

A map with topological minimal self-joinings in the sense of del Junco

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(Received 18 November 1988)

Abstract Andrés del Junco has proposed a definition of topological minimal self-joinings intended to parallel Dan Rudolph's measure-theoretic concept. By means of a rank-two 'cutting and stacking', this article constructs the first example of a system (a subshift) satisfying his proposed definition of 2-fold topological minimal self-joinings.

The second part of the article shows that 2-fold topological minimal self-joinings does not imply 3-fold and that no map has 4-fold topological minimal self-joinings. This latter result follows from a generalization of a theorem of Schwartzman.

0 Introduction

In 1979 Dan Rudolph showed that Don Ornstein's rank-1 mixing map, T , could be built to possess a measure-theoretic property which has come to be called *minimal self-joinings* of all orders. For order 2 (our tacit assumption, if no adjective is present) this says that whenever two copies of T sit simultaneously as factors of a third, ergodic, transformation then the two copies are either independent or are identified by some power of T . Equivalently, letting (X, μ) denote the space on which T acts, any $T \times T$ -invariant ergodic measure on $X \times X$ projecting to μ on each coordinate is either product measure $\mu \times \mu$ or else is supported on the graph of some T^n . One can similarly define K -fold minimal self-joinings by considering the possible ergodic measures living on $X^{\times K}$, the cartesian K th power of X . If T has minimal self-joinings of all orders then any automorphism of $T^{\times \mathbb{N}}$ is simply a cartesian product of powers of T composed with a permutation of the coordinates. Rudolph constructed such a map in $[\mathbf{R}]$ and used the automorphism property to fabricate a menagerie of counterexample transformations. Later investigations showed, $[\mathbf{K}, \mathbf{T}]$, that the maps with minimal self-joinings play the role of elementary building blocks for a class of maps containing the finite-rank mixing maps.

What analogue does this notion have in the topological category of a homeomorphism T of a compact metric space X ? Using minimal sets as the analogue of ergodic measures, Nelson Markley proposed in $[\mathbf{M}]$ a definition paralleling Rudolph's measure-theoretic definition. He and Joe Auslander proved in $[\mathbf{A}, \mathbf{M}]$ that this property for order 2 implies the property for all orders and powers. They then went

on to establish a structure theorem roughly analogous to the measure-theoretic one of [K]. However, an exact analogue of Rudolph's theory is not obtained because, for such transformations, the automorphism group of $T^{\times N}$ seems difficult to pin down.

In [J] Andrés del Junco took orbit closures as the natural analogue of ergodic measures, giving rise to a different definition of topological minimal self-joinings (below) which is appealing to measure-theorists because it permits an analogue of product measure. However, since no examples fulfilling his definition were known, he worked with a complex alternative definition designed to be fulfilled by the topological Chacón's transformation and for it obtained an analogue of a large part of the Rudolph theory.

This note constructs a map satisfying Andrés' original 'natural' definition, in the hope of resuscitating interest in it. The method by which the map is constructed seems also to be of independent interest.

Section 2 strengthens an old theorem of Schwartzman to show that 4-fold topological minimal self-joinings does not exist, thus answering negatively a question of [J]. By contrast, in the measure-theoretic setting, all known maps with 2-fold minimal self-joinings have *msj* of all orders and indeed '2-fold \Rightarrow N -fold' is a central open question related to the (in)famous problem of whether 2-fold mixing implies N -fold mixing. What is known and will appear elsewhere, is that 4-fold minimal self-joinings implies N -fold, in the measure-theoretic category.

Nomenclature. We use $[n, m)$ to indicate a half-open 'interval of integers' $\mathbb{Z} \cap [n, m)$. A similar convention holds for closed and open intervals of integers.

Having fixed a finite alphabet A , a *name* denotes a doubly infinite string of letters from A , i.e. is a point $x \in A^{\mathbb{Z}}$. Use $x[n]$ to denote the n th letter in x and let $x[n, m)$ denote the substring

$$x[n]x[n+1]x[n+2] \dots x[m-1]$$

with the obvious meaning if $m = \infty$. A *word* means a finite string of letters from A . Words are always indexed from zero, i.e. if word W is of length h then $W = W[0, h)$. The general purpose term 'string' may denote a finite, half-infinite, or bi-infinite string of letters.

Let $\#X$ denote the cardinality of the set X . Agree to use $\stackrel{\Delta}{=}$ to mean 'is defined to be', in $a \stackrel{\Delta}{=} b$ the expression b defines the symbol a .

Topological notions. This paper concerns a homeomorphism $T: X \rightarrow X$ of a compact metric space with $d, |\cdot|$ denoting the metric. Use $\mathcal{O}_T(x)$ or just $\mathcal{O}(x)$ to represent the orbit $\{T^n x\}_{n \in \mathbb{Z}}$ of x and let $\bar{\mathcal{O}}(x)$ denote its orbit closure. An $x \in X$ is a *transitive point* if $\bar{\mathcal{O}}(x) = X$. Distinct points x and y are *proximal* if

$$\inf_{n \in \mathbb{Z}} |T^n x, T^n y| = 0,$$

equivalently, there exist a point z and times $n(i)$ such that $T^{n(i)} x \rightarrow z$ and $T^{n(i)} y \rightarrow z$. Points x and y are future (past) *asymptotic* if $|T^n x, T^n y|$ goes to zero as $n \rightarrow +\infty$ ($n \rightarrow -\infty$).

