone over the "minimal" M_n.

The group $\mathbb{C}^*$ acts on M_n by weighted homogeneous scaling

$$\big(x_{-1}, \{x_{*,0}\}, \{x_{*,1}\}, \dots, \{x_{*,n-5}\}\big) \to \big(\lambda^{-1}x_{-1}, \{x_{*,0}\}, \lambda\{x_{*,1}\}, \dots, \lambda^{n-5}\{x_{*,n-5}\}\big),$$

$\lambda \in \mathbb{C}^*$. This is seen directly; the defining equations are equivariant. The points of each $\mathbb{C}^*$ orbit correspond to isomorphic Lie algebras, as the action is induced by the rescaling of the generators, $e_i \to \lambda^{i-1}e_i$.

Accordingly, the set of homogeneous coordinates splits into the subsets $\{x_{*,s}\}_s$, coordinates of the same $\mathbb{C}^*$-weights. The loci $\mathsf{M}_n \cap \{x_{*,i} = 0 \text{ for } i \ne j\}$ are invariant under this $\mathbb{C}^*$ action. The parameter space gets decomposed into 'elementary pieces' as described in the proposition below.

Proposition 2.1 *Let $n \ge 9$.*

(i) *For n odd, the open dense part $\mathsf{M}_n \setminus \{x_{*,0} = 0\}$ deforms (homotopically) to $\mathsf{M}_n \cap \{x_{*,i} = 0 \text{ for } i > 0\}$. For n even, $\mathsf{M}_n \setminus \{x_{-1} = 0\}$ deforms (homotopically) to $\mathsf{M}_n \cap \{x_{*,i} = 0 \text{ for } i \ge 0\}$, which is a point. In both cases the open dense part $\mathsf{M}_n \setminus \{x_{*,n-5} = 0\}$ deforms (homotopically) to $\mathsf{M}_n \cap \{x_{*,i} = 0 \text{ for } i < n-5\}$.*

(ii) *The topological Euler characteristic of M_n can be computed as*

$$\chi(\mathsf{M}_n) = \sum_{j=-1}^{n-5} \chi\big(\mathsf{M}_n \cap \{x_{*,i} = 0 \text{ for } i \ne j\}\big).$$

(iii) *For n odd, $\mathsf{M}_n \cap \big\{x_{*,i} = 0 \text{ for } 0 \le i \le \frac{n-9}{2}\big\}$ is isomorphic to the projective space with homogeneous coordinates*

$$\{x_{j,k}\}_{2 \le j \le \frac{n-1}{2}}^{\frac{n-9}{2}+1 \le k \le n-5}.$$

(iv) $\mathsf{M}_n \cap \{x_{*,i} = 0 \text{ for } i \ne [\frac{n-9}{2}]\} = \{F_{2,3,2[\frac{n-9}{2}]} = 0\} \subset \mathbb{P}(\{x_{j,[\frac{n-9}{2}]}\}_{j=2\dots[\frac{n-1}{2}]})$.

(v) *For $n > 10$, $\mathsf{M}_n \cap \{x_{*,i} = 0 \text{ for } i \ne [\frac{n-9}{2}] - 1\} = \{F_{2,3,n-11} = 0 = F_{2,4,n-11}\}$ for n odd and $\{F_{2,3,n-12} = 0 = F_{2,4,n-12} = F_{3,4,n-12}\}$ for n even.*

Proof (i): The case of odd n. Suppose at least one of $x_{*,0}$ is not zero. Consider the flow $(x_{*,0}, \lambda x_{*,1}, \dots, \lambda^{n-5}x_{*,n-5})$, for $\lambda \in \mathbb{C}^*$. This is a continuous flow for $\lambda \in \mathbb{C}$, well-defined on $\mathsf{M}_n \setminus \{x_{*,0} = 0\}$. It provides the homotopy from $\mathsf{M}_n \setminus \{x_{*,0} = 0\}$, for $\lambda = 1$, to $\mathsf{M}_n \cap \{x_{*,i} = 0 \text{ for } i > 0\}$, for $\lambda = 0$. The other statements in (i) are proved similarly.

(ii): Suppose n is odd (the proof for n even is similar). We shall use (i) and similar statements:

- $(\mathsf{M}_n \cap \{x_{*,0} = 0\}) - \{x_{*,1} = 0\}$ deforms homotopically to
$$\mathsf{M}_n \cap \{x_{*,i} = 0 \text{ for } i \neq 1\},$$
- $(\mathsf{M}_n \cap \{x_{*,0} = 0, x_{*,1} = 0\}) - \{x_{*,2} = 0\}$ deforms homotopically to
$$\mathsf{M}_n \cap \{x_{*,i} = 0 \text{ for } i \neq 2\},$$

etc.

By the additivity of the Euler characteristic,

$$\begin{aligned}\chi(\mathsf{M}_n) &= \chi(\mathsf{M}_n \cap \{x_{*,0} = 0\}) + \chi(\mathsf{M}_n \backslash \{x_{*,0} = 0\}) \\ &= \chi(\mathsf{M}_n \cap \{x_{*,0} = 0 = x_{*,1}\}) \\ &\quad + \chi((\mathsf{M}_n \cap \{x_{*,0} = 0\}) \backslash \{x_{*,1} = 0\}) + \chi(\mathsf{M}_n \cap \{x_{*,i} = 0 \text{ for } i > 0\}) \\ &= \sum_{j=0}^{n-5} \chi(\mathsf{M}_n \cap \{x_{*,i} = 0 \text{ for } i \neq j\})\end{aligned}$$

(iii): The polynomial $F_{j,q,r}$ consists of monomials $x_{*,t}x_{*,r-t}$, hence if $x_{*,i} = 0$ for $0 \leq i \leq \left[\frac{n-9}{2}\right]$ and $r \leq n-9$, then all the relevant polynomials vanish. So, $\mathsf{M}_n \cap \{x_{*,i} = 0 \text{ for } 0 \leq i \leq [\frac{n-9}{2}]\}$ is the projective space, spanned by the remaining coordinates.

(iv) and (v) follow by direct check of the defining equations. ■

2.2 The Parameter Space for $n = 9, 10, 11$

In this subsection we quickly discuss M_9, after which we compute Betti numbers of the components of M_n for $n = 10, 11$ and discuss geometric structure of these components. Some facts that we use are summarized in Appendix A.

2.2.1 $n = 9$

The subvariety $\mathsf{M}_9 \subset \mathbb{P}^2$ is defined by the equation

$$2x_{2,0}x_{4,0} - 3x_{3,0}^2 + x_{3,0}x_{4,0} = 0, \tag{2.1}$$

where $x_{2,0}, x_{3,0}, x_{4,0}$ are the coefficients that appear in the deformation cocycle.

Remark: (2.1) is consistent with the first equation in [M1, (22)]. In the notation of [M1], this is the equation $F_{2,3,0} = 0$.

This subvariety (an algebraic curve of degree 2) is smooth. (Its singular locus is $x_{2,0} = 0 = x_{4,0} = x_{3,0}$ *i.e.,* it is empty.) By the genus formula the genus is $(2-1)(2-2)/2 = 0$, thus M_9 is isomorphic to $\mathbb{P}^1$.

2.2.2 $n = 10$

The subvariety $\mathrm{M}_{10} \subset \mathbb{P}(x_{-1}, \{x_{j,s}\})$ is defined by

$$\begin{aligned} F_{2,3,0}\colon\ & 2x_{2,0}x_{4,0} - 3x_{3,0}^2 + x_{3,0}x_{4,0} = 0 \\ F_{2,3,1} + xG_{2,3,1}\colon\ & -2x_{2,0}x_{4,1} + 7x_{3,0}x_{3,1} - x_{3,0}x_{4,1} - 3x_{4,0}x_{2,1} - 3x_{4,0}x_{3,1} \\ & + x_{-1}(2x_{2,2} + x_{3,2}) = 0 \\ x_{-1}G_{2,4,-1}\colon\ & x_{-1}(2x_{2,0} - x_{3,0} - x_{4,0}) = 0, \end{aligned}$$

where $x_{i,j}$ are the coefficients that appear in the deformation cocycle.

Denote by $\mathrm{M}_{10}^{(0)}$ the component of M_{10} that corresponds to $x_{-1} = 0$. Its equations are

$$\begin{aligned} 2x_{2,0}x_{4,0} - 3x_{3,0}^2 + x_{3,0}x_{4,0} &= 0 \\ -2x_{2,0}x_{4,1} + 7x_{3,0}x_{3,1} - x_{3,0}x_{4,1} - 3x_{4,0}x_{2,1} - 3x_{4,0}x_{3,1} &= 0. \end{aligned}$$

Note: the first equation is the same as (2.1), and the second equation ($F_{2,3,1} = 0$ in notations of [M1]) coincides with the third equation of [M1, Example 4.6].

To simplify the formulas we make a linear change of coordinates:

$$\begin{aligned} z_{0,0} &= 2x_{2,0} + x_{3,0}, & z_{1,0} &= x_{3,0}, & z_{2,0} &= x_{4,0}, \\ z_{0,1} &= 3(x_{2,1} + x_{3,1}), & z_{1,1} &= x_{3,1}, & z_{2,1} &= x_{4,1}. \end{aligned}$$

In the new coordinates, $\mathrm{M}_{10}^{(0)}$ is defined as

$$\{x_{-1} = z_{0,0}z_{2,0} - 3z_{1,0}^2 = 0 = -z_{0,0}z_{2,1} + 7z_{1,0}z_{1,1} - z_{2,0}z_{0,1}\} \subset \mathbb{P}^6_{x_{-1},z_{*,*}}.$$

Denote by $\mathrm{M}_{10}^{(1)}$ the component of M_{10} that corresponds to $x_{-1} \neq 0$. For it we get the system

$$\begin{aligned} 2x_{2,0}x_{4,0} - 3x_{3,0}^2 + x_{3,0}x_{4,0} &= 0 \\ -2x_{2,0}x_{4,1} + 7x_{3,0}x_{3,1} - x_{3,0}x_{4,1} - 3x_{4,0}x_{2,1} - 3x_{4,0}x_{3,1} + x_{-1}(2x_{2,2} + x_{3,2}) &= 0 \\ 2x_{2,0} - x_{3,0} - x_{4,0} &= 0. \end{aligned}$$

As before, we change the coordinates:

$$\begin{aligned} z_{0,0} &= 2x_{2,0} + x_{3,0}, & z_{1,0} &= x_{3,0}, & z_{2,0} &= x_{4,0}, & z_{0,1} &= 3(x_{2,1} + x_{3,1}), \\ z_{1,1} &= x_{3,1}, & z_{2,1} &= x_{4,1}, & z_{0,2} &= x_{3,2}, & z_{1,2} &= 2x_{2,2} + x_{3,2}. \end{aligned}$$

Then the equations become

$$\begin{aligned} \{z_{0,0}z_{2,0} - 3z_{1,0}^2 = 0 = -z_{0,0}z_{2,1} + 7z_{1,0}z_{1,1} - z_{2,0}z_{0,1} + x_{-1}z_{1,2} \\ = z_{0,0} - 2z_{1,0} - z_{2,0}\} \subset \mathbb{P}^7_{x_{-1},z_{*,*}}. \end{aligned}$$

We eliminate $z_{0,0}$ using the linear equation to get the equivalent presentation

$$\begin{aligned} \mathrm{M}_{10}^{(1)} = \{&(3z_{1,0} + z_{2,0})(z_{1,0} - z_{2,0}) = 0, \\ &(2z_{1,0} + z_{2,0})z_{2,1} = 7z_{1,0}z_{1,1} - z_{2,0}z_{0,1} + x_{-1}z_{1,2}\} \subset \mathbb{P}^6. \end{aligned} \tag{2.2}$$

Theorem 2.2 (i) $\mathsf{M}_{10}^{(0)} \subset \mathbb{P}^6$ *is a subvariety of dimension* 3*; its singular locus is the plane*

$$\{x_{-1} = z_{0,0} = 0 = z_{1,0} = z_{2,0}\} \subset \mathbb{P}^6.$$

The singularities of $\mathsf{M}_{10}^{(0)}$ *are resolved in one blowup, and the resolution* $\widetilde{\mathsf{M}_{10}^{(0)}}$ *is a projective bundle over* $\mathbb{P}^1$*. More explicitly,*

$$\widetilde{\mathsf{M}_{10}^{(0)}} = \mathbb{P}(T^*_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1))|_{C_2},$$

where $T^*_{\mathbb{P}^2}$ *is the cotangent bundle of the plane and* $|_{C_2}$ *denotes the restriction of the bundle onto a smooth conic* $C_2 \subset \mathbb{P}^2$*.*

$\mathsf{M}_{10}^{(0)}$ *admits an algebraic cell structure,* $\mathsf{M}_{10}^{(0)} = \mathbb{C}^3 \cup (2\mathbb{C}^2) \cup \mathbb{C}^1 \cup \mathbb{C}^0$*. In particular, its odd homologies vanish, while the even homologies are:*

$$H_0(\mathsf{M}_{10}^{(0)}, \mathbb{Z}) = \mathbb{Z} = H_6(\mathsf{M}_{10}^{(0)}, \mathbb{Z}), \quad H_2(\mathsf{M}_{10}^{(0)}, \mathbb{Z}) = \mathbb{Z}, \quad H_4(\mathsf{M}_{10}^{(0)}, \mathbb{Z}) = \mathbb{Z}^{\oplus 2}.$$

(ii) $\mathsf{M}_{10}^{(1)}$ *is isomorphic to the union of two quadric* 3*-folds in* $\mathbb{P}^5$*, intersecting along* $\{z_{0,0} = 0 = z_{1,0} = z_{2,0} = x_{-1}z_{1,2}\}$*. Each quadric is the cone over a smooth quadric surface in* $\mathbb{P}^3$*.*

Proof (i): We consider everything inside the hyperplane $\{x_{-1} = 0\} = \mathbb{P}^5 \subset \mathbb{P}^6$. The singular locus of $\mathsf{M}_{10}^{(0)}$ is defined by the maximal minors of the Jacobian matrix (the matrix of all the partials of the defining equations):

$$\begin{pmatrix} z_{2,0} & -6z_{1,0} & z_{0,0} & 0 & 0 & 0 \\ -z_{2,1} & 7z_{1,1} & -z_{0,1} & -z_{2,0} & 7z_{1,0} & -z_{0,0} \end{pmatrix}$$

Thus this singular locus is $Z := \{z_{0,0} = 0 = z_{1,0} = z_{2,0}\}$. (We consider the singular locus as a set, omitting the multplicities.) Note that the singular locus is of codimension one in $\mathsf{M}_{10}^{(0)}$; *i.e.,* the variety is not normal. To compute the normalization, *i.e.,* the normal variety $\widetilde{\mathsf{M}_{10}^{(0)}}$ with the finite surjective birational morphism, $\widetilde{\mathsf{M}_{10}^{(0)}} \to \mathsf{M}_{10}^{(0)}$, we blowup along the singular locus:

$$\widetilde{\mathsf{M}_{10}^{(0)}} := Bl_Z \mathsf{M}_{10}^{(0)} = \Big\{ (z_{0,0}, z_{1,0}, z_{2,0}) \sim (\sigma_0, \sigma_1, \sigma_2),\ \sigma_0\sigma_2 = 3\sigma_1^2, \\ -\sigma_0 z_{2,1} + 7\sigma_1 z_{1,1} - \sigma_2 z_{0,1} = 0 \Big\} \subset \mathbb{P}^5_{z_{*,*}} \times \mathbb{P}^2_{\sigma_*}$$

By construction, $\widetilde{\mathsf{M}_{10}^{(0)}}$ has two natural projections: $\widetilde{\mathsf{M}_{10}^{(0)}} \stackrel{\pi_\sigma}{\to} \mathbb{P}^2_{\sigma_*}$ and $\widetilde{\mathsf{M}_{10}^{(0)}} \stackrel{\pi_z}{\to} \mathsf{M}_{10}^{(0)}$.

Consider the fibres of π_z. If $(z_{0,0}, z_{1,0}, z_{2,0}) \neq (0,0,0)$, then the condition $(z_{0,0}, z_{1,0}, z_{2,0}) \sim (\sigma_0, \sigma_1, \sigma_2)$ determines the point $(\sigma_0, \sigma_1, \sigma_2) \in \mathbb{P}^2_{\sigma_*}$ uniquely. If $(z_{0,0}, z_{1,0}, z_{2,0}) = (0,0,0)$ then $(z_{0,1}, z_{1,1}, z_{2,1}) \neq (0,0,0)$, and therefore there are two equations on $(\sigma_0, \sigma_1, \sigma_2)$, linear and quadratic. Geometrically, for a fixed triple $(z_{0,1}, z_{1,1}, z_{2,1})$ we have a line and a conic in the plane $\mathbb{P}^2_{\sigma_*}$; they intersect in two points (counted with multiplicity). Therefore the projection π_z is *finite* and is a 2:1 cover over the singular locus.

Consider the projection π_σ. For a fixed point $(\sigma_0, \sigma_1, \sigma_2) \in \mathbb{P}^2_{\sigma_*}$ the conditions on $z_{*,*}$ are *linear* and linearly independent. Therefore this projection equips $\widetilde{\mathsf{M}_{10}^{(0)}}$ with the structure of $\mathbb{P}^2$ bundle over its image, the conic $\{\sigma_0\sigma_2 = 3\sigma_1^2\} \subset \mathbb{P}^2_{\sigma_*}$. In particular, it follows that $\widetilde{\mathsf{M}_{10}^{(0)}}$ is smooth, hence the map $\widetilde{\mathsf{M}_{10}^{(0)}} \xrightarrow{\pi_z} \mathsf{M}_{10}^{(0)}$ is not only a normalization but also a resolution of singularities. Finally note that $\pi_\sigma(\widetilde{\mathsf{M}_{10}^{(0)}}) \subset \mathbb{P}^2$ is a smooth conic $C \subset \mathbb{P}^2_\sigma$, therefore $\widetilde{\mathsf{M}_{10}^{(0)}}$ is a $\mathbb{P}^2$ bundle over $\mathbb{P}^1$.

