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THE TRANSLATION PLANES OF DEMPWOLFF

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1. Introduction. In [**2**], Dempwolff constructs three translation planes of order 16 using sharply 2-transitive sets of permutations in S_{16}. That is, if $\mathscr{T} \subseteq S_n$ acting on Λ is a sharply 2-transitive set of permutations then an affine plane of order n may be defined as follows: The set of points $= \{(x, y) | x, y \in \Lambda\}$ and the lines $= \{(x, y) | y = xg$ for fixed $y \in \mathscr{T}\}$, $\{(x, y)|x = c\}$, $\{(x, y)|y = c\}$ for $c \in \Lambda$.

Let V be a vector space of dimension k over $F \cong GF(p^r)$. A translation plane may be defined by finding a set M of $p^{kr} - 1$ linear transformations such that xy^{-1} is fixed point free on $V - \{\mathscr{O}\}$ for all $x \neq y$ in M.

Notice that if we allow V to act on itself $(v \xrightarrow{u} u + v)$ then MV is a sharply 2-transitive set on V if and only if xy^{-1} is fixed point free on $V - \{\mathscr{O}\}$ for all $x \neq y$ in M. If V_4 is 4-dimensional over $GF(2)$, $|M| = 15$. Note $GL(V_4) \cong A_8$ in this case. Aided by a computer program, Dempwolff [**2**] determined three sets M in $GL(V_4)$ which give rise to distinct translation planes. One of these planes admits the group $\mathbf{Z}_3 \times (\mathbf{Z}_3 \times A_4) \cdot \mathbf{Z}_2$ as a collineation group. In [**1**], Dempwolff constructed an infinite class of translation planes (orders 2^{4r}, r odd) which contain the order 16 plane mentioned above. Both the manner of construction and the collineation groups of the resulting planes are of interest.

In Section 2, we generalize Dempwolff's construction and in Section 3 consider the planes so constructed. In particular, Dempwolff's class mentioned above may be constructed by the derivation of a particular class of Knuth semifield planes.

2. The Dempwolff construction and generalization.

(2.1) *Definition.* Let V be a $2k$-dimensional vector space over $F \cong GF(p^r)$, p a prime. Let $\mathscr{G}$, K be subgroups of $GL_F(V)$ and let $h \in GL_F(V)$. Let $h^{\mathscr{G}} = \{g^{-1}hg | g \in \mathscr{G}\}$ and assume $\mathscr{G}$ normalizes K. If $(h^{\mathscr{G}} \cup K)V$ is sharply 2-transitive on V then a translation plane π may be defined: The points of $\pi = V \oplus V$ and the lines $\{(x, y)|x = c\}$, $\{(x, y)|y = c\}$ and $\{(x, y)|y = xg; g \in h^{\mathscr{G}} \cup K\}$. π is said to be of type $(h, \mathscr{G}, K)$.

If π is of type $(h, \mathscr{G}, K)$ we define the action of $\mathscr{G}$ on π by

$$(x, y) \xrightarrow{g} (xg, yg)$$

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for $g \in \mathscr{G}$. Since $g \in \mathscr{G}$ maps $y = xk$ onto $y = x(g^{-1}kg)$, $\mathscr{G}$ is a collineation group of π which fixes the net $\mathscr{N}_{K^+}$ whose lines consist of $x = c$, $y = c$, $y = xk$ for $k \in KV$. We shall use K^+ to denote $K \cup \{\mathscr{O}\}$.

(2.2) *Notes.* (a) If we define a multiplication in a type $(h, \mathscr{G}, K)$ translation plane we have $y = xk = x \cdot \bar{k}$ for all $k \in h^{\mathscr{G}} \cup K$ where $\bar{k}$ depends only on k. We shall call K^+ a *subquasifield* if $\bar{K}^+$ is a subquasifield of a quasifield coordinatizing π.

(b) If K^+ is a subquasifield then it is also a subnearfield since K is a subgroup of $GL_F(V)$. If K^+ is a field of matrices then K^+ is also a subfield of a corresponding quasifield of π. Let π_{K^+} denote the nearfield subplane coordinatized by K^+ if K^+ is a subquasifield.

(c) Any type $(h, \mathscr{G}, K)$ plane π admits the collineation group $\mathscr{G}$ which fixes the net $\mathscr{N}_{K^+}$ and acts transitively on the lines through $(0, 0)$ of $\pi - \mathscr{N}_{K^+}$ $(y = xh^g$ for $g \in \mathscr{G})$.

(d) Any generalized Hall plane of type i (see [**6**]) or semitransitive plane (see [**4**]) is of type $(h, \mathscr{G}, K)$ for certain h, $\mathscr{G}$ and K.