Map T is *topologically transitive* if every invariant non-empty open set is dense, in the case of a metric space, this is equivalent to X possessing a transitive point. The map is topologically *weak-mixing* if $T \times T$ is topologically transitive. If every point of T is transitive then T is said to be *minimal*. Finally, a minimal map T is *proximal orbit dense* if for every pair of points x, y in distinct orbits there exists $n \neq 0$ such that x and $T^n y$ are proximal. This property is properly weaker than the following

DEFINITION A map $T: X \rightarrow X$ has *topological minimal self-joinings* in the sense of del Junco if two conditions hold

- (i) Every (non-zero) power of T is minimal
- (ii) For any pair of points $x, y \in X$ not in the same orbit, the pair $\langle x, y \rangle$ is a transitive point for $T \times T$

Condition (ii) says that any pair which *could* be transitive under $T \times T$ is transitive. Evidently if x and y are in the same orbit, $y = T^3 x$ for instance, then the $T \times T$ orbit closure of $\langle x, y \rangle$ is the third *off-diagonal*

$$D_3 \triangleq \{ \langle z, T^3 z \rangle \mid z \in X \}$$

and thus certainly cannot be all of $X \times X$ □

Condition (ii) of topological minimal self-joinings is equivalent to the seemingly weaker condition that whenever points x and y are in different orbits, then they are proximal. For this along with the minimality of T yields

$$\bar{\mathcal{O}}_{T \times T}(x, y) \supset D_0,$$

whenever x and y are points in different orbits. But for any k the points x and $T^{-k} y$ are in different orbits and so the orbit closure of $\langle x, T^{-k} y \rangle$ contains D_0 , equivalently, $\bar{\mathcal{O}}_{T \times T}(x, y)$ contains D_k . Thus

$$\bar{\mathcal{O}}_{T \times T}(x, y) \supset \text{Closure} \{ \langle z, T^k z \rangle \mid z \in X, k \in \mathbb{Z} \} \stackrel{\text{note}}{=} X \times X$$

This last equality follows by the minimality of T □

The point(s) of difficulty. The topological Chacón’s map is proximal orbit dense but does not have topological minimal self-joinings. The difficulty comes from the existence of a pair $\langle x, y \rangle$ of future asymptotic points. In the context of a symbolic space this means that there is some N for which $x[N, \infty) = y[N, \infty)$.

For a minimal T , any asymptotic pair x and y must be in distinct orbits. For if $y = T^p x$, say, then any limit point $z \triangleq \lim_{i \rightarrow \infty} T^{n(i)} x$ is periodic – since $\lim_{i \rightarrow \infty} T^{n(i)} y$ must equal both z and $T^p z$. But T is minimal hence X has no periodic points (a periodic minimal map has no asymptotic pairs). Thus p must have been zero.

Now topological minimal self-joinings says that for a pair $\langle x, y \rangle$ the orbit closure is either an off-diagonal or is all of $X \times X$. But suppose we have two points x and y which are future-asymptotic under some shift k (i.e. x and $T^k y$ are future-asymptotic) and past-asymptotic under a different shift s . Then the $T \times T$ orbit

closure of $\langle x, y \rangle$ consists of two off-diagonals, $D_k \cup D_s$. (Indeed, after shifting y so that neither k nor s is zero, the pair $\langle x, y \rangle$ is not proximal.)

The exceptional points of Chacón’s map. This map fails to achieve topological minimal self-joinings by the slimmest of margins – there is one bad pair of orbits. There are names x and y for which $x(-\infty \ 0) = y(-\infty \ 0)$ and $x[0 \ \infty) = y[1 \ \infty)$. These names look like

$$\begin{aligned} x &= \tilde{p}\tilde{q}, \\ y &= \tilde{p}s\tilde{q}, \end{aligned} \tag{0 1}$$

where \tilde{p} is a past string, \tilde{q} a future string, and s is a letter from the alphabet (see [J, K])

What to avoid. Asymptoticity cannot be avoided – any expansive system $T : X \rightarrow X$ has future-asymptotic pairs and past-asymptotic pairs (this well-known fact follows from Theorem 2.1). What must be prevented is that a pair $\langle x, y \rangle$ possesses, upto a shift, both future and past asymptoticity.

Say that a string is ‘valid’ if it is a substring of some x in X . Suppose there existed pairs of valid words V_n and W_n , whose lengths went to infinity, such that the concatenations

$$V_n W_n \quad \text{and} \quad V_n s W_n$$

were both valid strings. By taking a weak limit, the space X would be forced to have points x and y as in (0.1).

To prevent this, we will construct X so that when a word V is sufficiently long, it determines the letter which must follow it.

1 The construction

How might one go about building a shift-invariant symbolic space? Fix a finite alphabet A . Suppose \mathcal{V} is some collection of strings over A – strings which are finite, left-infinite, right-infinite, or bi-infinite. Define $CI(\mathcal{V})$ to be the set of names $x \in A^{\mathbb{Z}}$ for which

$$\text{Every finite substring of } x \text{ is a substring of some } V \in \mathcal{V}$$

Since no mention of time is made in its definition, $CI(\mathcal{V})$ is a shift-invariant subset of $A^{\mathbb{Z}}$. A fortiori $CI(CI(\mathcal{V})) = CI(\mathcal{V})$ and consequently

$$CI(\mathcal{V}) \text{ is a closed shift-invariant subset of } A^{\mathbb{Z}}$$

The shift-left map on this forms a topological dynamical system.

Building the space X . Our alphabet will be $\{a, b, s\}$, ‘ s ’ will be employed as a spacer between n -blocks. For $n = 0, 1, 2, \dots$ we build two types of n -block

$$H_n^0 \quad \text{and} \quad H_n^1,$$

whose common length is denoted h_n . It will be convenient that the $\{h_n\}$ be even numbers, we leave verification to the reader that the operations done below do not prevent our arranging this

Concatenation will be indicated by juxtaposition. However, we use \otimes to write iterated concatenation. For example, if $\text{Letter}(i)$ denotes the i th Roman letter then $\otimes_{i=1}^5 \text{Letter}(i)$ means abcde. The expression $3 \times \text{'ab'}$ indicates ababab.

In the sequel, the symbols α and β take on the values 0 and 1. When not desiring to specify the type of an n -block we will write H_n .

STEP A Set $H_0^0 \triangleq \text{'a'}$ and $H_0^1 \triangleq \text{'b'}$

STEP B At stage n , with H_n^0 and H_n^1 known, pick a K much larger than h_n . For each α pick a sequence of gap sizes

$$\{g^\alpha(k)\}_{k=0}^{K-1}$$

with each $g^\alpha(k)$ either 0 or 1 [We have suppressed the subscript n in K_n and in g_n^α]

STEP C Define the two $(n+1)$ -blocks as follows

$$H_{n+1}^\alpha \triangleq \left(\otimes_{k=0}^{K-1} H_n^{g^\alpha(k)} [g^\alpha(k) \times \text{'s'}] \right) H_n^\alpha$$

Thus each n -block of type 1 is followed by a spacer, those of type 0 are followed by no spacer. The type of the rightmost n -block must agree with the type of its enclosing $(n+1)$ -block.