It remains to understand the $\mathbb{P}^2$ bundle structure. First, consider the particular locus in $\widetilde{\mathsf{M}_{10}^{(0)}}$, where $z_{0,0} = z_{1,0} = z_{2,0} = 0$. We claim that

$$\widetilde{\mathsf{M}_{10}^{(0)}}|_{z_{0,0}=z_{1,0}=z_{2,0}=0} = \mathbb{P}T^*_{\mathbb{P}^2}|_C.$$

Indeed, after rescaling the coordinates $(z_{2,1}, z_{1,1}, z_{0,1})$, we get the defining equation $\{(\sigma_0, \sigma_1, \sigma_2) \times (z_{2,1}, z_{1,1}, z_{0,1}) = 0\} \subset \mathbb{P}^2_z \times \mathbb{P}^2_\sigma$. Geometrically, this can be interpreted as the variety of pairs: a point in $\mathbb{P}^2_\sigma$ and the lines passing through this point. (Recall that a line in $\mathbb{P}^2$ is defined by a 1-form.) The set of lines passing through a point, *i.e.,* the set of non-zero 1-forms up to scaling, is naturally the projectivization of the cotangent space to $\mathbb{P}^2_\sigma$ at this point. Therefore $\widetilde{\mathsf{M}_{10}^{(0)}}|_{z_{0,0}=z_{1,0}=z_{2,0}=0} = \mathbb{P}T^*_{\mathbb{P}^2}|_C$.

As the other extremal case, consider the locus $z_{2,1} = z_{1,1} = z_{0,1} = 0$. Then $(z_{0,0}, z_{1,0}, z_{2,0}) \neq (0,0,0)$ and the fibre over $(\sigma_0, \sigma_1, \sigma_2)$ is one point: the projectivization of the tautological bundle, $\mathcal{O}_{\mathbb{P}^2}(-1)$.

Finally, for any fixed point $\sigma \in \mathbb{P}^2_\sigma$ the remaining equations are linear. Hence each fibre over σ is the span of $\mathbb{P}T^*_{\mathbb{P}^2,\sigma}$ and $\mathbb{P}\mathcal{O}_{\mathbb{P}^2}(-1)_\sigma$, *i.e.,* each fibre is $\mathbb{P}(T^*_{\mathbb{P}^2,\sigma} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)_\sigma)$. As this identification is canonical, we get the statement.

(ii): Follows from the presentation in equation (2.2). ■

Remark 2.3 **Singular points of $\mathsf{M}_{10}^{(0)}$:** as mentioned above, the singular points of $\mathsf{M}_{10}^{(0)}$ are

$$[z_0 : z_1 : z_2 : z_3 : z_4 : z_5]$$

such that $z_0 = z_1 = z_2 = 0$ and z_3, z_4 and z_5 are any (but not all zero at the same time). This means that $x_{2,0} = x_{3,0} = x_{4,0} = 0$ and $x_{2,1}, x_{3,1}, x_{4,1}$ take any values (not all zeros). Therefore, the deformation cocycle Ψ of $\mathfrak{m}_0(10)$ (see definition of $\mathfrak{m}_0(n)$ in Section 3) is of the form

$$\Psi = x_{2,1}\Psi_{2,1} + x_{3,1}\Psi_{3,1} + x_{4,1}\Psi_{4,1},$$

where $\Psi_{2,1}$, $\Psi_{3,1}$ and $\Psi_{4,1}$ are closed cocycles from $H^2_+(\mathfrak{m}_0(10), \mathfrak{m}_0(10))$ defined in [M1, p. 183] by explicit formulas. Then, for $i, j > 1$ $\Psi(e_i, e_j) = \alpha_{ij} e_{i+j+1}$ for appropriate scalars α_{ij}, and the cocycle vanishes on the remaining vector pairs. Let us re-name basis elements as follows

$$f_1 = e_1, \quad f_{i+1} = e_i, \quad i = 2, \dots, 10.$$

This is an $\mathbb{N}$-graded basis for a filiform Lie algebra $\mathfrak{g}$ corresponding to Ψ. Indeed,

$$[f_i, f_j] = [e_{i-1}, e_{j-1}] = \alpha_{i-1,j-1} e_{i+j-1} = \alpha_{i-1,j-1} f_{i+j},$$

where $i, j > 1$ and

$$[f_1, f_i] = [e_1, e_{i-1}] = e_i = f_{i+1},$$

where $i > 1$. As shown in Section 3, $\mathfrak{g}$ must be one of the following algebras: $\mathfrak{m}^3_{0,1}(11)$ (the central extension of $\mathfrak{m}^3_0(10)$), $\mathfrak{m}^3_{0,3}(11)$ (the third central extension of $\mathfrak{m}^3_0(8)$), $\mathfrak{m}^3_{0,5}(11)$ (the fifth central extension of $\mathfrak{m}^3_0(6)$).

2.3 $n = 11$

The equations of M_{11} are [M1]:

$$
\begin{aligned}
&F_{2,3,0}\colon 2x_{2,0}x_{4,0} - 3x_{3,0}^2 + x_{3,0}x_{4,0} = 0,\\
&F_{2,3,1}\colon -2x_{2,0}x_{4,1} + 7x_{3,0}x_{3,1} - x_{3,0}x_{4,1} - 3x_{4,0}x_{2,1} - 3x_{4,0}x_{3,1} = 0,\\
&F_{2,4,0}\colon -2x_{2,0}x_{5,0} + 4x_{3,0}x_{4,0} - 6x_{4,0}^2 + x_{3,0}x_{5,0} + x_{4,0}x_{5,0} = 0,\\
&F_{2,3,2}\colon -2x_{2,0}x_{4,2} + 8x_{3,0}x_{3,2} - x_{3,0}x_{4,2} - 4x_{4,0}x_{2,2} - 6x_{4,0}x_{3,2}\\
&\qquad\qquad + 2x_{5,0}x_{2,2} + x_{5,0}x_{3,2} - 3x_{2,1}x_{4,1} + 4x_{3,1}^2 - 3x_{3,1}x_{4,1} = 0.
\end{aligned}
$$

As was pointed out earlier, our expressions for $F_{2,3,0}$ and $F_{2,3,1}$ are consistent with [M1]. Our equation $F_{2,4,0} = 0$ is the same as the corresponding equation in [M1, Example 4.6]. Our equation $F_{2,3,2} = 0$ differs from that in [M1, Example 4.6] by the coefficient at the $x_{2,2}x_{4,0}$ term, but we checked our calculation several times and are confident that our coefficient is correct.

Apply the following change of variables:

$$
\begin{array}{llll}
z_{0,0} = 2x_{2,0} + x_{3,0}, & z_{1,0} = x_{3,0}, & z_{2,0} = x_{4,0}, & z_{3,0} = x_{5,0} - 6x_{4,0},\\
z_{0,1} = 3(x_{2,1} + x_{3,1}), & z_{1,1} = x_{3,1}, & z_{2,1} = x_{4,1}, & \\
z_{0,2} = 2x_{2,2} + x_{3,2}, & z_{1,2} = x_{3,2}, & z_{2,2} = x_{4,2}. &
\end{array}
$$

Then we get the following equations:

$$
\begin{aligned}
&F_1\colon z_{0,0}z_{2,0} - 3z_{1,0}^2 = 0,\\
&F_2\colon -z_{0,0}z_{2,1} + 7z_{1,0}z_{1,1} - z_{2,0}z_{0,1} = 0,\\
&F_3\colon z_{3,0}(2z_{1,0} - z_{0,0} + z_{2,0}) + z_{2,0}(16z_{1,0} - 6z_{0,0}) = 0\\
&F_4\colon -z_{0,0}z_{2,2} - z_{0,1}z_{2,1} + 4z_{1,1}^2 + 8z_{1,0}z_{1,2} + 4z_{2,0}(z_{0,2} - z_{1,2}) + z_{3,0}z_{0,2} = 0.
\end{aligned}
$$

Theorem 2.4 (i) *The parameter space has two irreducible components, X and Y, both of dimension five, as a set $\mathsf{M}_{11} = X \cup Y$.*

(ii) *The component $X = \{z_{0,0} = 0 = z_{1,0} = z_{2,0} = 4z_{1,1}^2 - z_{0,1}z_{2,1} + z_{3,0}z_{0,2}\} \subset \mathbb{P}^9$ enters with generic multiplicity 2. The singular locus of (reduced) X is*

$$\mathrm{Sing}(X) = \{z_{0,0} = 0 = z_{1,0} = z_{2,0} = z_{1,1} = z_{0,1} = z_{2,1} = z_{3,0} = z_{0,2}\} = \mathbb{P}^1_{z_{1,2}z_{2,2}}.$$

X admits the algebraic cell structure: $\mathbb{C}^5 \cup \mathbb{C}^4 \cup \mathbb{C}^3 \cup \mathbb{C}^2 \cup \mathbb{C}^1 \cup \mathbb{C}^0$. In particular, its odd cohomologies vanish, while all the even cohomologies are $\mathbb{Z}$.

(iii) *The component Y is reduced, it is the topological closure of the affine part of M_{11} in $\mathbb{C}^9 = \{z_{0,0} \neq 0\} \subset \mathbb{P}^9$. (Thus, the defining equations of $Y \cap \mathbb{C}^9$ are obtained from the equations above by setting $z_{0,0} = 1$.) The affine part $Y \cap \{z_{0,0} \neq 0\}$ is smooth. The intersection with the infinite hyperplane $Y \cap \{z_{0,0} = 0\}$ is defined, as a set, by $\{z_{0,0} = 0 = z_{1,0} = z_{3,0} = z_{0,1} = z_{1,1}^2 + z_{0,2}(z_{0,2} - z_{1,2})\}$.*

The affine part, $Y \cap \{z_{0,0} \neq 0\} \subset \mathbb{C}^9$, is isomorphic to the $\mathbb{C}^4$-bundle over the line with two punctures, $\mathbb{C}^1_{z_{1,0}} \setminus \{(z_{1,0}+1)(3z_{1,0}-1) = 0\}$.

Also, $H_{2i}(Y,\mathbb{Z}) = \mathbb{Z}$, for $0 \leq 2i \leq 10$ and $H_9(Y,\mathbb{Z}) = \mathbb{Z}^{\oplus 2}$; all the other cohomologies vanish.

Proof (i) Consider the part of M_{11} at infinity, $\mathsf{M}_{11} \cap \{z_{0,0} = 0\}$. We get (omitting multiplicities) $z_{0,0} = 0 = z_{1,0} = z_{2,0} z_{0,1}$. Hence this part splits; by direct check we get two components:

$$\mathsf{M}_{11} \cap \{z_{0,0} = 0\} = \left\{ z_{0,0} = 0 = z_{1,0} = z_{2,0} = 4z_{1,1}^2 - z_{0,1}z_{2,1} + z_{3,0}z_{0,2} \right\} \\ \cup \left\{ z_{0,0} = 0 = z_{1,0} = z_{3,0} = z_{0,1} = z_{1,1}^2 + z_{2,0}(z_{0,2} - z_{1,2}) \right\}$$

Note that neither of them lies inside the other, *e.g.*, they are distinguished by $z_{2,0} = 0$ and $z_{0,1} = 0$. By direct check, the first component is of codimension 4, the second is of codimension 5. Note that the whole space, M_{11} is defined by four equations, therefore at each point of M_{11} the (local) codimension is at most 4. Therefore, the first component is an honest component of M_{11} (we call it X), while the second component belongs to the intersection of Y with the infinite hyperplane $z_{0,0} = 0$.

(ii) To check the generic multiplicity of X, fix some generic values of $z_{0,2}$, $z_{1,2}$, $z_{2,2}$, $z_{3,0}$, $z_{0,1}$, $z_{1,1}$ and let $z_{0,0}, z_{1,0}, z_{2,0}$ vary near zero, while $z_{2,1}$ varies near a root of $4z_{1,1}^2 - z_{0,1}z_{2,1} + z_{3,0}z_{0,2} = 0$. This corresponds to the transverse intersection of the generic point of X by a linear space of the complementary dimension. The generic multiplicity of X is the multiplicity of this local intersection, *i.e.,* (algebraically) the length of the Artinian ring $\mathbb{C}\{z_{0,0}, z_{1,0}, z_{2,0}, z_{2,1}\}/\langle F_1, F_2, F_3, F_4\rangle$. Note that for the generic values of $z_{0,2}, z_{1,2}, z_{2,2}, z_{3,0}, z_{1,1}$, the variable $z_{2,1}$ enters linearly in equation F_4, hence can be eliminated. Similarly, the variable $z_{0,0}$ can be eliminated using F_3, while $z_{2,0}$ can be eliminated using F_2. Thus the ring $\mathbb{C}\{z_{0,0}, z_{1,0}, z_{2,0}, z_{2,1}\}/\langle F_1, F_2, F_3, F_4\rangle$ is isomorphic to $\mathbb{C}\{z_{1,0}\}/Q$, where the expansion of Q in $z_{1,0}$ begins with a quadratic term. Hence the intersection multiplicity (*i.e.*, the length of this ring) is two.

To understand the cell structure, consider the affine part, $X \cap \{z_{3,0} \neq 0\}$, and the part at infinity, $X \cap \{z_{3,0} = 0\}$. By direct check: $X \cap \{z_{3,0} \neq 0\} \approx \mathbb{C}^5_{z_{0,1}z_{1,1}z_{2,1}z_{1,2}z_{2,2}}$. Continue to "cut" the infinite part:

$$X \cap \{z_{3,0} = 0\} = \underbrace{X \cap \{z_{3,0} = 0 = z_{0,1}\}}_{\mathbb{P}^3_{z_{1,1}z_{2,1}z_{1,2}z_{2,2}}} \coprod \underbrace{X \cap \{z_{3,0} = 0,\ z_{0,1} \neq 0\}}_{\mathbb{C}^4_{z_{0,1}z_{1,1}z_{1,2}z_{2,2}}}.$$

Finally, we use the standard cell decomposition $\mathbb{P}^3 = \mathbb{C}^3 \coprod \mathbb{C}^2 \coprod \mathbb{C}^1 \coprod \mathbb{C}^0$. This gives the algebraic cell decomposition of X.

(iii) The defining equations of the affine part of $Y|_{z_{0,0} \neq 0}$ are

$$z_{2,0} = 3z_{1,0}^2, \quad z_{2,1} = 7z_{1,0}z_{1,1} - z_{2,0}z_{0,1},$$
$$z_{3,0}(2z_{1,0} - 1 + z_{2,0}) + 3z_{1,0}^2(16z_{1,0} - 6) = 0,$$
$$z_{2,2} = -z_{0,1}z_{2,1} + 4z_{1,1}^2 + 8z_{1,0}z_{1,2} + 4z_{2,0}(z_{0,2} - z_{1,2}) + z_{3,0}z_{0,2}.$$

Therefore Y projects isomorphically onto

$$\{z_{3,0}(2z_{1,0} - 1 + 3z_{1,0}^2) + 3z_{1,0}^2(16z_{1,0} - 6) = 0\} \subset \mathbb{C}^6_{z_{1,0}z_{3,0}z_{0,1}z_{1,1}z_{0,2}z_{1,2}}.$$

Note that the defining equation does not contain variables $z_{0,1}z_{1,1}z_{0,2}z_{1,2}$, so this is a $\mathbb{C}^4$ bundle over the curve $\{z_{3,0}(2z_{1,0} - 1 + 3z_{1,0}^2) + 3z_{1,0}^2(16z_{1,0} - 6) = 0\} \subset \mathbb{C}^2_{z_{1,0}z_{3,0}}$. Finally, this curve projects isomorphically onto $\mathbb{C}^1_{z_{1,0}}$ outside the locus $(3z_{1,0} - 1)(z_{1,0} + 1) = 0$, *i.e.,* with two points punctured.

Finally, we use Section A.3 for the pair:

$$Y|_{z_{0,0}=0} \subset Y: \cdots \to H_i(Y|_{z_{0,0}=0}, \mathbb{Z}) \to H_i(Y, \mathbb{Z}) \to H_i(Y, Y|_{z_{0,0}=0}, \mathbb{Z}) \to \cdots .$$

Note that $Y \setminus Y|_{z_{0,0}=0}$ is smooth of real dimension 10, thus by Proposition A.5, $H_i(Y, Y|_{z_{0,0}=0}, \mathbb{Z}) = H^{10-i}(Y \setminus Y|_{z_{0,0}=0}, \mathbb{Z})$. As established above, the homotopy type of $Y \setminus Y|_{z_{0,0}=0}$ is that of $\mathbb{C}^1$ with two punctured points. Therefore, $H_i(Y|_{z_{0,0}=0}, \mathbb{Z}) \xrightarrow{\sim} H_i(Y, \mathbb{Z})$ for $i < 8$ and

$$0 \to H_9(Y, \mathbb{Z}) \to H_1(\mathbb{C}^1 \setminus \{2 \text{ pts}\}, \mathbb{Z}) \to H_8(Y|_{z_{0,0}=0}, \mathbb{Z}) \to H_8(Y, \mathbb{Z}) \to 0$$

As shown above, $Y|_{z_{0,0}=0}$ is a singular quadric in $\mathbb{P}^5$, and its cell structure is $\mathbb{C}^4 \cup \mathbb{C}^3 \cup \mathbb{C}^2 \cup \mathbb{C}^1 \cup \mathbb{C}^0$. In particular, all its even cohomologies are $\mathbb{Z}$, while all the odd cohomologies vanish. Note also that $H_8(Y, \mathbb{Z})$ contains (at least) one factor of $\mathbb{Z}$, being a projective hypersurface. Therefore, from $H_8(Y|_{z_{0,0}=0}, \mathbb{Z}) \to H_8(Y, \mathbb{Z}) \to 0$, we get $H_8(Y, \mathbb{Z}) = \mathbb{Z}$, and the last map is an isomorphism. Thus, $0 \to H_9(Y, \mathbb{Z}) \to H_1(\mathbb{C}^1 \setminus \{2pts\}, \mathbb{Z}) \to 0$, proving the statement. ■

A remark about the first component of M_{11}

The component X of M_{11} contains the $\mathbb{Z}$-graded algebra $\mathfrak{g}$ of type $\mathfrak{m}_{0,1}(11)$ defined in [M2, p. 268], which would correspond to $z_3 = z_4 = z_5 = z_7 = z_8 = z_9 = 0$ but $z_6 \neq 0$. Besides, X also contains algebras corresponding to $z_7 = z_8 = z_9 = z_6 = 0$. The corresponding deformation cocycle is

$$\Psi(e_i, e_j) = \alpha_{ij} e_{i+j+1},$$

where $i, j > 1$, α_{ij} are scalars. Besides, Ψ vanishes on the other pairs of vectors from the standard basis $\{e_1, e_2, \ldots, e_{11}\}$ of $\mathfrak{m}_0(11)$. Renaming basis elements $f_1 = e_1$ and $f_{i+1} = e_i$ where $i = 2, \ldots, 11$, we obtain an $\mathbb{N}$-graded basis for $\mathfrak{g}$ corresponding to Ψ. As would follow from results of Section 3, $\mathfrak{g}$ should be of one of the following types: $\mathfrak{m}^3_{0,2}(12)$ (the second central extension of $\mathfrak{m}^3_0(10)$), $\mathfrak{m}^3_{0,3}(12)$ (the third central extension of $\mathfrak{m}^3_0(9)$), $\mathfrak{m}^3_{0,5}(12)$ (the fifth central extension of $\mathfrak{m}^3_0(7)$), $\mathfrak{m}^3_{0,6}(12)$ (the sixth central extension of $\mathfrak{m}^3_0(6)$).