Conversely, we have:

(2.3) THEOREM. *Let π be a translation plane of type $(h, \mathscr{G}, K)$ where K^+ is a subquasifield. If $\mathscr{G}$ centralizes K and leaves π_{K^+} invariant then π is a semitransitive plane.*

Proof. If $\mathscr{G}$ centralizes K then $\mathscr{G}$ fixes $\mathscr{N}_{K^+}$ componentwise. If $\mathscr{G}$ leaves π_{K^+} invariant then π is semitransitive (see Jha's definition [**4**]).

(2.4) COROLLARY. *Let π be a translation plane of type $(h, \mathscr{G}, K)$ where* kern $\neq GF(2)$. *Assume K^+ is a subquasifield and $\mathscr{G}$ centralizes K. Either π_{K^+} is not invariant by $\mathscr{G}$ or K^+ is a field of order $\sqrt{\text{order } \pi}$ so that π_{K^+} is a Desarguesian Baer subplane.*

Proof. If kern $\neq GF(2)$ and the order of π is 16, then π is Desarguesian, Hall or a semifield plane (see [**7**]). In this case only the Hall plane is of type $(h, \mathscr{G}, K)$.

If the order of $\pi \neq 16$, then by [**4**] π is derivable and π_{K^+} is a Desarguesian Baer subplane.

(2.5) THEOREM. *Let V be a vector space of dimension s over $F \cong GF(p^r = q)$. Let $K \cup \{\mathscr{O}\} = K^+$ be a field $\cong GF(p^m)$ where $K \subseteq GL(s, q)$. Then the net $\mathscr{N}_{K^+}$ in $\pi = V \oplus V$ whose components are $y = xg$, $x = 0$ for $g \in K^+$ is a Desarguesian net. $\mathscr{N}_{K^+}$ is a replaceable net if and only if $|K^+| = \sqrt{q^s}$.*

Proof. V is a vector space over K^+ of dimension rs/m so π has dimension $2^{rs}m$ over K^+. Let

$$\pi_{V_0} = \{(V_0y, V_0h) | y, h \in K^+ \text{ for all } V_0 \in V\}.$$

Then clearly

$$\bigcup_{V_0 \in V} \pi_{V_0} = (X = 0) \bigcup_{g \in K^+} y = xg.$$

For $\mathcal{N}_{K^+}$ to be replaceable, $|\pi_{V_0}| = p^{rs} = p^{2m}$. Thus, $p^m = p^{rs_2}$. In this context we may overcome the restriction on the kern in (2.4).

(2.6) THEOREM. *Let π be a translation plane of order $\neq 16$ and of type $(h, \mathcal{G}, K)$. Let K^+ be a field of matrices and assume $\mathcal{G}$ centralizes K and leaves π_{K^+} invariant. Then $\mathcal{N}_{K^+}$ is a derivable net and π_{K^+} is a Baer subplane.*

Proof. Choosing coordinates within π_{K^+}, we have that $\bar{K}^+$ is a subfield of the (left) quasifield Q coordinatizing π and Q is a right vector space over $\bar{K}^+$ (since $\mathcal{N}_{K^+}$ is a Desarguesian net by (2.5)). From here we can use Jha's results to obtain the result. Actually, he used kern $\neq GF(2)$ to obtain that $\bar{K}^+$ is a subfield under the assumptions that it was a subquasifield.

On the other hand, if $\mathcal{N}_{K^+}$ is derivable:

(2.7) THEOREM. *Let π be a translation plane of type $(h, \mathcal{G}, K)$ where K^+ is a field of order $\sqrt{\text{order } \pi}$ and $\mathcal{G}$ centralizes K. If $\mathcal{G}$ does not leave π_{K^+} invariant then π is a Hall plane.*

Proof. Clearly $\mathcal{G} \subseteq GL(2, K^+)$ acting on any component of $\mathcal{N}_{K^+}$. Let $\sigma \in \mathcal{G}$ of order p where the characteristic of π is p. Then σ fixes pointwise a 1-space over K^+ on each of $x = 0$, $y = 0$ and $y = x$. Thus, σ fixes a 2-space $F(\sigma)$ over K^+ pointwise. Clearly, $F(\sigma) = \pi_{V_0}$ for some V_0 in V (see (2.5)). Thus, $F(0)$ is a Baer subplane in $\mathcal{N}_{K^+}$. Clearly, each Sylow p-subgroup S must also fix a Baer subplane $F(S)$ in $\mathcal{N}_{K^+}$ pointwise. π is of order q^{2k} and $\mathcal{G}$ acts transitively on the $q^k(q^k - 1)$ components of $\pi - \mathcal{N}_{K^+}$ (see (2.2) (a)). Thus, $q^k \mid |S|$ so there must be exactly $q^k + 1$ Sylow p-subgroups corresponding to the $q^k + 1$ Baer subplanes of $\mathcal{N}_{K^+}$ which are also subspaces. Since $\mathcal{G} \subseteq GL(2, K^+)$, clearly $SL(2, K^+) \subseteq \mathcal{G}$. If the order of π is not 16, π is Hall by [**3**]. If the order of π is 16, π is Hall or one of the planes constructed by Dempwolff (see [**7**]). However, since $\mathcal{G}$ fixes $\mathcal{N}_{K^+}$ componentwise this latter possibility cannot occur.