Finally, let T denote the shift-left map on X where $X \triangleq \text{Cl}\{H_n^0 \mid n \in \mathbb{N}\}$

DEFINITION Say that a word W is neatly n -blocked if it can be written in the form

$$W = \left(\otimes_{l=0}^{L-1} H_n^{\beta(l)} [\beta(l) \times \text{'s'}] \right) H_n^{\beta(L)}, \tag{1.1}$$

where each $\beta(l)$ is 0 or 1

Let $\text{Spacer}_n(W)$ denote $\beta(L)$. This is a slight abuse of notation since $\beta(L)$ depends on the righthand side of (1.1) - which we have not bothered to show unique. However, this will cause no difficulty where we need it, in the following lemma.

CONSISTENCY LEMMA For any pair $M \geq N$ and each α

$$H_M^\alpha \text{ is neatly } N\text{-blocked}$$

Thus for any $x \in X$, each substring $x[a \quad b)$ can be extended to a larger enclosing substring $x[a' \quad b')$ which is neatly N -blocked.

Remark Consequently, given any position i on x there is a position j with

$$i - \frac{1}{2}h_N \leq j \leq i + \frac{1}{2}h_N,$$

such that $x[j \quad j + h_N) = H_N$

Proof For any n and β the word H_n^β is neatly $(n-1)$ -blocked, by definition. Evidently if words U and V are neatly $(n-1)$ -blocked and $k \triangleq \text{Spacer}_{n-1}(U)$ then the concatenation

$$U[k \times 's']V$$

is neatly $(n-1)$ -blocked. But STEP C implies $\text{Spacer}_n(H_n^\beta) = \text{Spacer}_{n-1}(H_n^\beta)$ and so for any word W

$$W \text{ neatly } n\text{-blocked} \Rightarrow W \text{ neatly } (n-1)\text{-blocked}$$

By induction on n from M down to $N+1$, the word H_M^α is neatly N -blocked \square

Creating topological properties

The properties of T depend on our algorithm for choosing, at stage n , the parameters K and $\{g^\alpha(k)\}_{k=0}^{K-1}$. Below we give a first approximation of these gap sequences, this will be refined by later lemmas, whose proofs will require successive modifications to the sequences. These modifications will not affect properties previously obtained.

FIRST VERSION At stage n , set $K \triangleq 4\kappa$ for some large κ . Let $\{g^\alpha(k)\}_{k=0}^{K-1}$ equal the following

$$\begin{array}{rcccl}
 & \underbrace{\hspace{1.5cm}} & \underbrace{\hspace{1.5cm}} & \underbrace{\hspace{1.5cm}} & \underbrace{\hspace{1.5cm}} \\
 \text{For } \alpha = 0 & 000 & 00 & 111 & 11 & 000 & 00 & 111 & 11 \\
 \text{For } \alpha = 1 & 101010 & & & & & & & 1010
 \end{array} \tag{1 2a}$$

$\underbrace{\hspace{10cm}}_{\text{Length } K}$

Each row has the same number of 1's. Thus H_{n+1}^0 and H_{n+1}^1 indeed have the same length.

The next two lemmas need some notation. Given two words V and W of some length h and an integer s , let ' $s+W$ ' indicate the word W shifted s units to the left. The phrase

'*the intersection $V \cap [s+W]$ contains word U* '

is to mean that there exists a position i such that

$$V[i \quad i+u) = U \quad \text{and} \quad W[i+s \quad i+s+u) = U,$$

where u denotes the length of word U . (Thus i must satisfy $0 \leq i$ and $i+u \leq h$ as well as $0 \leq i+s$ and $i+s+u \leq h$.)

OPPOSITE TYPES LEMMA For any shift s satisfying $|s| \leq \frac{1}{2}h_n$, the intersection

$$H_n^0 \cap [s + H_n^1]$$

contains a copy of H_{n-1}^0 .

Proof Let h denote h_{n-1} . Since the $\alpha = 1$ row of (1 2a) has an equal number of 1's in its first half as in its second half, shifting H_n^1 by at most $\frac{1}{2}h_n$ corresponds to a horizontal shift of the bottom row of (1 2a), by at most half its length, relative to the top row. The interval where the rows overlap will contain κ many 0's sitting above the periodic pattern **0101 01**. Thus the upstairs word H_n^0 will contain a periodic pattern of period h sitting above a periodic pattern in H_n^1 with period $2h + 1$ (the length of the string $H_{n-1}^0 H_{n-1}^1$ s which arises from the **01** gap pattern).

These two numbers h and $2h + 1$ are relatively prime. Consequently if we have chosen

$$\kappa \gg h (2h + 1), \tag{1 2b}$$

then there will be some position, i , where the upstairs and downstairs periodic patterns start in synchronism. Starting there,

$$H_n^0[i \quad i + h) = H_{n-1}^0 \quad \text{and} \quad [s + H_n^1][i \quad i + 2h + 1) = H_{n-1}^0 H_{n-1}^1 s,$$

which gives the desired conclusion □

The above lemma finds a common substring between two n -blocks, their relative shift is essentially arbitrary (just not too big) but their *types* had to be different. The lemma below obtains a similar conclusion where the restriction has been moved from the types to the shift.

ANY TYPES LEMMA *Given any shift t for which $\frac{1}{2}h_n < |t| \leq \frac{1}{2}h_{n+1}$, given any types α and β , the intersection*

$$H_{n+1}^\alpha \cap [t + H_{n+1}^\beta]$$

contains the word H_{n-1}^0

Proof Without loss of generality t is positive. Suppress the subscript and let K and h denote K_n and h_n . Given a type $\gamma \in \{0, 1\}$ let $i^\gamma(k)$ denote the position in H_{n+1}^γ commencing the k th copy, traversing left to right, of an n -block. Thus

$$H_{n+1}^\gamma [i^\gamma(k) \quad i^\gamma(k) + h) = H_n \quad \text{for } k = 0, 1, \dots, K$$

Note that $i^\gamma(0)$ equals zero.