There are also algebras in X corresponding to $z_3 = z_4 = z_5 = z_6 = 0$. In this case, the deformation cocycle is $\Psi(e_i, e_j) = \alpha_{ij} e_{i+j+2}$ for $i, j > 1$, and it equals zero on the remaining vector-pairs. Changing notation of the basis elements of $\mathfrak{m}_0(11)$: $f_1 = e_1$, $f_{i+2} = e_i$, $i = 2, \ldots, 11$, we obtain an $\mathbb{N}$-graded basis for $\mathfrak{g}$ corresponding to Ψ. Thus, $\mathfrak{g}$ is obtained by taking one-dimensional central extensions of $\mathfrak{m}^4_0(8)$. Therefore, X contains algebras of the following types: $\mathfrak{m}^4_{0,1}(13)$ (one-dimensional central extension of $\mathfrak{m}^4_0(12)$), $\mathfrak{m}^4_{0,3}(13)$ (the third central extension of $\mathfrak{m}^4_0(10)$), $\mathfrak{m}^4_{0,5}(13)$ (the fifth central extension of $\mathfrak{m}^4_0(8)$).

3 Structure of $\mathbb{N}$-graded Filiform Lie Algebras

The classification of nilpotent Lie algebras is a difficult problem widely discussed in literature. Nilpotent Lie algebras up to dimension 5 are well known. In [CGS] the authors gave a full classification of 6-dimensional nilpotent Lie algebras over arbitrary fields. In higher dimensions, there are infinite families of pairwise nonisomorphic nilpotent Lie algebras. In dimension 7 each infinite family can be parameterized by a single parameter. Many papers on classification of 7-dimensional nilpotent Lie algebras have been published, but the most complete list of such Lie algebras was obtained by Ming-Peng Gong (see [G]).

In this section we are concerned with those nilpotent Lie algebras whose nil-index is $n-1$ for a given dimension n over *an algebraically closed field F of zero characteristic*. Such Lie algebras will be called *filiform*. An infinite-dimensional analog of a filiform Lie algebra is a so-called Lie algebra of *maximal class* (or of coclass 1). Namely, a residually nilpotent Lie algebra is called a Lie algebra of *maximal class* (or coclass 1) if $\sum_{i\geq 1}(\dim \mathfrak{g}^i/\mathfrak{g}^{i+1} - 1) = 1$, where $\{\mathfrak{g}^i\}$ is the lower central series of $\mathfrak{g}$. In [V] M. Vergne has shown that an arbitrary filiform Lie algebra is isomorphic to some deformation of the graded filiform Lie algebra $\mathfrak{m}_0(n)$ defined by its basis $e_1, \dots, e_n$ and nontrivial Lie products: $[e_1, e_i] = e_{i+1}$, $i = 2, \dots, n-1$.

Example 3.1 Let $\mathfrak{m}_0$ be a linear space with a basis $\{e_1, e_2, \dots\}$. Define the Lie product on $\mathfrak{m}_0$ by $[e_1, e_i] = e_{i+1}$ for $i > 1$, and let the other products be zero. Note that we can introduce two types of $\mathbb{N}$-gradings:

Type 1: $\mathfrak{m}_0 = \bigoplus_{i\geq 1} L_i$, where $L_1 = \operatorname{span}\{e_1, e_2\}$ and $L_i = \operatorname{span}\{e_{i+1}\}$.
Type 2: $\mathfrak{m}_0 = \bigoplus_{i\geq 1} \widetilde{L}_i$, where $\widetilde{L}_1 = \operatorname{span}\{e_1\}$ and $\widetilde{L}_i = \operatorname{span}\{e_i\}$.

In the case of infinite-dimensional $\mathbb{N}$-graded Lie algebras of maximal class, Vergne proved the following theorem.

Theorem 3.2 *Let $L = \bigoplus_{i\in\mathbb{N}} L_i$ be an infinite-dimensional $\mathbb{N}$-graded Lie algebra of maximal class and suppose $L = \langle L_1\rangle$. Then $L \cong \mathfrak{m}_0$ (with Type 1 grading).*

By taking quotients of $\mathfrak{m}_0$ we obtain finite-dimensional filiform Lie algebras $\mathfrak{m}_0(n) = \mathfrak{m}_0/I_n$ where $I_n = \operatorname{span}\{e_{n+1}, e_{n+2}, \dots\}$. We next introduce other important examples of infinite-dimensional Lie algebras of maximal class.

Example 3.3 The Lie algebra $\mathfrak{m}_2$ is defined by its basis $\{e_1, e_2, \dots\}$ with multiplication table as follows

$$[e_1, e_i] = e_{i+1},\ i \geq 2,\ [e_2, e_i] = e_{i+2},\ i \geq 3,$$

with the remaining products all zero.

Example 3.4 The Lie algebra W (the Witt algebra) is defined by its basis $\{e_1, e_2, \dots\}$ with multiplication table as follows

$$[e_i, e_j] = (j-i)e_{i+j}, \quad i, j \geq 1.$$

In [F] the classification of $\mathbb{N}$-graded Lie algebras of maximal class $L = \bigoplus_{i\in\mathbb{N}} L_i$ generated by L_1, L_2 was obtained. Namely, the following theorem holds.

Theorem 3.5 *Let $L = \bigoplus_{i\in\mathbb{N}} L_i$ be an infinite-dimensional $\mathbb{N}$-graded Lie algebra of maximal class and suppose $L = \langle L_1, L_2\rangle$. Then one of the following holds:*

(i) $L \cong \mathfrak{m}_0$;

(ii) $L \cong \mathfrak{m}_2$;
(iii) $L \cong W$.

(Note that this result was also obtained 14 years later in [SZ], and it also follows from [M2, Theorem 5.17] 2004.)

Let us now consider $\mathbb{N}$-graded Lie algebras of maximal class that are generated by graded components of degrees 1 and q where $q > 2$. Hence,

$$\mathfrak{g} = \bigoplus_{i=1,q}^{\infty} \mathfrak{g}_i.$$

We give some examples of such algebras below.

Example 3.6 The Lie algebra $\mathfrak{m}_0^q$ is defined by its basis $e_1, e_q, \dots$ with multiplication table as follows

$$[e_1, e_i] = e_{i+1}, \quad i \geq q,$$

and the remaining products are zero. The basis as above will be called the *standard* basis for $\mathfrak{m}_0^q$.

By taking quotients of $\mathfrak{m}_0^q$ we obtain finite-dimensional filiform Lie algebras $\mathfrak{m}_0^q(n) = \mathfrak{m}_0^q/I_n$, where $I_n = \operatorname{span}\{e_{n+1}, e_{n+2}, \dots\}$ also generated by components of degrees 1 and q.

Example 3.7 The Lie algebra $\mathfrak{m}_q$ has the basis $e_1, e_q, \dots$ and the following multiplication table:

$$[e_1, e_i] = e_{i+1}, \quad i \geq q,$$
$$[e_q, e_i] = e_{q+i}, \quad i \geq q+1,$$

with the other products zero.

Example 3.8 The Lie algebra W^q is given by its basis $e_1, e_q, \dots$ with the following multiplication table:

$$[e_i, e_j] = (j - i)e_{i+j},$$

with the remaining products all zero.

Notice that W^q is a *nonsolvable* Lie algebra of maximal class. It is not known yet whether there are nonsolvable Lie algebras of maximal class other than algebras described in the preceding example. The isomorphism classes of solvable Lie algebras of maximal class were given in [B] (see also [L]).

Here is the main conjecture.

Conjecture *Let $\mathfrak{g}$ be an $\mathbb{N}$-graded Lie algebra of maximal class generated by graded components of degrees 1 and q. Then $\mathfrak{g}$ is isomorphic (as a graded algebra) to one of the following three algebras:* $\mathfrak{m}_0^q$, $\mathfrak{m}_q$, W^q.

Later we will see that this conjecture is actually equivalent to the conjecture from [M1, p. 190].

For $q > 2$, we show the following theorem.

https://doi.org/10.4153/CJM-2014-008-x Published online by Cambridge University Press

Theorem 3.9 *Let* $\mathfrak{g}$ *be an* $\mathbb{N}$*-graded Lie algebra of maximal class generated by nonzero graded components* $\mathfrak{g}_1$ *and* $\mathfrak{g}_q$*, where* $q > 2$*, and let* $\mathfrak{g}_{q+2} \neq \{0\}$*. If*

$$[\mathfrak{g}_q, \mathfrak{g}_{q+1}] = \cdots = [\mathfrak{g}_{2q}, \mathfrak{g}_{2q+1}] = 0,$$

then $\mathfrak{g} \cong \mathfrak{m}_0^q$.

In some sense this result is similar to Theorem 3.2. If $\mathfrak{g}$ is generated by two graded components as above, then under some technical condition there can only be one isomorphism type.

Besides, we prove the conjecture for $q = 3$ using some general results on central extensions of $\mathfrak{m}_0^q(n)$ obtained in Subsections 3.1 and 3.2.

3.1 Central Extensions

Let L be a Lie algebra and V a vector space with a skew-symmetric bilinear form $\theta\colon L \times L \mapsto V$, *i.e.,* $\theta(x, x) = 0$ for all $x \in L$. Then θ, as above, satisfying

$$\theta([x, y], z) + \theta([z, x], y) + \theta([y, z], x) = 0,$$

where $x, y, z \in L$, is said to be a *cocycle*. If $\theta\colon L \times L \mapsto V$ is a cocycle, then $L_\theta = L \oplus V$ with the product defined by

$$[x + v, y + w]' = [x, y] + \theta(x, y)$$

is a Lie algebra. Then L_θ is said to be a *central extension* of L by V. Note that V is central in L_θ. If both L and $L_\theta = L \oplus V$ are filiform, then $\theta \neq 0$. Otherwise, $L_\theta^2 = L^2$ and

$$L_\theta / L_\theta^2 = (L \oplus V)/L^2 = L/L^2 \oplus V.$$

Then

$$\dim L_\theta / L_\theta^2 = \dim L/L^2 + \dim V \geq 3,$$

since $\dim L/L^2 = 2$ (this fact holds for any filiform Lie algebra). Therefore, L_θ cannot be filiform, a contradiction. Furthermore, if $L_\theta = L \oplus V$ is a one-dimensional filiform central extension of a filiform L, *i.e.,* $\dim V = 1$, then L_θ is generated by L. Indeed, since $\dim V = 1$, $V = \operatorname{span}\{w\}$, $w \neq 0$. As noted above, $\theta \neq 0$, *i.e.,* there are two $x, y \in L$ such that $\theta(x, y) \neq 0$. Thus, $[x, y]' = [x, y] + \theta(x, y) = [x, y] + \alpha w$ for some $\alpha \neq 0$. Hence, $w = \alpha^{-1}[x, y]' - \alpha^{-1}[x, y] \in \langle L \rangle$, and, therefore, $V \subseteq \langle L \rangle$.

Let $\mathfrak{g}$ be an $\mathbb{N}$-graded filiform Lie algebra generated by nonzero $\mathfrak{g}_1$ and $\mathfrak{g}_q$, $q > 2$. Then $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_q \oplus \cdots \oplus \mathfrak{g}_n$ for some n. Without any loss of generality we assume that $\mathfrak{g}_n \neq \{0\}$, otherwise, we discard it.

Lemma 3.10 *Let* $\mathfrak{g}$ *be an* $\mathbb{N}$*-graded filiform Lie algebra generated by nonzero* $\mathfrak{g}_1$ *and* $\mathfrak{g}_q$*. Additionally, assume that* $\mathfrak{g}_{2+q} \neq \{0\}$*. Then every* $\mathfrak{g}_i$*,* $i = 1, q, \ldots, n$ *is a nonzero component of dimension one.*

Proof We first want to prove that if $\mathfrak{g}_i \neq \{0\}$, then $\dim \mathfrak{g}_i = 1$. Since $\mathfrak{g} = \langle \mathfrak{g}_1, \mathfrak{g}_q \rangle$, $\mathfrak{g}_{2+q} = [\mathfrak{g}_1, [\mathfrak{g}_1, \mathfrak{g}_q]] \neq 0$. Hence, $[\mathfrak{g}_1, \mathfrak{g}_q] \neq 0$ and $\mathfrak{g}_{1+q} = [\mathfrak{g}_1, \mathfrak{g}_q] \neq \{0\}$. Then we can write $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_q \oplus \mathfrak{g}_{q+1} \oplus \mathfrak{g}_{q+2} \oplus \mathfrak{g}_{i_1} \oplus \cdots \oplus \mathfrak{g}_{i_s},$$

where $\mathfrak{g}_{i_1}, \dots, \mathfrak{g}_{i_s}$ are the remaining nonzero graded components. Hence, $\dim \mathfrak{g} \geq 4 + s$ (the total number of nonzero components). Since $\mathfrak{g}$ is filiform, its nil-index $m = \dim \mathfrak{g} - 1 \geq 3 + s$. Directly computing components of the lower central series of $\mathfrak{g}$ we obtain the following:

$$\begin{aligned}
\mathfrak{g}^2 &\subseteq \mathfrak{g}_{q+1} + \cdots + \mathfrak{g}_{i_s},\\
\mathfrak{g}^3 &\subseteq \mathfrak{g}_{q+2} + \cdots + \mathfrak{g}_{i_s},\\
\mathfrak{g}^4 &\subseteq \mathfrak{g}_{i_1} + \cdots + \mathfrak{g}_{i_s},\\
&\vdots\\
\mathfrak{g}^{3+r} &\subseteq \mathfrak{g}_{i_r} + \cdots + \mathfrak{g}_{i_s},\\
&\vdots\\
\mathfrak{g}^{3+s-1} &\subseteq \mathfrak{g}_{i_{s-1}} + \mathfrak{g}_{i_s},\\
\mathfrak{g}^{3+s} &\subseteq \mathfrak{g}_{i_s},\\
\mathfrak{g}^{4+s} &= \{0\}.
\end{aligned}$$

This means that nil-index $m \leq 3+s$. Therefore, $m = \dim \mathfrak{g} - 1 = 3+s$, and $\dim \mathfrak{g} = 4+s$. Since there are exactly $4+s$ nonzero graded components, each component must be one-dimensional. Since $\dim \mathfrak{g}/\mathfrak{g}^2 = 2$ and $\dim \mathfrak{g}^i/\mathfrak{g}^{i+1} = 1$, $i \geq 2$, all inclusions above become equalities.