(2.8) THEOREM. *Let π be a translation plane of type $(h, \mathcal{G}, K)$ where $K, \mathcal{G} \subseteq GL(2k, p^r)$. If K^+ is a field of matrices of order p^{kr} centralized by $\mathcal{G}$ and $\mathcal{G}$ is the full normalizer of a Sylow p-subgroup within $GL(2, K^+)$, then π is a plane derived from a semifield plane of Knuth type II.*

Proof. By (2.4) there is a $\mathcal{G}$-invariant Baer subplane $F(S)$ which is pointwise fixed by a Sylow p-subgroup S. $F(S) \cap X = \mathcal{O}$ and $F(S) \cap Y = \mathcal{O}$ are l-spaces over K. By appropriate basis choice on $X = \mathcal{O}$, $\mathcal{G}$

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has the form

$$\left\{\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \middle| ac \neq \mathcal{O}, a, b, c \in K\right\}$$

(recall $\mathcal{G}$ is the full normalizer of S and we may choose $S = \left\{\begin{bmatrix} I & a \\ 0 & I \end{bmatrix} \middle| a \in K\right\}$ on $X = \mathcal{O}$). Thus, $\mathcal{G}$ contains the subgroups

$$\left\{\begin{bmatrix} I & \mathcal{O} \\ \mathcal{O} & a \end{bmatrix} \middle| a \in K - \mathcal{O}\right\} = {}_aH$$

and

$$\left\{\begin{bmatrix} a & \mathcal{O} \\ \mathcal{O} & I \end{bmatrix} \middle| a \in K - \mathcal{O}\right\} = {}^aH.$$

Considering the action on π, ${}_aH$ and ${}^aH$ both fix 2-spaces over K pointwise. (${}^aH$ and S both fix the same Baer subplane of $\mathcal{N}_K$ pointwise.) By [**6**], π may be derived from a semifield plane coordinatized by a semifield which is of dimension 2 over its right and middle nuclei simultaneously. That is, the right equals the middle nucleus. So, π is of Knuth type II (see [**5**] and Knuth [**8**]; also see (3.4) (1) and (2)).

(2.9) COROLLARY. *Let π be a translation plane of type $(h, \mathcal{G}, K)$ where K^+ is a field of matrices of order p^{kr} centralized by $\mathcal{G}$, $\mathcal{G}, K \subseteq GL(2k, p^r)$, and $\mathcal{G}$ is generated by Baer collineations. Then π is either Hall or a derived semifield plane of Knuth type II.*

Proof. By (2.4), we may assume $\mathcal{G} \subseteq GL(2, K^+)$ normalizes a Sylow p-subgroup. So, on $X = \mathcal{O}$,

$$\mathcal{G} \subseteq \left\{\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \middle| ac \neq \mathcal{O}, a, b, c \in K\right\} = \left\{\begin{bmatrix} a & \mathcal{O} \\ \mathcal{O} & a \end{bmatrix} \middle| a \in K - \mathcal{O}\right\}$$
$$\cdot \left\{\begin{bmatrix} I & a \\ \mathcal{O} & I \end{bmatrix} \middle| a \in K\right\} \cdot \left\{\begin{bmatrix} a & \mathcal{O} \\ \mathcal{O} & I \end{bmatrix} \middle| a \in K - \mathcal{O}\right\}.$$

If $g \in \mathcal{G}$ fixes a Baer subplane pointwise then g must fix pointwise an l-space over K on each of $X = \mathcal{O}$, $Y = \mathcal{O}$, and $Y = X$. If on $X = \mathcal{O}$ this l-space is not $F(S) \cap X = \mathcal{O}$, we may choose it to be $\{(h, \mathcal{O}) | h \in K\}$. That is, $g = \begin{bmatrix} I & 0 \\ 0 & c \end{bmatrix}$ (for $c \in K$) on $X = \mathcal{O}$. Notice the l-spaces over K^+ on $X = \mathcal{O}$ and $\neq F(X) \cap X = \mathcal{O}$ are in an orbit under S. Thus, if $\bar{g} \in \mathcal{G}$ fixes an l-space pointwise, $\bar{g}^l$ fixes $\{(h, \mathcal{O}) | h \in K\}$ pointwise for $l \in S$. The point of this is that any element of $\mathcal{G}$ may be written as a product of elements that pointwise fix either $F(S) \cap X = \mathcal{O}$ or

$\{(h, \mathscr{O})|h \in K\}$ on $X = \mathscr{O}$. That is, a typical element has the form

$$\begin{bmatrix} I & a \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} I & a \\ 0 & I \end{bmatrix} \begin{bmatrix} d & 0 \\ 0 & I \end{bmatrix} \cdots .$$

Since $S \lhd \mathscr{G}$, the elements of $\mathscr{G}$ may be written in the form

$$\begin{bmatrix} I & a \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} c & 0 \\ 0 & I \end{bmatrix}$$

for $a, b, c \in K^+$.