Picture the word H_{n+1}^α written horizontally above the word H_{n+1}^β which has been shifted left by t positions. Let \hat{k} denote the rightmost n -block in word $t + H_{n+1}^\beta$ which overlaps, by at least $\frac{1}{2}h$, the leftmost n -block in H_{n+1}^α . In other words, pick \hat{k} largest such that $|s| \leq \frac{1}{2}h$ where

$$s \triangleq i^\beta(\hat{k}) - i^\alpha(0) - t$$

The remark after the *Consistency lemma* (and that h is even) shows that such a \hat{k} exists. Moreover $\hat{k} \geq 1$, since the given shift t strictly exceeds $\frac{1}{2}h$.

Suppose, for the sake of contradiction, that our lemma fails. The substrings

$$H_{n+1}^\alpha [i^\alpha(0) \quad i^\alpha(0) + h) \quad \text{and} \quad H_{n+1}^\beta [i^\beta(\hat{k}) \quad i^\beta(\hat{k}) + h)$$

are n -blocks, upstairs and downstairs respectively, with a relative shift of s . Were their types different, our lemma would be proved by an application of the *Opposite types lemma* to these n -blocks. Their types are consequently the same and so

$$g_n^\alpha(0) = g_n^\beta(\hat{k})$$

But this implies that $i^\beta(\hat{k} + 1) - i^\alpha(1) - t$, the relative shift between the succeeding n -blocks upstairs and downstairs, also equals s . Again we must be unable to apply the *Opposite types lemma* and so their following gaps must also be equal. By inductively stepping across the upstairs and downstairs $(n + 1)$ -blocks in this fashion we conclude that $g_n^\alpha(j)$ equals $g_n^\beta(\hat{k} + j)$ for $j = 0, 1, \dots, K - \hat{k} - 1$.

Periodicity in the gap sequence. An upper bound on \hat{k} arises from its definition in that $i^\beta(\hat{k})$ must be dominated by $t + \frac{1}{2}h$ and hence by $\frac{1}{2}h_{n+1} + \frac{1}{2}h_n$. Now each row of (1.2a) has about the same number of 1's in its first half as in its second half. Hence whether β is 0 or 1 we conclude that \hat{k} is essentially dominated by $\frac{1}{2}K$. Certainly by magnanimously only asserting that

$$\hat{k} < \frac{2}{3}K, \tag{1.2c}$$

we have absorbed any errors.

Evidently α must equal β . Otherwise we could have applied the *Opposite types lemma* to the given $(n + 1)$ -blocks and concluded that their intersection contains an H_n^0 , hence *a fortiori* contains an H_{n-1}^0 .

The preceding two paragraphs rewrite the conclusion of the 'inductively stepping' paragraph as

$$\text{For } 0 \leq j \leq \frac{1}{3}K_n \quad g_n^\alpha(j) = g_n^\alpha(\hat{k} + j) \tag{1.3}$$

This is a periodicity condition on a gap sequence. Thus our proof will be completed by altering the definition of a gap sequence so as to make (1.3) impossible. This simply requires inserting a few distinct 'marker' words

$$U \triangleq 1001 \quad V \triangleq 10001 \quad W \triangleq 100001,$$

which appear nowhere else in a gap sequence.

SECOND VERSION At stage n , with H_n^0 and H_n^1 known. Pick an even integer $\kappa \gg h_n$. ($2h_n + 1$) Define $\{g^\alpha(k)\}_{k=0}^{K-1}$ as follows

For $\alpha = 0$	$\underbrace{0000}_{\kappa \text{ many}}$	$\underbrace{00U}_{\kappa \text{ many}}$	$\underbrace{1111}_{\kappa \text{ many}}$	$\underbrace{11V}_{\kappa \text{ many}}$	$\underbrace{0000}_{\kappa \text{ many}}$	$\underbrace{00W}_{\kappa \text{ many}}$	$\underbrace{1111}_{\kappa \text{ many}}$	$\underbrace{11}_{\kappa \text{ many}}$
For $\alpha = 1$	1010	10U	1010	10V	1010	10W	1010	10
	$\text{Length } K$							

Any subsequence of $\{g^\alpha(k)\}_{k=0}^{K-1}$ taking up at least a third of it, must contain one of the three marker words. But a particular marker word appears nowhere else in the gap sequence. This prohibits (1.3), since \hat{k} is not zero. □

Proving proximality of points in different orbits. Here starts the proof that T has topological minimal self-joinings. Choose points $x, y \in X$ which are *not* proximal. In several steps, the lemmas above will imply that x and y must be in the same orbit.

Obtaining the shift progression $\{s_n\}_{n=0}^\infty$ By the remark following the *Consistency lemma* there exists a large interval containing, say, $[-10h_n, 10h_n]$ such that

$$x[a, b] \text{ is neatly } n\text{-blocked,} \tag{1.4}$$

where $a \leq -10h_n < 10h_n \leq b$. Thus there exists an index

$$t \in [-\frac{3}{2}h_n, -\frac{1}{2}h_n] \tag{1.5}$$

such that $x[t, t + h_n]$ is an n -block. Similarly, by applying the lemma to y we can find an index $j \in [t - \frac{1}{2}h_n, t + \frac{1}{2}h_n]$ such that $y[j, j + h_n] = H_n$. Let

$$s_n \triangleq t - j$$

denote the relative shift between these two n -blocks.

The limit supremum of $|s_n|$ is finite. We show that there does not exist a sequence of $n \in \mathbb{N}$ along which s_n gets arbitrarily large. Fix n and set $t \triangleq s_n = t - j$. By its definition, $|t|$ is dominated by $\frac{1}{2}h_n$. We would like to apply the *Any types lemma* to the two n -blocks at t and j . Indeed we could, if we knew that $|t|$ exceeded $\frac{1}{2}h_{n-1}$. Since, however, it might not, simply pass to a smaller value of the subscript. For note that for any $m < n$

$$x[t, t + h_{m+1}] = x[t, t + h_n][0, h_{m+1}] = H_n[0, h_{m+1}] = H_{m+1}$$

Similarly, $y[j, j + h_{m+1}]$ is an $(m + 1)$ -block of some type. Pick the value m less than n such that

$$\frac{1}{2}h_m < |t| \leq \frac{1}{2}h_{m+1}$$

By the *Any types lemma* there now exists a position p such that

$$x[p, p + h_{m-1}] = y[p, p + h_{m-1}],$$

because they both equal H_{m-1}^0 .