We next show that there is no 'gap' in the grading from $q+1$ to n. This means that all $\mathfrak{g}_i$, $i = q+1, \dots, n$ must be nonzero. Assume the contrary; *i.e.,* there exists s, $q < s < n$ such that $\mathfrak{g}_s = \{0\}$. Let s be the smallest number satisfying this condition. Clearly, $s > 2+q$. Let $\mathfrak{g}_{s+t}$, $t \geq 1$, $s+t \leq n$ be the first nonzero component following $\mathfrak{g}_{s-1}$. Consider $\widetilde{\mathfrak{g}} = \mathfrak{g}/J$, where $J = \bigoplus_{j>s+t} \mathfrak{g}_j$ is the ideal of $\mathfrak{g}$. Then $\widetilde{\mathfrak{g}}$ is also filiform, and

$$\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_1 \oplus \widetilde{\mathfrak{g}}_q \oplus \cdots \oplus \widetilde{\mathfrak{g}}_{s-1} \oplus \widetilde{\mathfrak{g}}_{s+t},$$

where $\widetilde{\mathfrak{g}}_i = \mathfrak{g}_i + J$, $\dim \widetilde{\mathfrak{g}}_i = \dim \mathfrak{g}_i = 1$. Besides, $\widetilde{\mathfrak{g}}$ is also generated by $\widetilde{\mathfrak{g}}_1$ and $\widetilde{\mathfrak{g}}_q$. We next choose a basis: $e_1, e_q, \dots, e_{s-1}, e_{s+t}$ such that $[e_1, e_q] = e_{q+1}$, and $[e_1, e_{q+1}] = e_{q+2}$. Since $\widetilde{\mathfrak{g}}_s = \{0\}$, $[e_1, e_{s-1}] = 0$. It is known that a filiform Lie algebra $\widetilde{\mathfrak{g}}$ has an *adapted* basis: $f_1, f_2, \dots, f_k$, $k = s - q + 2 = \dim \widetilde{\mathfrak{g}}$ such that $[f_1, f_i] = f_{i+1}$, $i = 2, \dots, k-1$, and $[f_i, f_j] \in \operatorname{span}\{f_{i+j}, \dots, f_k\}$. Moreover,

$$\begin{aligned}
\widetilde{\mathfrak{g}}/\widetilde{\mathfrak{g}}^2 &= \operatorname{span}\{f_1, f_2\} + \widetilde{\mathfrak{g}}^2 = \operatorname{span}\{e_1, e_q\} + \widetilde{\mathfrak{g}}^2,\\
&\vdots\\
\widetilde{\mathfrak{g}}^i/\widetilde{\mathfrak{g}}^{i+1} &= \operatorname{span}\{f_{i+1}\} + \widetilde{\mathfrak{g}}^{i+1} = \operatorname{span}\{e_{q+i-1}\} + \widetilde{\mathfrak{g}}^{i+1},\\
&\vdots\\
\widetilde{\mathfrak{g}}^{s-q}/\widetilde{\mathfrak{g}}^{s-q+1} &= \operatorname{span}\{f_{s-q+1}\} + \widetilde{\mathfrak{g}}^{s-q+1} = \operatorname{span}\{e_{s-1}\} + \widetilde{\mathfrak{g}}^{s-q+1},\\
\widetilde{\mathfrak{g}}^{s-q+1}/\widetilde{\mathfrak{g}}^{s-q+2} &= \operatorname{span}\{f_{s-q+2}\} = \operatorname{span}\{e_{s+t}\}
\end{aligned}$$

Therefore, $e_1 = \lambda_1 f_1 + \lambda_2 f_2 + h$, where $h \in \operatorname{span}\{f_3, \dots, f_k\}$ and $e_{s-1} = \mu f_{k-1} + \beta f_k$, $\mu \neq 0$, $k = s - q + 2$. Then

$$0 = [e_1, e_{s-1}] = [\lambda_1 f_1 + \lambda_2 f_2 + h, \mu f_{k-1} + \beta f_k] = \lambda_1 \mu [f_1, f_{k-1}] = \lambda_1 \mu f_k.$$

Since $\mu \neq 0$ we have that $\lambda_1 = 0$. Hence, $e_1 = \lambda_2 f_2 + h$. Write $e_{q+1} = \gamma f_3 + h'$, $\gamma \neq 0$, $h' \in \operatorname{span}\{f_4, \dots, f_k\}$, and $e_{q+2} = \delta f_4 + h''$, $\delta \neq 0$, $h'' \in \operatorname{span}\{f_5, \dots, f_k\}$. Therefore,

$$e_{q+2} = [e_1, e_{q+1}] = [\lambda_2 f_2 + h, \gamma f_3 + h'] = \lambda_2 \gamma f_5 + \bar{h},$$

where $\bar{h} \in \operatorname{span}\{f_6, \dots, f_k\}$. Comparing with $e_{q+2} = \delta f_4 + h''$ we obtain that $\delta = 0$, a contradiction. This means that there cannot be any 'gap' in the grading of $\mathfrak{g}$. The proof is complete. ∎

Lemma 3.11 *Let $\mathfrak{g}$ be an $\mathbb{N}$-graded filiform Lie algebra generated by nonzero $\mathfrak{g}_1, \mathfrak{g}_q$, and let $\mathfrak{g}_{2+q} \neq \{0\}$. Then there is a basis for $\mathfrak{g} : e_1, e_q, \dots, e_n$ such that $\mathfrak{g}_i = \operatorname{span}\{e_i\}$ and $[e_1, e_i] = e_{i+1}$, $i = q, \dots, n-1$.*

Proof As follows from the previous lemma, each component is of dimension one. Therefore, it suffices to show that $[\mathfrak{g}_1, \mathfrak{g}_i] \neq 0$ for any $i = 1, q, \dots, n-1$. We know that $[\mathfrak{g}_1, \mathfrak{g}_q] \neq 0$, $[\mathfrak{g}_1, \mathfrak{g}_{q+1}] \neq 0$. Assume that there exists i, $n > i > q+1$ such that $[\mathfrak{g}_1, \mathfrak{g}_i] = 0$. Consider $\widetilde{\mathfrak{g}} = \mathfrak{g}/J$, where $J = \bigoplus_{j > i+1} \mathfrak{g}_j$. Then

$$\widetilde{\mathfrak{g}} = \widetilde{\mathfrak{g}}_1 \oplus \widetilde{\mathfrak{g}}_q \oplus \cdots \oplus \widetilde{\mathfrak{g}}_i \oplus \widetilde{\mathfrak{g}}_{i+1},$$

where $\widetilde{\mathfrak{g}}_l = \mathfrak{g}_l + J$, $l = 1, q, \dots, i+1$. Then $\widetilde{\mathfrak{g}}^{i-q+1} = \widetilde{\mathfrak{g}}_i \oplus \widetilde{\mathfrak{g}}_{i+1}$ and $\widetilde{\mathfrak{g}}^{i-q+2} = [\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}_i \oplus \widetilde{\mathfrak{g}}_{i+1}] = \{0\}$. This means that $\dim \widetilde{\mathfrak{g}}^{i-q+1}/\widetilde{\mathfrak{g}}^{i-q+2} = 2$. This contradicts the fact that $\widetilde{\mathfrak{g}}$ is filiform. Therefore, $[\mathfrak{g}_1, \mathfrak{g}_i] \neq 0$ for any $i = 1, q, \dots, n-1$. It is now easy to see that we can choose a required basis for $\mathfrak{g}$. ∎

The following corollaries are immediate consequences of the above lemmas.

Corollary 3.12 *Let $\mathfrak{g}$ be an $\mathbb{N}$-graded filiform Lie algebra generated by nonzero $\mathfrak{g}_1, \mathfrak{g}_q$, and let $\mathfrak{g}_{2+q} \neq \{0\}$. If $n < 2q+1$, then $\mathfrak{g} \cong \mathfrak{m}_0^q(n)$.*

Corollary 3.13 *Let $\mathfrak{g}$ be an $\mathbb{N}$-graded Lie algebra of maximal class generated by both graded components $\mathfrak{g}_1$ and $\mathfrak{g}_q$, and let $\mathfrak{g}_{q+2} \neq \{0\}$. Then each graded component is one-dimensional. Moreover, there exists a basis $e_1, e_q, \dots$ for $\mathfrak{g}$ such that $\mathfrak{g}_i = \operatorname{span}\{e_i\}$ and $[e_1, e_i] = e_{i+1}$, $i > 1$.*

Definition 3.14 Let $\mathfrak{g}$ be an $\mathbb{N}$-graded filiform Lie algebra as above. Any basis $\{e_1, e_q, \dots, e_n\}$ of $\mathfrak{g}$ satisfying $\mathfrak{g}_i = \operatorname{span}\{e_i\}$, and $[e_1, e_i] = e_{i+1}$, $i > 1$, will be called *canonical.*

3.2 Central Extensions of $\mathfrak{m}_0^q(n)$

In this section we discuss one-dimensional $\mathbb{N}$-graded filiform central extensions of $\mathfrak{m}_0^q(n)$. The following lemma is similar to [M2, Corollary 5.3]. As was noted earlier if $\mathfrak{g}$ is an $\mathbb{N}$-graded one-dimensional filiform central extension of $\mathfrak{m}_0^q(n)$, then $\mathfrak{g}$ is generated by $\mathfrak{g}_1$ and $\mathfrak{g}_q$, since $\mathfrak{m}_0^q(n) = \langle \mathfrak{g}_1, \mathfrak{g}_q \rangle$ (see the beginning of Subsection 3.1).

Lemma 3.15 *Let* $\mathfrak{g}$ *be a one-dimensional* $\mathbb{N}$*-graded filiform central extension of* $\mathfrak{m}_0^q(n)$.

(i) *If* $n = 2k+1$, *then* $\mathfrak{g} \cong \mathfrak{m}_0^q(2k+2)$.
(ii) *If* $n = 2k$, *then either* $\mathfrak{g} \cong \mathfrak{m}_0^q(2k+1)$ *or* $\mathfrak{g} \cong \mathfrak{m}_{0,1}^q(2k+1)$ *defined by the basis* $e_1, e_q, \dots, e_{2k}, e_{2k+1}$ *and structure relations:*

$$[e_1, e_i] = e_{i+1}, \quad i = q, \dots, 2k \quad \text{and} \quad [e_r, e_{2k+1-r}] = (-1)^{r-k} e_{2k+1}, \quad r = q, \dots, k.$$

Proof First we consider the case of odd $n = 2k+1$. If $k < q$, then $n = 2k+1 < 2q+1$ and $n+1 = 2k+2 < 2q+1$. By Corollary 3.12, $\mathfrak{g} \cong \mathfrak{m}_0^q(2k+2)$. Now let $k \geq q$. Let us choose the *standard basis* for $\mathfrak{m}_0^q(2k+1)$:

$$e_1, e_q, \dots, e_{2k+1},$$

where $[e_i, e_j] = \lambda_{ij} e_{i+j}$, $\lambda_{1i} = 1$, $i \geq q$ and $\lambda_{ij} = 0$, $i, j \geq q$. Let $\mathfrak{g}$ denote an $\mathbb{N}$-graded one-dimensional filiform central extension of $\mathfrak{m}_0^q(2k+1)$. By Lemma 3.11, the standard basis can be extended to the following canonical basis $e_1, e_q, \dots, e_{2k+1}, e_{2k+2}$ of $\mathfrak{g}$ such that

$$[e_i, e_{2k+2-i}] = \lambda_{i,2k+2-i} e_{2k+2}, \quad i = q, \dots, k.$$

Note that if $i + j < 2k+2$, then the products $[e_i, e_j]$ are exactly the same as in $\mathfrak{m}_0^q(2k+1)$. Let us find unknown structure constants $\lambda_{i,2k+2-i}$, $i = q, \dots, k$. We know that $J(e_1, e_r, e_{2k+1-r}) = 0$, where $r = q, \dots, k$ and $J(\ )$ is the Jacobian. This equation can be rewritten in terms of λ's as follows:

$$\lambda_{1r}\lambda_{1+r,2k+1-r} + \lambda_{r,2k+1-r}\lambda_{2k+1,1} + \lambda_{2k+1-r,1}\lambda_{2k+2-r,r} = 0. \tag{3.1}$$

Note that $\lambda_{1r} = 1$, $\lambda_{2k+1-r,1} = -1$, and $\lambda_{r,2k+1-r} = 0$. Therefore, (3.1) becomes

$$\lambda_{1+r,2k+1-r} + \lambda_{r,2k+2-r} = 0, \quad r = q, \dots, k,$$

and $\lambda_{k+1,k+1} = 0$. Clearly, this system has a unique solution: $\lambda_{r,2k+2-r} = 0$, $r = q, \dots, k$. Thus, $\mathfrak{m}_0^q(2k+2)$ is the only central extension of $\mathfrak{m}_0^q(2k+1)$.

Let us now assume that $n = 2k$. If $k < q$, then $2k < 2q$ and $2k+1 < 2q+1$. By Corollary 3.12, $\mathfrak{g} \cong \mathfrak{m}_0^q(2k+1)$. Now let $k \geq q$. Choose the standard basis for $\mathfrak{m}_0^q(2k)$:

$$e_1, e_q, \dots, e_{2k},$$

where $[e_i, e_j] = \lambda_{ij} e_{i+j}$, $\lambda_{1i} = 1$, $i \geq q$, and $\lambda_{ij} = 0$, $i, j \geq q$. Let $\mathfrak{g}$ be an $\mathbb{N}$-graded one-dimensional filiform central extension of $\mathfrak{m}_0^q(2k)$. By Lemma 3.11, the standard basis can be extended to the canonical basis $e_1, e_q, \dots, e_{2k+1}, e_{2k+2}$ of $\mathfrak{g}$ such that where $[e_r, e_{2k+1-r}] = \lambda_{r,2k+1-r} e_{2k+1}$, $r = q, \dots, k$, and the remaining products are exactly the same as in $\mathfrak{m}_0^q(2k)$. Let us find unknown structure constants $\lambda_{r,2k+1-r}$, $r = q, \dots, k$. Since $\mathfrak{g}$ is a Lie algebra, we have that for every $r = q, \dots, k$,

$$J(e_1, e_r, e_{2k-r}) = 0.$$

Therefore,

$$\lambda_{1r}\lambda_{1+r,2k-r} + \lambda_{r,2k-r}\lambda_{2k,1} + \lambda_{2k-r,1}\lambda_{2k+1-r,r} = 0,$$

where $\lambda_{1r} = 1$, $\lambda_{2k,1} = \lambda_{2k-r,1} = -1$, and $\lambda_{r,2k-r} = 0$. Equivalently,

$$\lambda_{1+r,2k-r} + \lambda_{r,2k+1-r} = 0.$$

Set $\lambda_{k,k+1} = \beta$. Then $\lambda_{r,2k+1-r} = (-1)^{k-r}\beta$. For $\beta = 0$, g is isomorphic to $\mathrm{m}_0^q(2k+1)$. Assume that $\beta \neq 0$. Then introducing a new $\mathbb{N}$-graded basis $\{e'_1, e'_q, \dots, e'_{2k+1}\}$ such that $e'_1 = e_1$, $e'_i = \beta^{-1}e_i$, $i = q, \dots, 2k+1$, we obtain the following structure relations for g:

$$[e'_1, e'_i] = e'_{i+1}, \quad i = q, \dots, 2k,$$
$$[e'_r, e'_{2k+1-r}] = (-1)^{r-k}e'_{2k+1}.$$

This is a Lie algebra, since $J(e_i, e_j, e_r) = 0$ for any admissible $i < j < r$. Indeed, if $i+j+r < 2k+1$, then $J(e_i, e_j, e_r) = 0$, since $\mathrm{m}_0^q(2k)$ is a Lie algebra. If $i+j+r = 2k+1$, then the following two cases occur.

Case 1: $i \geq q$. Since $q > 1$ we have that $i + j < j + r < i + r < 2k + 1$. Consequently, $\lambda_{ij} = \lambda_{jr} = \lambda_{ir} = 0$. Thus,

$$J(e_i, e_j, e_r) = (\lambda_{ij}\lambda_{i+j,r} + \lambda_{jr}\lambda_{j+r,i} + \lambda_{ri}\lambda_{r+i,j})e_{i+j+r} = 0.$$

Case 2: $i = 1$. Then $j + r = 2k$, $r = 2k - j$. Then

$$\begin{aligned} J(e_1, e_j, e_{2k-j}) &= \lambda_{1j}\lambda_{1+j,2k-j} + \lambda_{j,2k-j}\lambda_{2k,1} + \lambda_{2k-j,1}\lambda_{2k+1-j,j} \\ &= \lambda_{1+j,2k-j} + \lambda_{j,2k+1-j} = 0. \end{aligned}$$

The proof is complete. ■

Definition 3.16 The basis $e_1, \dots, e_{2k+1}$ for $\mathrm{m}_{0,1}^q(2k+1)$ with multiplication table as in Lemma 3.15 will be called the *standard* basis.

Lemma 3.17 *Let* g *be a one-dimensional* $\mathbb{N}$*-graded filiform central extension of* $\mathrm{m}_{0,1}^q(2k+1)$. *Then* g *is isomorphic to* $\mathrm{m}_{0,2}^q(2k+2)$, *defined by its basis* $e_1, e_q, \dots, e_{2k+1}, e_{2k+2}$ *and structure relations*

$$[e_1, e_i] = e_{i+1}, \quad i = q, \dots, 2k+1, \quad [e_l, e_{2k+1-l}] = (-1)^{l-k}e_{2k+1}, \quad l = q, \dots, k,$$
$$[e_r, e_{2k+2-r}] = (-1)^{r-k}(k+1-r)e_{2k+2}, \quad r = q, \dots, k+1.$$

Proof First of all, we determine all $\mathbb{N}$-graded one-dimensional central extensions of $\mathrm{m}_{0,1}^q(2k+1)$ in the same way as we did in Lemma 3.15. Let $e_1, e_q, \dots, e_{2k+1}$ denote the standard basis for $\mathrm{m}_{0,1}^q(2k+1)$. Then its one-dimensional $\mathbb{N}$-graded filiform central extension g can be defined by the canonical basis: $e_1, e_q, \dots, e_{2k+1}, e_{2k+2}$ (see Lemma 3.11). Arguing in the same way as in Lemma 3.15 we obtain that $J(e_1, e_r, e_{2k+1-r}) = 0$ and $r = q, \dots, k$ is equivalent to

$$\lambda_{1r}\lambda_{1+r,2k+1-r} + \lambda_{r,2k+1-r}\lambda_{2k+1,1} + \lambda_{2k+1-r,1}\lambda_{2k+2-r,r} = 0. \tag{3.2}$$

Note that in (3.2), $\lambda_{1r} = 1$, $\lambda_{2k+1,1} = -1$, $\lambda_{2k+1-r,1} = -1$ and $\lambda_{r,2k+1-r} = (-1)^{r-k}$. Hence, (3.2) takes the form

$$\lambda_{1+r,2k+1-r} + \lambda_{r,2k+2-r} = (-1)^{r-k}.$$

This yields $\lambda_{r,2k+2-r} = (-1)^{r-k}(k+1-r)$. Therefore, g has the same multiplication table as $\mathrm{m}_{0,2}^{q}(2k+2)$ does. We next show that $\mathrm{m}_{0,2}^{q}(2k+2)$ is indeed a Lie algebra. Consider any $i, j, r = 1, q, \dots, 2k+2$ such that $i < j < r$ and $i + j + r = 2k+2$. The following two cases occur.