This m depends on t and so we write it as $m[t]$. If, along a subsequence, the shift progression $t(l) \triangleq s_{n(l)}$ goes to infinity (in absolute value) as $l \rightarrow \infty$, then $\lim_{l \rightarrow \infty} m[t(l)]$ equals infinity. Thus x and y have equal, aligned, substrings of arbitrarily large length – they would be proximal, contrary to our standing assumption.

The point y is in the orbit of x . The preceding says that there is an infinite subset $F \subset \mathbb{N}$ such that $n \mapsto s_n$ is constant, say, 17, on F . Fix an $n \in F$ and let t and $j \stackrel{\text{note}}{=} t - 17$ be as in the *Obtaining the shift progression* paragraph. Thus

$$x[t, t + h_n] = H_n^\alpha \text{ and } y[j, j + h_n] = H_n^\beta$$

for some α and β . We may have taken n large enough that x and y have no equal, aligned, substring of length equal to h_{n-1} . Consequently, the *Opposite types lemma* allows us to conclude that $\alpha = \beta$.

Since the type of a block determines the number of spacers following it,

$$x[t, t'] = y[j, j'],$$

where $i' \triangleq i + h_n + \alpha$ and $j' \triangleq j + h_n + \beta$. By (1.4) position i' commences an n -block on x and analogously for j' on y , say $x[i' \dots i' + h_n] = H_n^\alpha$ and $y[j' \dots j' + h_n] = H_n^\beta$. As above, the *Opposite types lemma* must fail to be applicable to this pair of blocks and so α' equals β' .

Putting this all together yields

$$x[i \dots i + L] = y[i - 17 \dots i - 17 + L],$$

where L denotes $h_n + \alpha + h_n$. Make explicit their dependence on n and write $i(n)$ and $L(n)$. From (1.5) we have that

$$i(n) \leq -\frac{1}{2}h_n \quad \text{and} \quad i(n) + L(n) \geq -\frac{3}{2}h_n + 2h_n = \frac{1}{2}h_n$$

Thus $i(n) \rightarrow -\infty$ and $i(n) + L(n) \rightarrow +\infty$ as $n \rightarrow \infty$ inside of F . Consequently, $y(-\infty \dots \infty)$ equals $x(-\infty \dots \infty)$ shifted left by 17.

Since y has been shown to be in the orbit of x , our map T fulfills the proximality condition for topological minimal self-joinings ◇

Minimality of T It suffices to show that given any valid word W and any $x \in X$, there exists a position i for which $x[i \dots i + \text{len}(W)]$ equals W . But X is the closure of $\{H_n^0\}_{n=1}^\infty$ and so without loss of generality $W = H_n^0$ for some n . By the *Consistency lemma*, the name x contains an $(n + 1)$ -block of some type. And *both* types of $(n + 1)$ -block contain H_n^0 .

Total minimality (T^n is minimal for all non-zero n) can be seen directly by a similar argument – but it also follows on general principles. The proximality condition of *tmsj* implies that T is (topologically) weak-mixing. And it is well-known [Ke], that a weak-mixing minimal map is totally minimal ◇

Remark It is not hard to see that the foregoing T is uniquely ergodic. We remark without proof that the gap sequences can be modified so that the map still has topological minimal self-joinings but now supports two ergodic measures.

An application

The cartesian square $T^{\times 2}$ is not conjugate to $T^{\times 3}$ First note that a point $\langle x, y, z \rangle$ is in a minimal set for $T^{\times 3}$ if and only if $\mathcal{O}(x) = \mathcal{O}(y) = \mathcal{O}(z)$. For the closure $\bar{\mathcal{O}}_{T \times T}(x, y)$ must be a minimal set for $T^{\times 2}$ (since the projection map is a homomorphism and the homomorphic image of a minimal set is minimal) and so $y \in \mathcal{O}(x)$. Similarly $z \in \mathcal{O}(x)$.

Discussions with Joe Auslander produced this proof that $T^{\times 3}$ is not a factor of $T^{\times 2}$. For suppose

$$\varphi : X \times X \rightarrow X \times X \times X$$

were a homomorphism of $T^{\times 2}$ onto $T^{\times 3}$. Set $\mathbf{a} \triangleq \langle x, x, y \rangle$ where $x, y \in X$ with $y \notin \mathcal{O}(x)$.

Set

$$\mathbf{a}' = \langle w, z \rangle \triangleq \varphi^{-1}(\mathbf{a})$$

Evidently $\mathcal{O}_{T \times T \times T}(\mathbf{a})$ is not dense, since the first two coordinates of \mathbf{a} are equal. Thus $\mathcal{O}_{T \times T}(\mathbf{a}')$ is not dense. Since T has topological minimal self-joinings this implies that w and z are in the same T -orbit, thus the orbit closure $\bar{\mathcal{O}}_{T \times T}(\mathbf{a}')$ is a minimal set. Hence the φ -image of this set,

$$\bar{\mathcal{O}}_{T \times T \times T}(\mathbf{a}),$$

is a minimal subset of $X \times X \times X$. But by the paragraph above, this would imply that y is in the T -orbit of x . \diamond

2 Higher order topological minimal self-joinings

The N -fold generalization of topological minimal self-joinings is that for any N points x_1, \dots, x_N inhabiting N different T -orbits, the tuple $\langle x_1, \dots, x_N \rangle$ is a transitive point for $T^{\times N}$. As before, this is equivalent to asking that every such N points be proximal under T .

Two-fold topological minimal self-joinings does not imply three-fold. Our example above fails to have 3-fold topological minimal self-joinings, which can be seen as follows. Notice, for the gap sequences chosen, that the sequence of type α begins with an α . Thus the $(n+1)$ -block of type $\mathbf{0}$ starts and ends with H_n^0 and the $(n+1)$ -block of type $\mathbf{1}$ begins and ends with an n -block of its same type. This will imply the existence of three distinct points $x, y, z \in X$ such that $\langle x, y \rangle$ are future-asymptotic and $\langle y, z \rangle$ are past-asymptotic. As argued earlier, minimality ensures that neither x nor z is in $\mathcal{O}(y)$. Nor could $z \in \mathcal{O}(x)$ since this implies that $\bar{\mathcal{O}}_{T \times T}(x, y)$ consists of two off-diagonals and thus is not dense. So the points $\{x, Ty, z\}$ are in three distinct orbits and yet they cannot be triply proximal.

To make these names it will be convenient to let $\tilde{\mathbf{0}}$ and $\tilde{\mathbf{1}}$ denote certain right-infinite strings, that is, points in $\mathbf{A}^{\mathbb{N}}$. Define them inductively by

$$\tilde{\mathbf{0}}[0 \ h_n) \triangleq H_n^0 \quad \text{and} \quad \tilde{\mathbf{1}}[0 \ h_n) \triangleq H_n^1 \quad n = 0, 1, 2,$$

This produces valid strings because H_{n+1}^α commences with an H_n^α . Similarly, since H_{n+1}^α ends with H_n^α , the left-infinite pasts $\tilde{\mathbf{0}}$ and $\tilde{\mathbf{1}}$ specified inductively as

$$\tilde{\mathbf{0}}[-h_n \ 0) \triangleq H_n^0 \quad \text{and} \quad \tilde{\mathbf{1}}[-h_n \ 0) \triangleq H_n^1,$$

are also well-defined. Concatenate the futures to the pasts as follows

$$\begin{aligned} x &\triangleq \tilde{\mathbf{1}}\tilde{\mathbf{s}}\tilde{\mathbf{0}} \quad \text{since } H_n^1\tilde{\mathbf{s}}H_n^0 \text{ is valid,} \\ y &\triangleq \tilde{\mathbf{0}}\tilde{\mathbf{0}} \quad \text{since } H_n^0H_n^0 \text{ is valid,} \\ z &\triangleq \tilde{\mathbf{0}}\tilde{\mathbf{1}} \quad \text{since } H_n^0H_n^1 \text{ is valid} \end{aligned}$$

These names x, y and z are points in X because the corresponding concatenations of blocks, listed at right, are valid words ◇

Four-fold minimal self-joinings cannot exist

In the case of symbolic systems, four-fold topological minimal self-joinings does not happen because of the existence of asymptotic points. Any expansive system $T: X \rightarrow X$ has a pair of distinct points $x, y \in X$ which are future-asymptotic and a pair of points $p, q \in X$ which are past-asymptotic. The $T^{\times 4}$ orbit of $\langle p, q, x, y \rangle$ is not dense.

To handle general not-necessarily-expansive maps, we need the following result, Theorem 10.30 in [G, H], due to S. Schwartzman. For completeness we include a demonstration: the neat proof below is slight variation of one due to Mike Boyle, Will Geller and Jim Propp. Say that two distinct points x and y are *future ϵ -bounded* if $|T^n x, T^n y| \leq \epsilon$ for all $n \geq 1$ and define ‘past ϵ -bounded’ analogously.

THEOREM 2.1 *Suppose X is infinite. For any positive ϵ there exists a future ϵ -bounded pair.*

Proof. Suppose, for the sake of contradiction, that there is no future ϵ -bounded pair. Let M be the supremum of those natural numbers N for which there exists a pair x, y with

$$|x, y| \geq \epsilon \quad \text{and} \quad \forall n \in [1, N) \quad |T^n x, T^n y| \leq \epsilon \tag{*}$$

If M is infinite then there exists a pair x_N, y_N fulfilling (*) and without loss of generality

$$x \stackrel{\Delta}{=} \lim_{N \rightarrow \infty} x_N \quad \text{and} \quad y \stackrel{\Delta}{=} \lim_{N \rightarrow \infty} y_N$$

exist. Evidently $|x, y| \geq \epsilon$. Moreover, for each $k \in \mathbb{Z}_+$

$$|T^k x, T^k y| = \lim_{N \rightarrow \infty} |T^k x_N, T^k y_N| \leq \lim_{N \rightarrow \infty} \epsilon = \epsilon,$$

since T^k is continuous. Thus one is forced to conclude that M is finite after all.

Any power of T is uniformly continuous, X being compact. So there exists δ such that

$$|x, y| < \delta \quad \Rightarrow \quad |T^m x, T^m y| \leq \epsilon \quad \text{for every } m \in [0, M)$$

and so the maximality of M for (*) implies that $|T^{-1} x, T^{-1} y| < \epsilon$. Applying this iteratively yields that

$$|T^{-i} x, T^{-i} y| < \epsilon \quad \text{for } i = 1, 2, 3,$$

(In other words the family $\{T^{-i}\}_{i=1}^\infty$ is equicontinuous.) Now cover X with finitely many, L , open balls of diameter δ . Let $F \subset X$ be any collection of $L+1$ points. For each integer N the set $T^N(F)$ has $L+1$ points and so by the pigeon-hole principle

there exists a pair $x, y \in F$ of distinct points such that $T^N x$ and $T^N y$ are in the same δ -ball Hence

$$\forall m < N \quad |T^m x, T^m y| < \varepsilon \tag{2.2}$$

Renaming x and y to x_N and y_N we can drop to a subsequence $N(i) \nearrow \infty$ for which

$$x_{N(1)} = x_{N(2)} = \dots \quad \text{and} \quad y_{N(1)} = y_{N(2)} = \dots$$

Call these two points x and y Now (2.2) holds for N replaced by any $N(i)$, hence for $N = \infty$ *A fortiori* x and y are future ε -bounded □

The next several paragraphs are devoted to strengthening this result to produce an ε -bounded pair x, y in *distinct* orbits, i.e. with $y \notin \mathcal{O}(x)$