Case 1: $i \geq q$. Then

$$i + j = 2k + 2 - r < 2k + 2 - q < 2k$$

since $q > 2$. Likewise, $j + r < 2k$ and $i + r < 2k$. Thus, $\lambda_{ij} = \lambda_{jr} = \lambda_{ir} = 0$, and

$$J(e_i, e_j, e_r) = (\lambda_{ij}\lambda_{i+j,r} + \lambda_{jr}\lambda_{j+r,i} + \lambda_{ri}\lambda_{r+i,j})e_{i+j+r} = 0.$$

Case 2: $i = 1$. Then $j + r = 2k + 1$, $r = 2k + 1 - j$. Then

$$\begin{aligned} J(e_1, e_j, e_{2k+1-j}) &= \lambda_{1j}\lambda_{1+j,2k+1-j} + \lambda_{j,2k+1-j}\lambda_{2k+1,1} + \lambda_{2k+1-j,1}\lambda_{2k+2-j,j} \\ &= \lambda_{1+j,2k+1-j} + \lambda_{j,2k+2-j} - (-1)^{j-k} = 0. \end{aligned}$$

Therefore, $J(e_i, e_j, e_r) = 0$ for any $i, j, r = 1, q, \dots, 2k+2$. This means that $\mathrm{m}_{0,2}^{q}(2k+2)$ is a Lie algebra. The proof is complete. ■

Definition 3.18 Let $\mathrm{m}_{0,3}^{q}(2k+3; \beta_1)$ denote an algebra spanned by $e_1, e_q, \dots, e_{2k+2}, e_{2k+3}$ with the following structure relations:

$$\begin{gathered} [e_1, e_i] = e_{i+1}, \quad i = q, \dots, 2k+2, \quad [e_l, e_{2k+1-l}] = (-1)^{l-k}e_{2k+1}, \quad l = q, \dots, k, \\ [e_j, e_{2k+2-j}] = (-1)^{j-k}(k+1-j)e_{2k+2}, \quad j = q, \dots, k+1, \\ [e_r, e_{2k+3-r}] = (-1)^{r-k}\left(\binom{k-r+2}{k-r} - \beta_1 \right) e_{2k+3},\ r = q, \dots, k+1, \end{gathered}$$

where β_1 is any scalar.

Definition 3.19 We inductively define algebras of type $\mathrm{m}_{0,s}^{q}(2k+s; \bar{\beta})$ where $s \geq 3$, $\bar{\beta} = (\beta_1, \dots, \beta_l)$ and $l = \left[\frac{s+1}{2}\right] - 1$. An algebra of type $\mathrm{m}_{0,3}^{q}(2k+3; \beta_1)$ was introduced above. Assume that $\mathrm{m}_{0,s}^{q}(2k+s; \bar{\beta})$ with a basis: $e_1, e_q, \dots, e_{2k+s}$ has been constructed.

(i) For an even s, $\mathrm{m}_{0,s+1}^{q}(2k+s+1; \bar{\beta}') = \operatorname{span}\{e_1, e_q, \dots, e_{2k+s}, e_{2k+s+1}\}$, where $\bar{\beta}' = (\beta_1, \dots, \beta_l, \beta_{l+1})$ (with additional parameter β_{l+1}) and

(3.3)

$$\begin{gathered} [e_r, e_{2k+s+1-r}] = (-1)^{k-r}\left(\binom{k-r+s}{k-r} + \sum_{i=1}^{l+1}(-1)^i \binom{k-r+s-i}{k-r+i}\beta_i \right) e_{2k+s+1}, \\ r = q, \dots, k + \left[\frac{s+1}{2}\right]. \end{gathered}$$

(ii) For an odd s, $\mathfrak{m}_{0,s+1}^{q}(2k+s+1;\bar{\beta}') = \operatorname{span}\{e_1, e_q, \dots, e_{2k+s}, e_{2k+s+1}\}$, where $\bar{\beta}' = \bar{\beta} = (\beta_1, \dots, \beta_l)$ and

(3.4)
$$[e_r, e_{2k+s+1-r}] = (-1)^{k-r}\left(\binom{k-r+s}{k-r} + \sum_{i=1}^{l}(-1)^i\binom{k-r+s-i}{k-r+i}\beta_i\right)e_{2k+s+1},$$
$$r = q, \dots, k + \left[\frac{s+1}{2}\right].$$

Additionally, $[e_1, e_{2k+s}] = e_{2k+s+1}$, and if $i + j \le 2k + s$, then $[e_i, e_j]$ remains the same as in $\mathfrak{m}_{0,s}^{q}(2k+s;\bar{\beta})$.

The basis $e_1, \dots, e_{2k+s+1}$ for $\mathfrak{m}_{0,s+1}^{q}(2k+s+1;\bar{\beta}')$ with the above multiplication table will be called the *standard* basis.

Lemma 3.20 *Let $\mathfrak{m}_{0,s}^{q}(2k+s;\bar{\beta})$ be a Lie algebra. Then $\mathfrak{m}_{0,s}^{q}(2k+s;\bar{\beta})$ is filiform. If $\mathfrak{g}$ is its one-dimensional $\mathbb{N}$-graded filiform central extension, then $\mathfrak{g}$ is isomorphic to $\mathfrak{m}_{0,s+1}^{q}(2k+s+1;\bar{\beta}')$ for some $\bar{\beta}'$.*

Proof By our assumption $\mathfrak{m}_{0,s}^{q}(2k+s;\bar{\beta})$ is a Lie algebra. As follows from Definition 3.19,
$$\mathfrak{m}_{0,s}^{q}(2k+s;\bar{\beta}) = \mathfrak{g}_1 \oplus \mathfrak{g}_q \oplus \cdots \oplus \mathfrak{g}_{2k+s},$$
where $\mathfrak{g}_i = \operatorname{span}\{e_i\}$ is an $\mathbb{N}$-grading. Since $[e_i, e_i] = e_{i+1}$, $i = q, \dots, 2k+s-1$, we have that
$$\mathfrak{g}^2 = \mathfrak{g}_{q+1} \oplus \cdots \oplus \mathfrak{g}_{2k+s}, \quad \text{and} \quad \mathfrak{g}^i = \mathfrak{g}_{q+i-1} \oplus \cdots \oplus \mathfrak{g}_{2k+s},$$
where $i > 2$. Hence, $\dim \mathfrak{g}/\mathfrak{g}^2 = 2$ and $\dim \mathfrak{g}^i/\mathfrak{g}^{i+1} = 1$, which is a necessary and sufficient condition for $\mathfrak{g}$ to be filiform. It is also easy to see that $\mathfrak{m}_{0,s}^{q}(2k+s;\bar{\beta})$ is generated by the first two graded components.

Let us now determine all $\mathbb{N}$-graded one-dimensional filiform central extensions of $\mathfrak{m}_{0,s}^{q}(2k+s;\bar{\beta})$. Let $e_1, e_q, \dots, e_{2k+s}$ be the standard basis for $\mathfrak{m}_{0,s}^{q}(2k+s;\bar{\beta})$. By Lemma 3.11 its one-dimensional $\mathbb{N}$-graded filiform central extension can be defined by the following *canonical* basis:
$$e_1, e_q, \dots, e_{2k+2l-1}, e_{2k+2l}.$$
Since $\mathfrak{g}$ is a Lie algebra, the Jacobian $J(e_1, e_r, e_{2k+s-r}) = 0$, $r = q, \dots, k + \left[\frac{s}{2}\right]$. Equivalently,

(3.5)
$$\lambda_{1r}\lambda_{1+r,2k+s-r} + \lambda_{r,2k+s-r}\lambda_{2k+s,1} + \lambda_{2k+s-r,1}\lambda_{2k+s+1-r,r} = 0$$
where $\lambda_{1r} = 1$, $\lambda_{2k+s,1} = \lambda_{2k+s-r,1} = -1$. Therefore, it can be rewritten as

(3.6)
$$\lambda_{1+r,2k+s-r} + \lambda_{r,2k+s+1-r} = \lambda_{r,2k+s-r}.$$
Consider the following two cases.

Case 1: $s = 2l+1$. Then the right side of (3.6) is
$$\lambda_{r,2k+s-r} = (-1)^{k-r}\left(\binom{k-r+2l}{k-r} + \sum_{i=1}^{l}(-1)^i\binom{k-r+2l-i}{k-r+i}\beta_i\right),$$

https://doi.org/10.4153/CJM-2014-008-x Published online by Cambridge University Press

$r = q, \dots, k+l$. Since $\lambda_{k+l,k+l} = 0$, this system of linear equations has a unique solution:

(3.7)

$$\lambda_{r,2k+s+1-r} = (-1)^{k-r}\left(\binom{k-r+2l+1}{k-r} + \sum_{i=1}^{l}(-1)^i\binom{k-r+2l+1-i}{k-r+i}\beta_i\right),$$

$r = q, \dots, k+l$. These structure constants define $\mathrm{m}_{0,s+1}^q(2k+s+1;\bar\beta')$, where $\bar\beta' = \bar\beta$.

Case 2: $s = 2l$. The right side of (3.6) is

$$\lambda_{r,2k+s-r} = (-1)^{k-r}\left(\binom{k-r+2l-1}{k-r} + \sum_{i=1}^{l-1}(-1)^i\binom{k-r+2l-1-i}{k-r+i}\beta_i\right).$$

Introducing a new parameter $\beta_l = \lambda_{k+l,k+l+1}$, we obtain

$$\lambda_{r,2k+s+1-r} = (-1)^{k-r}\left(\binom{k-r+2l}{k-r} + \sum_{i=1}^{l}(-1)^i\binom{k-r+2l-i}{k-r+i}\beta_i\right),$$

which defines a Lie algebra of type $\mathrm{m}_{0,2k+s+1}^q(2k+s+1;\bar\beta')$, where $\bar\beta' = (\bar\beta, \beta_l)$. This proves the lemma. ∎

Proposition 3.21 *For any value of multiparameter* $\bar\beta = (\beta_1, \dots, \beta_{[\frac{s+1}{2}]-1})$, $s = 1, \dots, q$, $\mathrm{m}_{0,s}^q(2k+s;\bar\beta)$ *is a Lie algebra.*

(i) *If $s < q$ is odd, then $\mathrm{m}_{0,s}^q(2k+s;\bar\beta)$ has a unique $\mathbb{N}$-graded one-dimensional central extension which is $\mathrm{m}_{0,s+1}^q(2k+s+1;\bar\beta)$ with the same multiparameter $\bar\beta$.*

(ii) *If $0 < s < q$ is even, then $\mathrm{m}_{0,s}^q(2k+s;\bar\beta)$ has infinitely many non-isomorphic $\mathbb{N}$-graded one-dimensional central extensions. Each such extension is of the form $\mathrm{m}_{0,s+1}^q(2k+s+1;\bar\beta')$ with multiparameter $\bar\beta' = (\bar\beta, \beta_{[\frac{s+1}{2}]})$. Moreover, for different values of $\beta_{[\frac{s+1}{2}]}$, we obtain non-isomorphic central extensions.*

Proof Let us prove by induction on s that $\mathrm{m}_{0,s}^q(2k+s;\bar\beta)$, $s = 1, \dots, q$ is a Lie algebra. Lemma 3.17 is a basis for induction when $s = 1$. Assume that for some $s < q$ $\mathrm{m}_{0,s}^q(2k+s;\bar\beta)$ is a Lie algebra for any $\bar\beta$. Consider $\bar{\mathfrak{g}} = \mathrm{m}_{0,s+1}^q(2k+s+1;\bar\beta')$. It follows from Definition 3.19 that $\bar{\mathfrak{g}}$ is obtained from an appropriate $\mathrm{m}_{0,s}^q(2k+s;\bar\beta)$ by extending its standard basis and adding relations (3.3) or (3.4). Let $e_1, e_q, \dots, e_{2k+s}, e_{2k+s+1}$ be the standard basis for $\bar{\mathfrak{g}}$. By our inductive assumption, $J(e_i, e_j, e_k) = 0$ if $i+j+r \le 2k+s$. Hence, we only need to consider the case when $i+j+r = 2k+s+1$, $i < j < r$.

If $i \ge q$, then $j + r \le 2k + (s-q) + 1 \le 2k$, because $s < q$. Since

$$i + j < i + r < j + r \le 2k,$$

we have that $\lambda_{i,j} = \lambda_{i,r} = \lambda_{j,r} = 0$. Therefore,

$$J(e_i, e_j, e_k) = (\lambda_{i,j}\lambda_{i+j,k} + \lambda_{j,k}\lambda_{j+k,i} + \lambda_{k,i}\lambda_{k+i,j})e_{i+j+k} = 0.$$

If $i = 1$, then $j + r = 2k + s$. In this case, $J(e_1, e_j, e_{2k+s-j}) = 0$ is equivalent to (3.5), and as was already shown, (3.7) is a solution to (3.5). Consequently, $\mathrm{m}_{0,2k+s+1}^q(2k+s+1;\bar\beta')$ is a Lie algebra for any values of $\bar\beta'$.

On the other hand, by Lemma 3.20 any one-dimensional filiform central extension $\mathfrak{m}_{0,s}^{q}(2k+s;\bar\beta)$ must be of type $\mathfrak{m}_{0,s+1}^{q}(2k+s+1;\bar\beta')$. For an odd s it is unique, while for an even s there is one-parameter family of them.

Let s be a positive even integer such that $1 < s < q$. It remains to show that for different values of the parameter $\beta_{[\frac{s+1}{2}]}$ we obtain non-isomorphic $\mathbb{N}$-graded Lie algebras. For this, we consider two one-dimensional central extensions of $\mathfrak{m}_{0,s}^{q}(2k+s;\bar\beta)$ corresponding to different values of $\beta_{[\frac{s+1}{2}]}$:

$$\mathfrak{g}_1 = \operatorname{span}\{e_1, e_q, \dots, e_{2k+s}, e_{2k+s+1}\},$$
$$[e_r, e_{2k+s+1-r}] = (f_{r,s} + (-1)^{k-r+\frac{s}{2}}\beta)e_{2k+s+1},$$

where

$$f_{r,s} = (-1)^{k-r}\left(\binom{k-r+s}{k-r} + \sum_{i=1}^{s/2-1}(-1)^i\binom{k-r+s-i}{k-r+i}\beta_i\right),$$

β is a particular value of $\beta_{[\frac{s+1}{2}]}$, and

$$\mathfrak{g}_2 = \operatorname{span}\{e'_1, e'_q, \dots, e'_{2k+s}, e'_{2k+s+1}\},$$
$$[e'_r, e'_{2k+s+1-r}] = (f_{r,s} + (-1)^{k-r+\frac{s}{2}}\beta')e'_{2k+s+1},$$

where $f_{r,s}$ is as above and β' is another value of $\beta_{[\frac{s+1}{2}]}$ such that $\beta \neq \beta'$. Notice that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ have the same structure constants λ_{ij} whenever $i+j \le 2k+s$. Let us now assume that $\mathfrak{g}_1 \cong \mathfrak{g}_2$ as $\mathbb{N}$-graded algebras. This means that there exists a graded isomorphism $\varphi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ defined by $\varphi(e_i) = \alpha_i e'_i$, where $i = 1, q, \dots, 2k+s+1$. Clearly, every α_i is a nonzero scalar. Then $\varphi([e_1, e_i]) = [\varphi(e_1), \varphi(e_i)]$, $i = 1, q, \dots, 2k+s$. Hence, $\alpha_{i+1} = \alpha_1 \cdot \alpha_i$, $i = q, q+1, \dots, 2k+s$, which means that

$$\alpha_i = \alpha_1^{i-q} \cdot \alpha_q, \tag{3.8}$$

where $i = q+1, \dots, 2k+s+1$. Next, we can always choose $i, j > 1$, $2k < i+j \le 2k+s$ such that $\lambda_{ij} \neq 0$. Then $\varphi([e_i, e_j]) = [\varphi(e_i), \varphi(e_j)]$, $\lambda_{ij}\alpha_{i+j} = \lambda_{ij}\alpha_i\alpha_j$, $\alpha_{i+j} = \alpha_i\alpha_j$. Using (3.8) we get $\alpha_q = \alpha_1^q$. It follows from multiplication tables of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ that for $r_0 = k + \frac{s}{2}$ we have

$$[e_{r_0}, e_{2k+s+1-r_0}] = \beta e_{2k+s+1}, \quad [e'_{r_0}, e'_{2k+s+1-r_0}] = \beta' e'_{2k+s+1}.$$

Therefore, $\varphi([e_{r_0}, e_{2k+s+1-r_0}]) = [\varphi(e_{r_0}), \varphi(e_{2k+s+1-r_0})]$, which means that

$$\beta\alpha_{2k+s+1} = \beta'\alpha_{r_0}\alpha_{2k+s+1-r_0}.$$

Using (3.8) we obtain $\beta\alpha_1^{2k+s+1} = \beta'\alpha_1^{2k+s+1}$, hence $\beta = \beta'$, which contradicts our original assumption. Thus, φ is not an isomorphism. The proof is complete. ∎

Remark 3.22 In order to simplify notation for the s-th central extension of $\mathfrak{m}_0^q(2k)$ we will omit $\bar\beta$ in $\mathfrak{m}_{0,2k+s}^{q}(2k+s;\bar\beta)$ whenever the value of $\bar\beta$ is not important and denote it by $\mathfrak{m}_{0,2k+s}^{q}(2k+s)$.

We next focus on studying m-th filiform central extensions of $\mathfrak{m}_0^q(2k)$ where $m > q$.

Lemma 3.23 *Let $k > q$, and let* $\mathfrak{g} = \mathfrak{m}^q_{0,q+s+1}(2k+q+s+1)$, $s \geq 1$, *be a Lie algebra. If* $\lambda_{q,2k+s} = 0$, *then either* $\lambda_{q,2k+s+1} = 0$ *or* $\lambda_{q+1,2k+s-q} + \lambda_{q,2k+s-q} = 0$.

Proof Since $k > q$, $2k > 2q + 1 = q + (q + 1)$. It follows from Definition 3.19 that the product $[e_q, e_{q+1}]$ in $\mathfrak{g}$ must be the same as in $\mathfrak{m}^q_0(2k)$. Therefore, $[e_q, e_{q+1}] = \lambda_{q,q+1}e_{2q+1} = 0$. Since $\mathfrak{m}^q_{0,q+s+1}(2k + q + s + 1)$ is a Lie algebra, we have that $J(e_q, e_{q+1}, e_{2k+s-q}) = 0$. Equivalently,

$$\lambda_{q+1,2k+s-q}\lambda_{2k+s+1,q} + \lambda_{2k+s-q,q}\lambda_{2k+s,q+1} = 0.$$

By Leibniz rule for derivations (see the beginning of Subsection 3.3),

$$\lambda_{q+1,2k+s} + \lambda_{q,2k+s+1} = \lambda_{q,2k+s}.$$

Hence, $\lambda_{q+1,2k+s} = \lambda_{q,2k+s} - \lambda_{q,2k+s+1} = -\lambda_{q,2k+s+1}$. Thus, the above equation takes the form

$$(\lambda_{q+1,2k+s-q} + \lambda_{q,2k+s-q})\lambda_{q,2k+s+1} = 0.$$

Hence, either $\lambda_{q+1,2k+s-q} + \lambda_{q,2k+s-q} = 0$ or $\lambda_{q,2k+s+1} = 0$ as required. ■

Lemma 3.24 *Let $k > q$. If both* $\mathfrak{m}^q_{0,q+1}(2k + q + 1)$ *and* $\mathfrak{m}^q_{0,q+2}(2k + q + 2)$ *are Lie algebras, then*

(i) $\lambda_{q,2k+1} = 0$,
(ii) $\lambda_{q,2k+2} = 0$ *if* $k > q + 1$.