The semigroup of continuous maps. Consider a compact metric space $(X, | \cdot |)$ Then $(\mathbf{C}, \| \cdot \|, | |)$ is a complete metric space, where \mathbf{C} denotes the set of continuous maps $f: X \rightarrow X$ under the supremum norm

$$\|f, g\| \triangleq \sup_{x \in X} |f(x) - g(x)|$$

Any $f \in \mathbf{C}$ is uniformly continuous and so there is a function $\varepsilon_f(\cdot)$, with $\varepsilon_f(r) \searrow 0$ as $r \searrow 0$, such that

$$|f(x) - f(y)| \leq \varepsilon_f(|x - y|)$$

for all $x, y \in X$ Letting fg denote composition $f \circ g$, the triangle inequality yields

$$\|f'g', fg\| \leq \|f', f\| + \varepsilon_f(\|g', g\|)$$

for any $f', g', f, g \in \mathbf{C}$ Thus $(f, g) \mapsto fg$ is a continuous map from $\mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$

Let $\mathbf{H} \subset \mathbf{C}$ denote the subcollection of invertible maps, necessarily homeomorphisms It is possible for a sequence of $f_n \in \mathbf{H}$ to have its limit in the complement $\mathbf{C} \setminus \mathbf{H}$ However

$$\text{If } g \triangleq \lim_{n \rightarrow \infty} f_n \text{ with each } f_n \in \mathbf{H} \text{ and } f_n f_k = f_k f_n \text{ for all } n \text{ and } k, \text{ then } g \in \mathbf{H}$$

For whenever two homeomorphisms f and F commute then

$$\|f^{-1}, F^{-1}\| = \|f^{-1} \circ (fF), F^{-1} \circ (fF)\| = \|F, f\|$$

Hence $\|f_n^{-1}, f_k^{-1}\| = \|f_n, f_k\|$ and so $\{f_n^{-1}\}_1^\infty$ is Cauchy and $G \triangleq \lim_n f_n^{-1}$ exists Letting I denote the identity map,

$$Gg = \lim_{n \rightarrow \infty} f_n^{-1} f_n = I = \lim_{n \rightarrow \infty} f_n f_n^{-1} = gG,$$

and one concludes that g is invertible

Define the norm $\|f\|$ to be $\|f, I\|$ and note that

$$\|fg\| \leq \|f, I\| + \varepsilon_f(\|g, I\|) = \|f\| + \|g\|$$

for all $f, g \in \mathbf{C}$ If $g \in \mathbf{H}$ then $\|g^{-1}\| = \|g^{-1} \circ g, I \circ g\| = \|g\|$ Thus

$$\|fg\| \geq \|f\| - \|g\|$$

Fix henceforth a $T \in \mathbf{H}$ and let $\mathbf{A}(T)$ denote its automorphism group consisting of those $S \in \mathbf{H}$ such that $ST = TS$. Here is a standard type of argument for normed groups

UNCOUNTABILITY LEMMA *Suppose there exist mutually commuting maps $\{P_n\}_1^\infty \subset \mathbf{A}(T)$, with $P_n \neq I$, such that $\|P_n\| \rightarrow 0$. Then for any positive ε the ball*

$$\{S \in \mathbf{A}(T) \mid \|S\| \leq \varepsilon\}$$

is uncountable

Proof By dropping to a subsequence of the $\{P_n\}$ we can arrange that $\sum_1^\infty \|P_n\| \leq \varepsilon$ and, for each K ,

$$\|P_K\| > \sum_{k=K+1}^\infty \|P_k\|$$

For a bit string $\vec{b} \in \{0, 1\}^\mathbb{N}$ define $R_K \in \mathbf{A}(T)$ by

$$R_K \triangleq P_1^{b(1)} P_2^{b(2)} \dots P_K^{b(K)}$$

Thus $\|R_N, R_{N+m}\| \leq \sum_{k=N+1}^{N+m} \|P_k\|$ which is less than $\|P_N\|$, hence $\{R_K\}_{K=1}^\infty$ is Cauchy. So $S_{\vec{b}} \triangleq \lim_{K \rightarrow \infty} R_K$ exists and $\|S_{\vec{b}}\| \leq \varepsilon$. Moreover, $S_{\vec{b}}$ commutes with T because each R_K commutes with T and composition is continuous.

Finally, if bit strings \vec{b} and \vec{c} are distinct, then $S_{\vec{b}} \neq S_{\vec{c}}$. For let K be smallest for which $b(K) \neq c(K)$, say $b(K) = 1$ and $c(K) = 0$. Then

$$\begin{aligned} \|S_{\vec{b}} S_{\vec{c}}^{-1}\| &= \left\| P_K \prod_{k=K+1}^\infty P_k^{b(k)} P_k^{-c(k)} \right\| \\ &\geq \|P_K\| - \sum_{k=K+1}^\infty \|P_k^{b(k)-c(k)}\| \geq \|P_K\| - \sum_{k=K+1}^\infty \|P_k\|, \end{aligned}$$

which is positive □

PROPOSITION 2.3a *Suppose $a \in X$ is a fixed point for T and there exists a sequence of points $z_n \rightarrow a$ and negative times $\{k(n)\}_1^\infty$ such that*

$$\inf_n |T^{k(n)} z_n, a| > 0$$

Then for all $\delta > 0$ there exists a point $y \neq a$ for which $\langle y, a \rangle$ is future δ -bounded

Proof Reduce δ to smaller than the above infimum and discard those z_n which are not within δ of a . By moving each $k(n)$ closer to zero, if need be, we may now assume that

$$\forall l \in (k(n) - 0] \quad |T^l z_n, a| \leq \delta \tag{2.4}$$

Without loss of generality $y \triangleq \lim_n T^{k(n)} z_n$ exists. Evidently $|y, a| \geq \delta$. If $\lim_n k(n) \neq -\infty$ then we could drop to a subsequence of $\{k(n)\}$ which was constant, say, always -7 . But then

$$|y, a| = \lim_{n \rightarrow \infty} |T^{k(n)} z_n, a| = \lim_{n \rightarrow \infty} |T^{-7} z_n, T^{-7} a| = 0,$$

which is a contradiction. Thus $k(n) \rightarrow -\infty$. So given any $m > 0$, eventually $m + k(n)$ is in $(k(n) - \delta, k(n) + \delta]$ for all large n . Consequently,

$$|T^m y, T^m a| = |T^{m+k(n)} y, a| \leq \delta$$

by inequality (2.4). Thus y and a are future δ -bounded. □

PROPOSITION 2.3b *Suppose $\#X = \infty$ and every minimal set is finite. Then for all ε there exists a future ε -bounded pair in distinct orbits.*

Proof. First suppose X consists only of periodic points. Then we can pick a convergent sequence $\{z_n\}_1^\infty$ of points in distinct orbits, with

$$\forall n \quad \mathcal{O}_T(z_n) \neq \mathcal{O}_T(a),$$

where $a \triangleq \lim_n z_n$. Thus a is a fixed point of $S \triangleq T^p$, where p denotes the least period of a . Choose δ sufficiently small that

$$\text{For all } y \in X \text{ and } m \in [0, p), \quad |y, a| \leq \delta \implies |T^m y, T^m a| \leq \varepsilon$$

Consequently, having taken δ smaller than the distance between any two of the p points of $\mathcal{O}_T(a)$, it will suffice to show that

There exists a point $y \neq a$ such that the pair $\langle y, a \rangle$ is future δ -bounded with respect to S . (2.5)

We may as well assume that no z_n can play the role of y and so there exist integers $\{k(n)\}$ such that

$$|S^{k(n)} z_n, a| \geq \delta$$

Since each z_n is a periodic point, we may take each $k(n)$ to be negative. The preceding proposition may now be applied to S to produce a point y which fulfills (2.5).

There exists an $x \in X$ with infinite T -orbit, is the other possibility. Pick $j(n) \nearrow \infty$ such that $a \triangleq T^{j(n)} x$ exists and is a point in a minimal set. Defining p , S and δ as above condition (2.5), it suffices to establish that condition

Dropping to a subsequence of $\{j(n)\}$ we can assume that all the $j(n)$ are congruent modulo p , say, to r . By replacing x by $T^{-r} x$ we can write

$$S^{-k(n)} x \xrightarrow{n} a,$$

where $k(n)$ is the negative number $-[j(n) - r]/p$. Setting $z_n \triangleq S^{-k(n)} x$ we have that

$$\inf_n |S^{k(n)} z_n, a| = |x, a| > 0,$$

and so we may again apply to S the preceding proposition. □

We now can obtain the desired strengthening of Schwartzman's Theorem

BOUNDEDNESS THEOREM 2.6 *Suppose T is a homeomorphism of an infinite compact metric space X . Then for every positive ε there exist a future ε -bounded pair $x, y \in X$ with $\mathcal{O}(x) \neq \mathcal{O}(y)$.*

Proof By the foregoing proposition we may assume that T has an infinite minimal set and so we may take T to be minimal. Suppose, for the sake of contradiction, that the conclusion fails for ε . By Schwartzman's theorem there exist distinct y and z which are future ε -bounded. Hence $z = T^l y$ for some integer l , and $T^l \neq I$. By minimality, for any point x there exists $N(l) \nearrow \infty$ such that $T^{N(l)} y \rightarrow x$ and thus $T^{N(l)} z \rightarrow T^l x$. Hence $|x, T^l x| \leq \varepsilon$ for all $x \in X$. In other words,

$$\|T^l\| \leq \varepsilon$$

Choosing a sequence $\varepsilon_n \searrow 0$ we obtain integers $l(n)$ such that

$$\|T^{l(n)}\| \rightarrow 0 \text{ as } n \rightarrow \infty$$

and $T^{l(n)} \neq I$. Thus we are entitled to the conclusion of the *Uncountability lemma*.

Fix some $x \in X$. For any S in the ε -ball in $A(T)$ we have that

$$|T^n x, T^n(Sx)| = |T^n x, S(T^n x)| \leq \|S\| \leq \varepsilon$$

for all $n \in \mathbb{Z}$. So the conclusion of our purported theorem can fail only if the point $S(x)$ lies in the T -orbit of x , a countable set. Since there are uncountably many S in the ε -ball, there must exist distinct $S_1, S_2 \in A(T)$ such that $S_1(x)$ equals $S_2(x)$. But then $S_1(T^n x) = S_2(T^n x)$ for every integer n and so S_1 and S_2 agree on the T -orbit of x , a dense set. Thus $S_1 = S_2$. □

No map has 4-fold topological minimal self-joinings. Suppose $T: X \rightarrow X$ is minimal and $\#X = \infty$. Fixing a small ε , the *Boundedness theorem* gives us future and past bounded pairs $f, f', p, p' \in X$ such that

$$|T^n f, T^n f'| \leq \varepsilon \text{ and } |T^{-n} p, T^{-n} p'| \leq \varepsilon \text{ for } n = 0, 1, 2,$$

with $\mathcal{O}(f) \neq \mathcal{O}(f')$ and $\mathcal{O}(p) \neq \mathcal{O}(p')$.

First suppose that the four points manage to inhabit only two orbits, say $p = T^t(f)$ and $p' = T^{t+s}(f')$ for some integers t and s . Any point $\langle x, y \rangle$ of $\bar{\mathcal{O}}_{T \times T}(f, f')$ must satisfy $|x, y| \leq \varepsilon$ or $|x, T^s y| \leq \varepsilon$, depending on whether it is in the future or past orbit closure of $\langle f, f' \rangle$. If T has even just 2-fold *tmsj* then $\bar{\mathcal{O}}_{T \times T}(f, f')$ equals $X \times X$ and so, fixing any point y , this implies

$$\mathbf{B}(y) \cup \mathbf{B}(T^s y) = X,$$

where $\mathbf{B}(y)$ is the closed radius- ε ball centered at y . But we could have taken ε to be so small that no two radius- ε balls can cover X .

Suppose instead that $\{f, f', p, p'\}$ inhabit three orbits, say $T^s(p') = f'$ and $p \notin \mathcal{O}(f)$. Then the triple $\langle f, f', T^s p \rangle$ belies 3-fold *tmsj* since its $T^{\times 3}$ orbit closure contains no triple of the form $\langle z, z', z \rangle$ where $|z, z'| > \varepsilon$.

Similarly, if $\{f, f', p, p'\}$ inhabit four orbits then

$$\langle z, z', z, z' \rangle \notin \bar{\mathcal{O}}_{T^{\times 4}}(f, f', p, p'),$$

which prohibits 4-fold topological minimal self-joinings.

Acknowledgement I thank Joe Auslander and Nelson Markley for cheerfully suffering and critically examining early versions of this argument. This work was supported by a National Science Foundation Postdoctoral Research Fellowship.

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