Proof Since $k > q$, $2k > 2q + 1 = q + (q + 1)$, and similarly to Lemma 3.23 we can show that $[e_q, e_{q+1}] = \lambda_{q,q+1}e_{2q+1} = 0$.

(i) By our assumption, $\mathfrak{m}^q_{0,q+1}(2k+q+1)$ is a Lie algebra. Thus, $J(e_q, e_{q+1}, e_{2k-q}) = 0$. Equivalently,

$$\lambda_{q+1,2k-q}\lambda_{2k+1,q} + \lambda_{2k-q,q}\lambda_{2k,q+1} = 0.$$

It follows from the multiplication table of $\mathfrak{m}^q_{0,q+1}(2k + q + 1)$ that $\lambda_{2k-q,q} = 0$ and $\lambda_{q+1,2k-q} = (-1)^{q+1-k} \neq 0$. Therefore, $\lambda_{q,2k+1} = 0$. Notice that $\lambda_{q,2k+1} = 0$ in $\mathfrak{m}^q_{0,q+2}(2k + q + 2)$ as well.

(ii) Since $\lambda_{q,2k+1} = 0$ we can use Lemma 3.23 for $s = 1$. Hence, either $\lambda_{q,2k+2} = 0$ or $\lambda_{q+1,2k+1-q} + \lambda_{q,2k+1-q} = 0$. Recall that $\lambda_{q+1,2k+1-q} = (-1)^{q+1-k}(k - q)$ and $\lambda_{q,2k+1-q} = (-1)^{q-k}$. Hence, if $k > q + 1$, then $\lambda_{q+1,2k+1-q} + \lambda_{q,2k+1-q} \neq 0$. Thus, $\lambda_{q,2k+2} = 0$, as required. ■

3.3 Proof of Theorem 3.9

Let $\mathfrak{g} = \operatorname{span}\{e_1, \dots, e_n\}$ be an $\mathbb{N}$-graded Lie algebra such that

$$[e_1, e_i] = e_{i+1}, \quad i = 1, \dots, n-1, \quad [e_i, e_j] = \lambda_{ij}e_{i+j}, \quad i, j > 1.$$

The Leibniz rule for derivation $\operatorname{ad}(e_1)$ yields

$$\lambda_{ij} = \lambda_{i+1,j} + \lambda_{i,j+1}. \tag{3.9}$$

Lemma 3.25 *Let $k > 2q$. If* $\mathfrak{g} = \mathfrak{m}^q_{0,2q}(2k + 2q)$ *is a Lie algebra, then we have*

$$\lambda_{2q-1,2k+1} = \lambda_{2q-2,2k+2} = \cdots = \lambda_{q,2k+q} = 0.$$

https://doi.org/10.4153/CJM-2014-008-x Published online by Cambridge University Press

Proof By the previous lemma we have that $\lambda_{q,2k+1} = \lambda_{q,2k+2} = 0$, since $k \neq q+1$.

Since $k > 2q$ we have that

$$\lambda_{q,q+1} = \lambda_{q+1,q+2} = \cdots = \lambda_{2q-1,2q} = 0.$$

Indeed, since $i+(i+1) = 2i+1 < 2k$, where $i = q, \dots, 2q-1$, all products $[e_i, e_{i+1}] = \lambda_{i,i+1} e_{2i+1}$ in $\mathfrak{g}$ must be the same as in $\mathfrak{m}_0^q(2k)$. Therefore, $[e_i, e_{i+1}] = \lambda_{i,i+1} e_{2i+1} = 0$, where $i = q, \dots, 2q-1$.

Let us now show that $\lambda_{q+2,2k+1} = \lambda_{q+1,2k+2} = \lambda_{q,2k+3} = 0$. Indeed,

$$J\big(e_{q+1}, e_{q+2}, e_{2k-q}\big) = 0.$$

Since $\lambda_{q+1,q+2} = 0$, we have that

$$\lambda_{q+2,2k-q}\lambda_{2k+2,q+1} + \lambda_{2k-q,q+1}\lambda_{2k+1,q+2} = 0,$$

where $\lambda_{q+2,2k-q} = (-1)^{q+2-k}(k-1-q)$ and $\lambda_{q+1,2k-q} = (-1)^{q+1-k}$. Hence,

$$(k-1-q)\lambda_{q+1,2k+2} + \lambda_{q+2,2k+1} = 0.$$

Using relation (3.9), we have

$$\lambda_{q+1,2k+2} + \lambda_{q+2,2k+1} = \lambda_{q+1,2k+1},$$

where $\lambda_{q+1,2k+1} = \lambda_{q,2k+1} - \lambda_{q,2k+2} = 0$ (by (3.9)).

Since $k > 2q$, $k-1-q \neq 1$, we have that

$$\lambda_{q+1,2k+2} = \lambda_{q+2,2k+1} = 0.$$

Finally, $\lambda_{q+1,2k+2} = \lambda_{q,2k+2} - \lambda_{q,2k+3}$ (by (3.9)). Hence, $\lambda_{q,2k+3} = 0$.

Let us now use induction on s. Assume that

$$\lambda_{q,2k+3} = \cdots = \lambda_{q,2k+s} = 0,$$
$$\lambda_{q+s-1,2k+1} = \lambda_{q+s-2,2k+2} = \cdots = \lambda_{q+1,2k+s-1} = 0,$$

where $2 \leq s < q$.

We know that

$$q+s-1 < q+s < 2k+2-q-s.$$

Besides,

$$(q+s-1) + (q+s) = 2q+2s-1 \leq 4q-1 < 2k.$$

Thus, $\lambda_{q+s-1,q+s} = 0$. Therefore, $J(e_{q+s-1}, e_{q+s}, e_{2k+2-q-s}) = 0$ is equivalent to

$$\lambda_{q+s,2k+2-q-s}\lambda_{2k+2,q+s-1} + \lambda_{2k+2-q-s,q+s-1}\lambda_{2k+1,q+s} = 0.$$

Since $\lambda_{q+s,2k+2-q-s} = (-1)^{q+s-k}(k+1-q-s)$ and $\lambda_{q+s-1,2k+2-q-s} = (-1)^{q+s-1-k}$, we have that

$$(k+1-q-s)\lambda_{q+s-1,2k+2} + \lambda_{q+s,2k+1} = 0.$$

By relation (3.9),

$$\lambda_{q+s-1,2k+2} + \lambda_{q+s,2k+1} = \lambda_{q+s-1,2k+1},$$

where $\lambda_{q+s-1,2k+1} = 0$ by inductive assumption. Since $k > 2q$, we have that $k+1-q-s \neq 1$ and $\lambda_{q+s-1,2k+2} = \lambda_{q+s,2k+1} = 0$. Next $\lambda_{q+s-1,2k+2} = \lambda_{q+s-2,2k+2} - \lambda_{q+s-2,2k+3}$. Also, $\lambda_{q+s-2,2k+2} = 0$ by inductive assumption. Thus, $\lambda_{q+s-2,2k+3} = 0$.

Likewise, $\lambda_{q+s-2,2k+3} = \lambda_{q+s-3,2k+3} - \lambda_{q+s-3,2k+4}$. Thus, $\lambda_{q+s-3,2k+4} = 0$. After a finite number of steps we get the following:

$$\lambda_{q+1,2k+s} = \lambda_{q,2k+s} - \lambda_{q,2k+s+1}.$$

Since $\lambda_{q+1,2k+s} = \lambda_{q,2k+s} = 0$, we have that $\lambda_{q,2k+s+1} = 0$, as required. Therefore,

$$\lambda_{q+s,2k+1} = \lambda_{q+s-1,2k+2} = \cdots = \lambda_{q,2k+s+1} = 0.$$

Finally, for $s = q - 1$ we obtain

$$\lambda_{2q-1,2k+1} = \lambda_{2q-2,2k+2} = \cdots = \lambda_{q,2k+q} = 0,$$

as required. The proof is complete. ■

Lemma 3.26 *Let $k > 2q$. Then $\mathfrak{g} = \mathfrak{m}^q_{0,2q-1}(2k + 2q - 1; \bar\beta)$, where $\bar\beta = (\beta_1, \ldots, \beta_{q-1})$, has no one-dimensional $\mathbb{N}$-graded filiform central extensions.*

Proof Assume that such a central extension exists. Then it must be of type $\mathfrak{m}^q_{0,2q}(2k + 2q; \bar\beta)$ and

$$[e_r, e_{2k+2q-r}] = (-1)^{k-r}\left(\binom{k-r+2q-1}{k-r} + \sum_{i=1}^{q-1}(-1)^i\binom{k-r+2q-1-i}{k-r+i}\beta_i\right)e_{2k+2q},$$

where $r = q, \ldots, k + q$ and the remaining products are the same as in $\mathfrak{m}^q_{0,2q-1}(2k + 2q - 1; \bar\beta)$. Since $k > 2q$, we can apply Lemma 3.25. Hence, we have that

$$\lambda_{q,2k+q} = \lambda_{q+1,2k+q-1} = \cdots = \lambda_{2q-2,2k+2} = \lambda_{2q-1,2k+1} = 0.$$

Equivalently, we have q linear equations:

$$\binom{k+q-1}{k-q} + \sum_{i=1}^{q-1}(-1)^i\binom{k+q-1-i}{k-q+i}\beta_i = 0$$

$$\binom{k+q-2}{k-q-1} + \sum_{i=1}^{q-1}(-1)^i\binom{k+q-2-i}{k-q-1+i}\beta_i = 0$$

$$\vdots$$

$$\binom{k}{k-2q+1} + \sum_{i=1}^{q-1}(-1)^i\binom{k-i}{k-2q+1+i}\beta_i = 0$$

Consider the following matrix:

$$A = \begin{pmatrix} \binom{k}{k-1} & \binom{k+1}{k-2} & \cdots & \binom{k+q-2}{k-q+1} & \binom{k+q-1}{k-q} \\ \binom{k-1}{k-2} & \binom{k}{k-3} & \cdots & \binom{k+q-3}{k-q} & \binom{k+q-2}{k-q-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \binom{k-q+1}{k-q} & \binom{k-q+2}{k-q-1} & \cdots & \binom{k-1}{k-2q+2} & \binom{k}{k-2q+1} \end{pmatrix}.$$

Dividing each row of A by its first entry (which is, of course, nonzero) we obtain the following matrix:

$$B = \begin{pmatrix} 1 & f_1(x_0) & f_2(x_0) & \dots & f_{q-1}(x_0) \\ 1 & f_1(x_1) & f_2(x_1) & \dots & f_{q-1}(x_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & f_1(x_{q-1}) & f_2(x_{q-1}) & \dots & f_{q-1}(x_{q-1}) \end{pmatrix},$$

where

$$f_1(x) = x(x+2),$$
$$f_2(x) = (x-1)x(x+2)(x+3), \dots, f_{q-1}(x) = (x-q+2)\cdots x(x+2)\cdots(x+q)$$

and

$$x_0 = k-1, \quad x_1 = k-2, \dots, x_{q-1} = k-q.$$

We next want to prove that B is a nonsingular matrix. Let

$$y = f_1(x),\ g_2(y) = y(y-3), \dots, g_{q-1}(y) = y(y-3)\cdots(y+2q-q^2).$$

Then

$$B = \begin{pmatrix} 1 & y_0 & g_2(y_0) & \dots & g_{q-1}(y_0) \\ 1 & y_1 & g_2(y_1) & \dots & g_{q-1}(y_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & y_{q-1} & g_2(y_{q-1}) & \dots & g_{q-1}(y_{q-1}) \end{pmatrix},$$

where

$$y_0 = x_0^2 + 2x_0, \quad y_1 = x_1^2 + 2x_1, \dots, y_{q-1} = x_{q-1}^2 + 2x_{q-1}.$$

Note that $\deg g_i = i$, $i = 2, \dots, q-1$. According to the statement on [SZ, p. 319], B is a nonsingular matrix. Therefore, A is also non-singular. This implies that the above system of linear equations is inconsistent. Hence, $\mathfrak{m}_{0,2q}^q(2k+2q)$ is not a Lie algebra. This implies that $\mathfrak{g}$ has no one-dimensional $\mathbb{N}$-graded filiform central extensions. The proof is complete. ∎

Let us now finish the proof of Theorem 3.9. Consider $\mathfrak{g} = \bigoplus_{i=1,q}^{\infty} \mathfrak{g}_i$ generated by both $\mathfrak{g}_1$ and $\mathfrak{g}_q$ that satisfies

$$[\mathfrak{g}_q, \mathfrak{g}_{q+1}] = [\mathfrak{g}_{q+1}, \mathfrak{g}_{q+2}] = \cdots = [\mathfrak{g}_{2q}, \mathfrak{g}_{2q+1}] = 0. \tag{3.10}$$

By Corollary 3.13, we can choose a basis for $\mathfrak{g} : \{e_1, e_q, e_{q+1}, \dots\}$ where $[e_1, e_i] = e_{i+1}$, and $\mathfrak{g}_i = \operatorname{span}\{e_i\}$, $i > 1$. Hence, condition (3.10) is equivalent to

$$\lambda_{q,q+1} = \lambda_{q+1,q+2} = \cdots = \lambda_{2q,2q+1} = 0,$$

where λ_{ij} are corresponding structure constants. Set $r = 2q$. By Corollary 3.12, $\mathfrak{g}(r)$ is isomorphic to $\mathfrak{m}_0^q(r)$. By Lemma 3.15 and condition (3.10) we have that

$$\mathfrak{g}(r+1) \cong \mathfrak{m}_0^q(r+1),$$
$$\mathfrak{g}(r+2) \cong \mathfrak{m}_0^q(r+2),$$
$$\vdots$$
$$\mathfrak{g}(r+2q+2) \cong \mathfrak{m}_0^q(r+2q+2).$$

Note that $\mathrm{m}_0^q(r+2q+2) = \mathrm{m}_0^q(4q+2) = \mathrm{m}_0^q(2(2q+1)) = \mathrm{m}_0^q(2k)$, where $k = 2q+1$. Thus, g is obtained by taking one-dimensional central extensions of $\mathrm{m}_0^q(2k)$, where $k > 2q$. If g is not of type m_0^q, then by taking one-dimensional central extensions at some point we obtain $\mathrm{m}_{0,1}^q(2l+1)$ where $l > 2q$. Then the $(2q-1)$-st filiform central extension will be of type $\mathrm{m}_{0,2q-1}^q(2l+2q-1)$, $l > 2q$, and by Lemma 3.26 it has no required central extensions, a contradiction. Therefore, g must be of type m_0^q. The proof is complete. ■

3.4 The Case of $q = 3$

The purpose of this section is to prove the following theorem.

Theorem 3.27 *Let* $\mathfrak{g} = \bigoplus_{i\in\mathbb{N}} \mathfrak{g}_i$ *be an infinite-dimensional* $\mathbb{N}$*-graded Lie algebra of maximal class and suppose* $\mathfrak{g} = \langle \mathfrak{g}_1, \mathfrak{g}_3\rangle$. *Then one of the following holds:*

(i) $\mathfrak{g} \cong \mathrm{m}_0^3$;
(ii) $\mathfrak{g} \cong \mathrm{m}_3$;
(iii) $\mathfrak{g} \cong W^3$.

The proof of this theorem will follow from the series of lemmas below.

Let g be an $\mathbb{N}$-graded Lie algebra of maximal class generated by both $\mathfrak{g}_1$ and $\mathfrak{g}_3$ (not generated by $\mathfrak{g}_1$ or $\mathfrak{g}_3$ only). Hence, it has the following $\mathbb{N}$-grading: $\mathfrak{g} = \bigoplus_{i=1,3}^{\infty} \mathfrak{g}_i$. At the beginning of Section 3 we introduced Lie algebras of types m_0^q, m_q, and W^q. For $q = 3$ we will show that these are the only $\mathbb{N}$-graded Lie algebras of maximal class generated by $\mathfrak{g}_1$ and $\mathfrak{g}_3$.

Recall that, by definition, $\mathrm{m}_3(l) = \mathrm{span}\{e_1, e_3, \dots, e_l\}$ such that $[e_1, e_i] = e_{1+i}$, $i = 3, \dots, l-1$ and $[e_3, e_j] = e_{3+j}$, $j = 4, \dots, l-3$. We can assume that $l \geq 7$ and write $l = 6 + n$, where $n \geq 1$.

Lemma 3.28 *Let* $\mathfrak{g}$ *be a filiform Lie algebra of type* $\mathrm{m}_3(6+n)$, *where* $n \geq 1$. *Then* $\mathfrak{g}$ *is isomorphic to* $\mathrm{m}_{0,n}^3(6+n;\bar\beta)$ *(n-th filiform central extension of* $\mathrm{m}_0^3(6)$*), where* $\bar\beta = (0,\dots,0)$.

Proof Let us compare the multiplication tables of both algebras. First, $\mathrm{m}_{0,n}^3(6+n;\bar\beta) = \mathrm{span}\{e_1, e_3, \dots, e_{6+n}\}$ where $\bar\beta = (0,\dots,0)$ has the following multiplication table:

$$[e_r, e_{6+l-r}] = (-1)^{3-r}\binom{3-r+l-1}{3-r} e_{6+l}, \quad r = 3,\dots,3+l/2, \quad l = 1,\dots,n.$$

If $r = 3$, then the above binomial coefficient is 1. If $r > 3$, then it is zero. Therefore, the multiplication table of $\mathrm{m}_{0,n}^3(6+n;\bar\beta)$ for $\bar\beta = (0,\dots,0)$ is exactly the same as that of $\mathrm{m}_3(6+n)$. Hence, they are isomorphic. ■

Lemma 3.29 *If* $n = 2l+1$, *then* $\mathrm{m}_3(6+2l+1)$ *has a unique one-dimensional* $\mathbb{N}$*-graded filiform central extension that is* $\mathrm{m}_3(6+2l+2)$. *If* $n = 2l$, *then* $\mathrm{m}_3(6+2l)$ *has a one-parameter family of* $\mathbb{N}$*-graded filiform central extensions of type* $\mathrm{m}_{0,2l+1}^3(6+2l+1;\bar\beta')$ *where* $\bar\beta' = (0,\dots,0,\beta_l)$, β_l *is a scalar.*

Proof If $n = 2l + 1$, then $\mathfrak{m}_3(6 + 2l + 1)$ is isomorphic to $\mathfrak{m}^3_{0,2l+1}(6 + 2l + 1; \bar{\beta})$, $\bar{\beta} = (0, \dots, 0)$. By Lemma 3.20, it has a unique one-dimensional $\mathbb{N}$-graded filiform central extension that is $\mathfrak{m}^3_{0,2l+2}(6 + 2l + 2; \bar{\beta})$, $\bar{\beta} = (0, \dots, 0)$ and, therefore, is isomorphic to $\mathfrak{m}_3(6+2l+2)$. If $n = 2l$, then $\mathfrak{m}_3(6+2l)$ is isomorphic to $\mathfrak{m}^3_{0,2l}(6+2l; \bar{\beta})$, $\bar{\beta} = (0, \dots, 0)$. By Lemma 3.20, it has a one-parameter family of the required central extensions $\mathfrak{m}^3_{0,2l+1}(6 + 2l + 1; \bar{\beta}')$, $\bar{\beta}' = (0, \dots, 0, \beta_l)$. ■

Lemma 3.30 *Let $\mathfrak{g} = \mathfrak{m}^3_{0,2l+1}(6 + 2l + 1; \bar{\beta}')$ (where $l \geq 3$, $\bar{\beta}' = (0, \dots, 0, \beta_l)$, $\beta_l \neq 0$) be a Lie algebra. Then it has no one-dimensional $\mathbb{N}$-graded filiform central extensions.*

Proof Assume the contrary, that is, a one-dimensional $\mathbb{N}$-graded filiform central extension of $\mathfrak{g}$ exists. Then by Lemma 3.20 it must be $\mathfrak{m}^3_{0,2l+2}(6 + 2l + 2; \bar{\beta}')$, where $\bar{\beta}' = (0, \dots, 0, \beta_l)$, and

$$[e_r, e_{6+2l+2-r}] = (-1)^{3-r}\left(\binom{3-r+2l+1}{3-r} + (-1)^l \binom{3-r+2l+1-l}{3-r+l}\beta_l \right) e_{8+2l}.$$

The corresponding structure constants are

$$\lambda_{r,8+2l-r} = (-1)^{3-r}\Big(\binom{3-r+2l+1}{3-r} + (-1)^l(3-r+l+1)\beta_l \Big).$$

This means that if $r = 3$, then $\lambda_{3,5+2l} = 1 + (-1)^l(l+1)\beta_l$. If $r > 3$, then $\lambda_{r,8+2l-r} = (-1)^{3-r+l}(3-r+l+1)\beta_l$. Since $\mathfrak{m}^3_{0,2l+2}(8 + 2l; \bar{\beta}')$ is a Lie algebra, $J(e_3, e_j, e_k) = 0$ where $3 + j + k = 8 + 2l$ and $3 < j < k$. In terms of structure constants this can be rewritten as:

$$\lambda_{3,j}\lambda_{3+j,k} + \lambda_{j,k}\lambda_{5+2l,3} + \lambda_{k,3}\lambda_{k+3,j} = 0.$$

Note that

$$3 + j < 3 + k < j + k = 5 + 2l < 6 + 2l.$$

Hence, the Lie products $[e_3, e_j]$, $[e_3, e_k]$, and $[e_j, e_k]$ are the same as in $\mathfrak{m}^3_{0,2l}(6+2l; \bar{\beta})$, where $\bar{\beta} = (0, \dots, 0)$, which is isomorphic to $\mathfrak{m}_3(6 + 2l)$. Thus, $\lambda_{3,j} = \lambda_{3,k} = 1$ and $\lambda_{j,k} = 0$, since both $j, k > 3$. The above equation takes the form $\lambda_{k,3+j} = \lambda_{j,k+3}$. Since $\beta_l \neq 0$, $(-1)^{3-k+l}(3-k+l+1) = (-1)^{3-j+l}(3-j+l+1)$. Since $j + k = 5 + 2l$, an odd number, one of j, k is even while the other one is odd. Therefore, $(3-k+l+1) = -(3-j+l+1)$, $k + j = 8 + 2l$, a contradiction. The lemma is proved. ■

Remark 3.31 It follows from the above lemmas that $\mathfrak{m}_3$ is the only infinite-dimensional Lie algebra that can be obtained by taking one-dimensional $\mathbb{N}$-graded filiform central extensions of $\mathfrak{m}_3(n)$, $n \geq 12$.

Next we will be interested in one-dimensional $\mathbb{N}$-graded filiform central extensions of $W^3(n)$. Recall that $W^q(n)$ is defined by its basis $\{e_1, e_q, \dots, e_n\}$ and relations $[e_i, e_j] = (j - i)e_{i+j}$ whenever $i + j \leq n$, and the products equal to 0, otherwise. By setting $y_1 = e_1$ and $y_i = 60(i-2)!e_i$ $q \leq i \leq n$, we have that $[y_1, y_i] = y_{i+1}$,

$i = q, \dots, n-1$, and $[y_i, y_j] = \lambda_{i,j} y_{i+j}$, $i + j \le n$, where

$$\lambda_{i,j} = \frac{60(i-2)!(j-2)!(j-i)}{(i+j-2)!}. \tag{3.11}$$

Lemma 3.32 *A Lie algebra of type $W^3(n)$, $n \ge 14$, has a unique one-dimensional $\mathbb{N}$-graded filiform central extension that is isomorphic to $W^3(n+1)$.*

Proof First of all, we note that $W^3(n+1)$ is a required central extension of $W^3(n)$. We only need to show that it is unique. Choose a basis for $W^3(n)$: $\{e_1, e_3, \dots, e_n\}$ with $[e_1, e_i] = e_{i+1}$, $i = 1, \dots, n-1$, and $[e_i, e_j] = \lambda_{i,j} e_{i+j}$, $i + j \le n$. If we set $I = \operatorname{span}\{e_7, \dots e_n\}$, then I is an ideal of $W^3(n)$ and $W^3(n)/I$ is isomorphic to $\mathrm{m}_0^3(6)$. Therefore, $W^3(n)$ is obtained from $\mathrm{m}_0^3(6)$ by taking one-dimensional $\mathbb{N}$-graded filiform central extensions. Let $n = 6 + r$, where $r \ge 1$. Then $W^3(6+r) \cong \mathrm{m}_{0,r}^3(6+r; \bar\beta)$, where $\bar\beta = (\beta_1, \dots, \beta_l)$, $l = \left[\frac{r+1}{2}\right] - 1$. If r is odd, then by Lemma 3.20 it has a unique one-dimensional $\mathbb{N}$-graded filiform central extension. If r is even, then by Lemma 3.20 a one-dimensional $\mathbb{N}$-graded filiform central extension of $\mathrm{m}_{0,r}^3(6+r; \bar\beta)$ is $\mathrm{m}_{0,r+1}^3(6+r+1; \bar\beta')$, where $\bar\beta' = (\beta_1, \dots, \beta_l, \beta_{l+1})$. Set $\beta_{l+1} = t$. To show the uniqueness we express t in terms of structure constants of $\mathrm{m}_{0,r}^3(6+r; \bar\beta)$ that are all known. As follows from (3.3), we have that

$$[e_s, e_{7+r-s}] = (-1)^{3-s}\binom{3-s+r}{3-s} + \sum_{i=1}^{l} (-1)^{3+i-s}\binom{3-s+r-i}{3-s+i}\beta_i + (-1)^{3-s+l} t.$$

Hence,

$$\lambda_{s,7+r-s} = (-1)^{3-s+l} t + A_s, \tag{3.12}$$

where A_s depends on parameters of $\mathrm{m}_{0,r}^3(6+r; \bar\beta)$, $s = 3, \dots, r/3 + 3$. Since $\mathrm{m}_{0,r+1}^3(6+r+1; \bar\beta')$ must be a Lie algebra, the Jacobi identity $J(e_3, e_4, e_r) = 0$ yields

$$\lambda_{3,4}\lambda_{7,r} + \lambda_{4,r}\lambda_{4+r,3} + \lambda_{r,3}\lambda_{r+3,4} = 0, \tag{3.13}$$

where $\lambda_{3,4} = 1$, and $\lambda_{4,r}$, $\lambda_{r,3}$ are known, since $3 + r < 4 + r < 7 + r$. Applying (3.12), $\lambda_{7,r} = (-1)^{l-4} t + A_7$, $\lambda_{3,4+r} = (-1)^l t + A_3$ and $\lambda_{4,r+3} = (-1)^{l-1} t + A_4$. Therefore, (3.13) gives rise to a linear equation in t. Applying (3.11), the coefficient of t in this equation is

$$1 + \lambda_{r,3} - \lambda_{4,r} = 1 + 60\frac{14 - r - r^2}{(r-1)r(r+1)(r+2)}.$$

When $r \ge 8$ it is nonzero. Thus, t is uniquely determined from (3.13). This completes the proof. ∎

Remark 3.33 It follows from the above lemmas that W^3 is the only infinite-dimensional Lie algebra that can be obtained by taking one-dimensional $\mathbb{N}$-graded filiform central extensions of $W^3(n)$, $n \ge 14$.

Remark 3.34 As follows from Lemma 3.26, m_0^3 is the only infinite-dimensional Lie algebra that can be obtained by considering one-dimensional $\mathbb{N}$-graded filiform central extensions of $\mathrm{m}_0^3(2k)$ with $k > 6$.

Further we have to consider the remaining cases: $k = 3, 4, 5$ or 6. We first deal with cases when $k = 4, 5,$ or 6.

Lemma 3.35 *Let $k = 4$. Then m_0^3 is the only infinite-dimensional Lie algebra obtained by taking one-dimensional $\mathbb{N}$-graded filiform central extensions of* $\mathrm{m}_0^3(8)$.

Proof Consider $\mathrm{g} = \mathrm{m}_0^3(8)$. By Lemma 3.15 its one-dimensional $\mathbb{N}$-graded filiform central extension is either $\mathrm{m}_0^3(9)$ or $\mathrm{m}_{0,1}^3(9)$. If it is $\mathrm{m}_0^3(9)$, then applying the same lemma again, its one-dimensional $\mathbb{N}$-graded filiform central extension must be of type $\mathrm{m}_0^3(10) = \mathrm{m}_0^3(2 \cdot 5)$, which leads to the case $k = 5$, which will be covered in the next lemma. Without loss of generality we assume that the second possibility holds. It follows from Lemma 3.24(i), that $\lambda_{3,9} = 0$ in $\mathrm{m}_{0,4}^3(12)$. Using formulas (3.3) and (3.4), we obtain

$$\lambda_{3,9} = (-1)^{3-4}\left(\binom{4-3+3}{4-3} - \binom{4-3+2}{4-3+1}\beta_1 \right) = 0.$$

Hence, $\beta_1 = \frac{4}{3}$. Next the Jacobi identity $J(e_3, e_4, e_7) = 0$ in $\mathrm{m}_{0,6}^3(14)$ is equivalent to

$$\lambda_{3,4}\lambda_{7,7} + \lambda_{4,7}\lambda_{11,3} + \lambda_{7,3}\lambda_{10,4} = 0, \tag{3.14}$$

where

$$\lambda_{7,7} = 0, \quad \lambda_{4,7} = (-1)^{4-4}\left(\binom{4-4+2}{4-4} - \beta_1 \right) = -\frac{1}{3}, \quad \lambda_{3,7} = -2,$$

$$\lambda_{3,11} = -\binom{6}{1} + \binom{5}{2}\beta_1 - \binom{4}{3}\beta_2 = -6 + 10\beta_1 - 4\beta_2,$$

$$\lambda_{4,10} = \binom{5}{0} - \binom{4}{1}\beta_1 + \binom{3}{2}\beta_2 = 1 - 4\beta_1 + 3\beta_2.$$

Hence, (3.14) gives rise to $-\frac{1}{3}(6-10\beta_1+4\beta_2)+2(-1+4\beta_1-3\beta_2) = 0$. Thus, $\beta_2 = \frac{50}{33}$. In $\mathrm{m}_{0,7}^3(15)$ we consider the Jacobi identity $J(e_3, e_4, e_8) = 0$, which is equivalent to

$$-\lambda_{4,8}\lambda_{3,12} + \lambda_{3,8}\lambda_{4,11} = 0, \tag{3.15}$$

where $\lambda_{3,8} = -\frac{5}{3}$, $\lambda_{4,8} = -\frac{5}{3}$, $\lambda_{4,11} = 1-5\beta_1+6\beta_2-\beta_3$, $\lambda_{3,12} = -7+15\beta_1-10\beta_2+\beta_3$. Then (3.15) implies that $\beta_3 = \frac{92}{33}$. In $\mathrm{m}_{0,8}^3(16)$ we consider the Jacobi identity $J(e_3, e_5, e_8) = 0$, which is equivalent to

$$-\lambda_{5,8}\lambda_{3,13} + \lambda_{3,8}\lambda_{5,11} = 0, \tag{3.16}$$

where $\lambda_{5,8} = -\frac{2}{11}$, $\lambda_{3,13} = -8 + 21\beta_1 - 20\beta_2 + 5\beta_3 = \frac{40}{11}$, $\lambda_{5,11} = \frac{40}{11}$. Substituting these values into (3.16) we obtain that the left side of it is nonzero, a contradiction.

The proof is complete. ∎

Lemma 3.36 *Let $k = 5$. Then m_0^3 is the only infinite-dimensional Lie algebra obtained by taking one-dimensional $\mathbb{N}$-graded filiform central extensions of* $\mathrm{m}_0^3(10)$.

Proof Consider $\mathrm{g} = \mathrm{m}_0^3(10)$. By Lemma 3.15, its one-dimensional $\mathbb{N}$-graded filiform central extension is either $\mathrm{m}_0^3(11)$ or $\mathrm{m}_{0,1}^3(11)$. If it is $\mathrm{m}_0^3(11)$, then applying the same lemma again, its one-dimensional $\mathbb{N}$-graded filiform central extension must be of type $\mathrm{m}_0^3(12) = \mathrm{m}_0^3(2 \cdot 6)$, which leads to the case $k = 6$ covered in the next lemma.

Without loss of generality we assume that the second possibility holds. It follows from Lemma 3.24, that $\lambda_{3,11} = 0$ in $\mathfrak{m}^3_{0,4}(14)$ and $\lambda_{3,12} = 0$ in $\mathfrak{m}^3_{0,5}(15)$. Using formulas (3.3), (3.4), we obtain

$$\lambda_{3,11} = (-1)^{3-5}\left(\binom{5-3+3}{5-3} - \binom{5-3+2}{5-3+1}\beta_1 \right) = 10 - 4\beta_1 = 0.$$

Hence, $\beta_1 = \frac{5}{2}$. Next,

$$\lambda_{3,12} = (-1)^{3-5}\left(\binom{5-3+4}{5-3} - \binom{5-3+3}{5-3+1}\beta_1 + \beta_2 \right) = 15 - 10\beta_1 + \beta_2 = 0.$$

Hence, $\beta_2 = 10$. The Jacobi identity $J(e_3, e_5, e_8) = 0$ implies

$$\lambda_{3,5}\lambda_{8,8} + \lambda_{5,8}\lambda_{13,3} + \lambda_{8,3}\lambda_{11,5} = 0 \tag{3.17}$$

where

$$\lambda_{8,8} = 0, \quad \lambda_{5,8} = (-1)^{5-5}\left(\binom{0+2}{0} - \beta_1 \right) = -\frac{3}{2}, \quad \lambda_{3,8} = 1,$$

$$\lambda_{3,13} = \binom{7}{2} - \binom{6}{3}\beta_1 + \binom{5}{4}\beta_2 = 21 - 20\beta_1 + 5\beta_2 = 21,$$

and $\lambda_{5,11} = (-1)^0 \left(\binom{0+5}{0} - \binom{0+4}{1}\beta_1 + \binom{3}{2}\beta_2\right) = 1 - 4\beta_1 + 3\beta_2 = 21$. However, the left side of (3.17) does not equal to 0, a contradiction. This completes the proof. ∎

Lemma 3.37 *Let $k = 6$. Then $\mathfrak{m}^3_0$ is the only infinite-dimensional Lie algebra obtained by taking one-dimensional $\mathbb{N}$-graded filiform central extensions of* $\mathfrak{m}^3_0(12)$.

Proof Consider $\mathfrak{g} = \mathfrak{m}^3_0(12)$. By Lemma 3.15 its one-dimensional $\mathbb{N}$-graded filiform central extension is either $\mathfrak{m}^3_0(13)$ or $\mathfrak{m}^3_{0,1}(13)$. If it is $\mathfrak{m}^3_0(13)$, then applying the same lemma again, its one-dimensional $\mathbb{N}$-graded filiform central extension must be of type $\mathfrak{m}^3_0(14) = \mathfrak{m}^3_0(2 \cdot 7)$, which leads to the case $k > 6$ covered in Remark 3.34. Without loss of generality we assume that the second possibility holds. It follows from Lemma 3.24, that $\lambda_{3,13} = 0$ in $\mathfrak{m}^3_{0,4}(16)$ and $\lambda_{3,14} = 0$ in $\mathfrak{m}^3_{0,5}(17)$. It follows from the proof of Lemma 3.25 that in this case $\lambda_{3,2\cdot 6+3} = \lambda_{3,15}$ is also zero. Using formulas (3.3), (3.4), we obtain

$$\lambda_{3,13} = (-1)^{3-6}\left(\binom{6-3+3}{6-3} - \binom{6-3+2}{6-3+1}\beta_1 \right) = -\binom{6}{3} + \binom{5}{4}\beta_1 = 0,$$

$$\begin{aligned}\lambda_{3,14} &= (-1)^{3-6}\left(\binom{6-3+4}{6-3} - \binom{6-3+3}{6-3+1}\beta_1 + \beta_2 \right)\\ &= -\binom{7}{3} + \binom{6}{4}\beta_1 - \beta_2 = 0,\end{aligned}$$

$$\begin{aligned}\lambda_{3,15} &= (-1)^{3-6}\left(\binom{6-3+5}{6-3} - \binom{6-3+4}{6-3+1}\beta_1 + \binom{6-3+3}{6-3+2}\beta_2 \right)\\ &= -\binom{8}{3} + \binom{7}{4}\beta_1 - 6\beta_2 = 0.\end{aligned}$$

However, this system of 3 linear equations in 2 variables is inconsistent. The proof is complete. ∎

Finally we study one-dimensional $\mathbb{N}$-graded filiform central extensions of $\mathfrak{m}_0^3(6)$. By Lemma 3.15 its one-dimensional $\mathbb{N}$-graded filiform central extension is either $\mathfrak{m}_0^3(7)$ or $\mathfrak{m}_{0,1}^3(7)$. If it is $\mathfrak{m}_0^3(7)$, then applying the same lemma again, its one-dimensional $\mathbb{N}$-graded filiform central extension must be of type $\mathfrak{m}_0^3(8) = \mathfrak{m}_0^3(2 \cdot 4)$, which leads to the case $k = 4$ covered in Lemma 3.35. Without loss of generality we assume that the second possibility holds. Taking one-dimensional filiform central extensions of $\mathfrak{m}_0^3(6)$, in ten steps we obtain $\mathfrak{m}_{0,10}^3(16; \beta_1, \beta_2, \beta_3, \beta_4)$. Since it must be a Lie algebra, we have that $J(e_3, e_4, e_5) = 0$, $J(e_3, e_5, e_6) = 0$ and also $J(e_3, e_4, e_8) = 0$. Rewriting these identities in terms of structure constants, and then using formulas (3.3), (3.4) to express each structure constant in terms of $\beta_1, \beta_2, \beta_3$, and β_4, we obtain the following relations:

$$(3.18)\quad \beta_2 = \frac{4\beta_1^2}{3(1+\beta_1)}, \quad \beta_3 = \frac{5\beta_1\beta_2 - 10\beta_2^2}{3 - 4\beta_2 - 2\beta_1}, \quad \beta_4 = \frac{-5\beta_2^2 + 6\beta_2\beta_3 + 4\beta_1\beta_3}{2\beta_1 + \beta_2}.$$

Considering the Jacobi identity $J(e_3, e_4, e_9) = 0$, we obtain

$$(3.19)\quad \beta_4 + (\beta_3 - 3\beta_2 + \beta_1)(-1 + 8\beta_1 - 21\beta_2 + 20\beta_3 - 5\beta_4) + (3\beta_2 - 4\beta_1 + 1)(\beta_1 - 6\beta_2 + 10\beta_3 - 4\beta_4) = 0.$$

Using (3.18), equation (3.19) can be rewritten as the following equation in one variable β_1:

$$245\beta_1^{10} + 238\beta_1^9 - 606\beta_1^8 + 270\beta_1^7 - 27\beta_1^6 = 0$$

that has the following roots: $\beta_1 = 0, \frac{3}{5}, \frac{1}{7}$, and $\frac{-6\pm3\sqrt{11}}{7}$. If $\beta_1 = \frac{3}{5}$ or $\frac{-6\pm3\sqrt{11}}{7}$, then it is easy to check that $J(e_3, e_5, e_8) \neq 0$. If $\beta_1 = 0$, then $\mathfrak{m}_{0,9}^3(15; \beta_1, \beta_2, \beta_3, \beta_4)$ is isomorphic to $\mathfrak{m}_3(15)$, and, by Remark 3.31, leads to infinite-dimensional Lie algebra $\mathfrak{m}_3$. If $\beta_1 = \frac{1}{7}$, then $\mathfrak{m}_{0,9}^3(15; \beta_1, \beta_2, \beta_3, \beta_4)$ is isomorphic to $W^3(15)$, and by Lemma 3.32, it leads to W^3.

Appendix A

Here we recall some facts about topology of varieties that are relevant to proofs in Section 2.

A.1 The Case of Smooth Varieties

Let $X_d \subset \mathbb{P}^n$ be a smooth complex projective hypersurface of degree d, *i.e.,* the set of points satisfying $\{f_d(x_0, \ldots, x_n) = 0\}$, where f_d is a homogeneous polynomial of degree d, defining X_d. (This in particular implies that f_d is an irreducible polynomial, X_d is an irreducible algebraic variety.) Any two such hypersurfaces (of the same degree) are diffeomorphic as smooth manifolds. In particular, their topological invariants (and some of their geometric invariants) are determined by the pair (d, n).

Proposition A.1 ([D, Ch. 5]) *The integral homologies are torsion free and satisfy* $H_{2i+1}(X_d, \mathbb{Z}) = 0$ *for* $2i + 1 \neq n - 1$, $H_{2i}(X_d, \mathbb{Z}) = \mathbb{Z}$ *for* $2n - 2 \geq 2i \geq 0$ *and*

$2i \neq n-1$ and

$$b_{n-1}(X_d) = \frac{(1-d)^{n+1}-1}{d} + \begin{cases} -1 & \text{for } n \text{ even,} \\ 2 & \text{for } n \text{ odd.} \end{cases}$$

The topological Euler characteristic is $\chi(X_d) = \frac{(1-d)^{n+1}-1}{d} + n + 1$.

Example A.2 For $n = 2$ we get a smooth plane curve of (topological) genus $g = \binom{d-1}{2}$ and with topological Euler characteristic $\chi(X_d) = 2 - 2g$.

Let $\{X_i\}_{i=1,\dots,k} \subset \mathbb{P}^n$ be some hypersurfaces. Assume that $X := \bigcap_{i=1}^k X_i$ is a complete intersection. Let d_j denote the degree of X_j.

As in the case of hypersurfaces, any two *smooth* complete intersections with the same multi-degree $(d_1, \dots, d_k)$ are diffeomorphic. So, the topological invariants (and some geometric invariants too) are completely determined by the numbers $(d_1, \dots, d_k, n)$.

Proposition A.3 *Let $X_{\underline{d}} \subset \mathbb{P}^n$ be a smooth complete intersection of multidegree $\underline{d} := (d_1, \dots, d_k)$. The integral homologies are torsion free and satisfy $H_{2i+1}(X_{\underline{d}}, \mathbb{Z}) = 0$ for $2i + 1 \neq n - k$, $H_{2i}(X_{\underline{d}}, \mathbb{Z}) = \mathbb{Z}$ for $2n - 2k \geq 2i \geq 0$ and $2i \neq n - k$ and* $\operatorname{rank}(H_{n-k}(X_{\underline{d}}, \mathbb{Z})) = \chi(X_{\underline{d}}) - (n-k)$ *for $(n-k)$ even,* $\operatorname{rank}(H_{n-k}(X_{\underline{d}}, \mathbb{Z})) = n - k + 1 - \chi(X_{\underline{d}})$ *for $(n-k)$ odd. Here, the topological Euler characteristic is*

$$\chi(X_{\underline{d}}) = \Big(\prod_{i=1}^k d_i\Big) \operatorname{Coeff}_{x^{n-k}} \frac{(1+x)^{n+1}}{\prod_{i=1}^k (1+d_i x)}.$$

References for this are [D, Ch. 5] and [Hi, Appendix 1]. The proof is based on the Lefschetz hyperplane section theorem. The Euler characteristic can be obtained *e.g.*, as the top Chern class $c_n(T_{X_{\underline{d}}})$ from the exact sequence $0 \to T_{X_{\underline{d}}} \to T_{\mathbb{P}^n}|_{X_{\underline{d}}} \to \mathcal{N}_{X_{\underline{d}}/\mathbb{P}^n} \to 0$ and the fact that $\mathcal{N}_{X_{\underline{d}}/\mathbb{P}^n} = \bigoplus \mathcal{O}_{\mathbb{P}^n}(d_i)|_{X_{\underline{d}}}$.

Note that in these statements the varieties are assumed to be smooth. For singular varieties the situation is more complicated and odd homologies can be nonzero.

A.2 Algebraic Cell Structure

Let X be a compact topological space and let $X = \coprod \sigma_\alpha$ be a cell decomposition (each cell σ_α is homeomorphic to some $\mathbb{R}^k$, the smaller cells are glued to the bigger ones by the boundary maps).

Example A.4

(i) $S^n = \mathbb{R}^n \cup R^0$.

(ii) For the complex projective space $\mathbb{P}^n$ we have the following cell decomposition: $\mathbb{P}^n = \mathbb{C}^n \cup \mathbb{C}^{n-1} \cup \cdots \cup \mathbb{C}^0$ (see *e.g.*, [GH, 1.5]).

If X is a complex algebraic variety (possibly singular), then it is natural to ask for an *algebraic* cell structure, so that each cell, σ_α, is itself a subvariety of X, isomorphic

https://doi.org/10.4153/CJM-2014-008-x Published online by Cambridge University Press

to $\mathbb{C}^n$. For example, the cell structure of $\mathbb{P}^n$ is of this type. Though such an algebraic cell decomposition is not always possible, it often occurs in our context.

Suppose a variety admits an algebraic cell decomposition $X = \coprod(k_i\mathbb{C}^i)$; here k_i is the number of cells of the given dimension. This fixes the homology. Indeed, all the cells are of even (real) dimensions. Thus all the boundary maps are zero. So, $H_{2i}(X,\mathbb{Z}) = \mathbb{Z}^{k_i}$ for $0 \le 2i \le \dim_{\mathbb{R}} X$ and $H_m(X,\mathbb{Z}) = 0$ for other m.

A.3 Exact Sequence of a Pair

For a pair of topological spaces, $A \subset X$, we have exact sequences

$$\cdots \to H_i(A,\mathbb{Z}) \to H_i(X,\mathbb{Z}) \to H_i(X,A,\mathbb{Z}) \to H_{i-1}(A,\mathbb{Z}) \cdots,$$
$$\cdots \to H^i(X,\mathbb{Z}) \to H^i(A,\mathbb{Z}) \to H^{i+1}(X,A,\mathbb{Z}) \to \cdots.$$

A.4 (Co)homology of Non-compact Spaces

Proposition A.5 ([FF, p.157]) *Let X be a compact topological space, let $A \subset X$ be its closed subspace such that $X \setminus A$ is smooth, orientable, without boundary, and of (real) dimension n. Then $H_i(X,A,\mathbb{Z}) \xrightarrow{\sim} H^{n-i}(X \setminus A,\mathbb{Z})$*

(This follows from the definition of Borel-Moore homology, via compactification: $H_i^{BM}(X \setminus A,\mathbb{Z}) = H_i(X,A,\mathbb{Z})$ and the Poincaré duality for non-compact manifolds: $H_i^{BM}(M,\mathbb{Z}) \xrightarrow{\sim} H^{n-i}(M,\mathbb{Z})$.)

Appendix B

Let us recall [M2, Proposition 5.16].

Proposition B.1 ([M2, Proposition 5.16]) *Let $2k+3 \ge 9$ then*

$$H^2_{(2k+4)}(\mathfrak{m}_{0,3}(2k+3)) = 0$$

and therefore $\mathfrak{m}_{0,3}(2k+3)$ has no filiform central extensions.

In the above proposition, $\mathfrak{m}_{0,3}(2k+3)$ denotes a $(2k+3)$-dimensional $\mathbb{N}$-graded filiform Lie algebra with the basis $e_1, e_2, \ldots, e_{2k+3}$ and multiplication table

$$\begin{aligned}
&[e_1, e_i] = e_{1+i}, && i = 2,\ldots,2k+2;\\
&[e_l, e_{2k+1-l}] = (-1)^{l+1} e_{2k+1}, && l = 2,\ldots,k;\\
&[e_j, e_{2k+2-j}] = (-1)^{j+1}(k-j+1)e_{2k+2}, && j = 2,\ldots,k;\\
&[e_m, e_{2k+3-m}] = (-1)^m\Big((m-2)k - \frac{(m-2)(m-1)}{2}\Big)e_{2k+3}, && m = 3,\ldots,k+1.
\end{aligned}$$

In fact, this proposition holds true for any $k > 3$. If $k = 3$, then $\mathfrak{m}_{0,3}(9)$ does have a filiform central extension that is $\mathfrak{m}_{0,4}(10)$ with the basis: $e_1, e_2, \ldots, e_9, e_{10}$ and multiplication table:

$$
\begin{aligned}
&[e_1, e_i] = e_{1+i}, \quad i = 2, \dots 9, \\
&[e_2, e_5] = -e_7, && [e_3, e_4] = e_7, \\
&[e_2, e_6] = -2e_8, && [e_3, e_5] = e_8, \\
&[e_3, e_6] = -2e_9, && [e_4, e_5] = 3e_9, \\
&[e_4, e_6] = 3e_{10}, && [e_3, e_7] = -5e_{10}, \\
&[e_2, e_8] = 5e_{10}.
\end{aligned}
$$

In its turn, $\mathrm{m}_{0,4}(10)$ has also filiform central extension, which is $\mathrm{m}_{0,5}(11) = \operatorname{span}\{e_1, e_2, \dots, e_{10}, e_{11}\}$ with the following multiplication table:

$$
\begin{aligned}
&[e_1, e_i] = e_{1+i}, \quad i = 2, \dots 10, \\
&[e_2, e_5] = -e_7, && [e_3, e_4] = e_7, \\
&[e_2, e_6] = -2e_8, && [e_3, e_5] = e_8, \\
&[e_3, e_6] = -2e_9, && [e_4, e_5] = 3e_9, \\
&[e_4, e_6] = 3e_{10}, && [e_3, e_7] = -5e_{10} \\
&[e_2, e_8] = 5e_{10}, && [e_3, e_8] = \frac{5}{2}e_{11}, \\
&[e_2, e_9] = \frac{5}{2}e_{11}, && [e_4, e_7] = -\frac{15}{2}e_{11}, \\
&[e_5, e_6] = \frac{21}{2}e_{11}.
\end{aligned}
$$

The latter algebra has no filiform central extensions.

Acknowledgments We are thankful to M. Vergne for telling us about reference [M1]. We thank A. Dhillon and M. Khalkhali for brief but helpful discussions. We are grateful to the referee for suggestions.

References

[B] F. Bratzlavsky. *Classification des algèbras de Lie nilpotentes de dimension n, de classe n − 1, dont l'idéal dérivé est commutatif.* Acad. Roy. Belg. Bull. Cl. Sci. **60**(1974), 858–865.

[CGS] S. Ciralo, W. A. de Graaf, and C. Schneider, *Six-dimensional nilpotent Lie algebras.* Linear Algebra Appl. **436**(2012), no. 1, 163–189. http://dx.doi.org/10.1016/j.laa.2011.06.037

[D] A. Dimca, *Singularities and topology of hypersurfaces.* Universitext, Springer-Verlag, New York, 1992.

[ENR] F. J. Echarte, J. Núñez, and F. Ramírez, *Description of some families of filiform Lie algebras.* Houston J. Math. **34**(2008), no. 1, 19–32.

[F] A. Fialowski, *On the classification of graded Lie algebras with two generators.* (Russian) Vestnik Moskov. Univ. Ser. I Mat. Mekh. **1983**, no. 2, 62–64.

[FF] A. T. Fomenko and D. B. Fuks, *Kurs gomotopicheskoi topologii.* (Russian) [A course in homotopic topology] "Nauka", Moscow, 1989.

[G] M.-P. Gong, *Classification of nilpotent Lie algebras of dimension 7 (over algebraically closed fields and R).* Ph.D. thesis, University of Waterloo, 1998.

[GH] P. Griffiths and J. Harris, *Principles of algebraic geometry.* John Wiley & Sons, Inc., New York, 1994.

[Ha] You. B. Hakimjanov, *Variété des lois d'algèbres de Lie nilpotentes.* Geom. Dedicata **40**(1991), no. 3, 269–295.

[Hi] F. Hirzebruch, *Topological methods in algebraic geometry.* Third enlarged ed., Springer-Verlag New York, Inc., New York, 1966.

[KN] A. Kirillov and Yu. Neretin, *The variety A_n of n-dimensional Lie algebra structures.* In: Some problems in modern analysis, Fourteen papers translated from Russian. AMS Transl., Ser. 2, 137, 1987, 21–30.

[L] C. Löfwall, *Solvable infinite filiform Lie algebras.* J. Commut. Algebra **2**(2010), no. 4., 429–436. http://dx.doi.org/10.1216/JCA-2010-2-4-429

[M1] D. Millionshchikov, *The variety of Lie algebras of maximal class.* Proc. Steklov Inst. Math. **266**(2009), no. 1, 177–194.

[M2] ———, *Graded filiform Lie algebras and symplectic nilmanifolds.* In: Geometry, topology, and mathematical physics, Amer. Math. Soc. Transl. Ser. 2, 212, American Mathematical Society, Providence, RI, 2004, pp. 259–279.

[SZ] A. Shalev and E. Zelmanov, *Narrow Lie algebras: a coclass theory and a characterization of the Witt algebra.* J. Algebra **189**(1997), no. 2, 294–331. http://dx.doi.org/10.1006/jabr.1996.6819

[T] Gr. Tsagas, *Lie algebras of dimension eight.* J. Inst. Math. Comput. Sci. Math. Ser. **12**(1999), no. 3, 179–183.

[V] M. Vergne, *Cohomologie des algèbres de Lie nilpotentes. Application à l'étude de la variété des algèbres de Lie nilpotentes.* Bull. Soc. Math. France **98**(1970), 81–116.

Department of Mathematics, University of Western Ontario, London, ON N6A 5B7
e-mail: tatyana.barron@uwo.ca

Department of Mathematics, Ben-Gurion University of the Negev, P.O.B. 653, Be'er Sheva 84105, Israel
e-mail: dmitry.kerner@gmail.com

Fields Institute for Research in Mathematical Sciences, Toronto, ON, M5T 3J1
e-mail: mtvalava@fields.utoronto.ca