Originally, $K \subseteq \mathscr{G}$. Thinking of $\mathscr{G} \subseteq GL(2, K^+)$ we have

$$\left\{ \begin{bmatrix} d & \\ & d \end{bmatrix} \middle| d \in K \right\} \subseteq \mathscr{G}.$$

Thus,

$$\begin{bmatrix} d & \\ & d \end{bmatrix} = \begin{bmatrix} I & a \\ 0 & I \end{bmatrix} \begin{bmatrix} c & 0 \\ 0 & b \end{bmatrix}$$

if and only if $a = \mathscr{O}$ and $c = b = d$. In other words, for each element of the form $\begin{bmatrix} I & 0 \\ 0 & d \end{bmatrix}$ in $\mathscr{G}$ there is a corresponding element of the form $\begin{bmatrix} d & 0 \\ 0 & I \end{bmatrix}$. Thus, $\mathscr{G}$ contains

$$\left\{ \begin{bmatrix} d & 0 \\ 0 & d^{-1} \end{bmatrix} \middle| d \in K - \mathscr{O} \right\},$$

so $\mathscr{G}$ is the full normalizer of S within $GL(2, K^+)$.

3. The Dempwolff planes of order 2^{4r}, r odd, and other constructions.

(3.1) *The Dempwolff Planes.* In [**1**], Dempwolff constructs an infinite class of translation planes of type $(h, \mathscr{G}, K)$: Let V be 4-dimensional over $F \cong GF(2^r)$ where r is odd. Let

$$h = \begin{bmatrix} \mathscr{O} & j \\ j & I_2 \end{bmatrix}$$

where

$$j = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Let D_{2^r+1} be a dihedral group of order $2(2^r + 1)$ in $GL(2, 2^r)$ containing j and let φ_{2^r+1} denote the cyclic stem. Let $Z(GL(2, 2^r))$ denote the center of $GL(2, 2^r)$. Let

$$\bar{K} = Z(GL(2, 2^r)) \cdot \varphi_{2^r+1}.$$

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Dempwolff takes

$$K = \left\{ \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} \middle| \alpha \in \bar{K} \right\}$$

and

$$\mathscr{G} = K\left\{ \begin{bmatrix} I_2 & \alpha \\ 0 & I_2 \end{bmatrix} \middle| \alpha \in \bar{K} \right\} \left\{ \begin{bmatrix} \alpha & \mathscr{O} \\ \mathscr{O} & \alpha^{-1} \end{bmatrix} \middle| \alpha \in \bar{K} \right\}.$$

$\mathscr{G} \subseteq GL(4, 2^r)$ centralizes K and K is a field of order 2^{2r}. Also $\mathscr{G}$ is the full normalizer within $GL(2, \bar{K}^+)$ of

$$\left\{ \begin{bmatrix} I_2 & \mathscr{O} \\ \alpha & I_2 \end{bmatrix} \middle| \alpha \in \bar{K} \right\}.$$

Therefore, by (2.5) the Dempwolff planes may be derived from a semifield plane of Knuth type II. However, (2.6) does not adequately describe the corresponding semifields nor does it specify the kernel (kern) of π.

We shall give an alternate proof that $(h, \mathscr{G}, K)$ defined above does give rise to a sharply 2-transitive set $(h^{\mathscr{G}} \cup K)V$. In the process we extend Dempwolff's construction so as to obtain all of the derived Knuth type II planes of maximal kern.

(3.2) LEMMA. *Let $P \cong GF(p^r)$ and assume $x^2 + fx - g$ is irreducible over P for fixed $f, g \in P$. Then*

$$\left\{ \begin{bmatrix} \alpha & \beta \\ \beta g, & \alpha + \beta f \end{bmatrix} \middle| \alpha, \beta \in P \right\}$$

is a field of order p^{2r}.

Proof. Note

$$\left| \begin{bmatrix} \alpha & \beta \\ \beta g & \alpha + \beta f \end{bmatrix} \right| = 0$$

if and only if

$$\alpha(\alpha + \beta f) - \beta^2 g = 0$$

if and only if

$$(\alpha\beta^{-1})^2 + (\alpha\beta^{-1})f - g = 0 \quad \text{for } \beta \neq 0.$$

This set of matrices is additive and multiplicative, so (3.2) is proved.

(3.3) *The Knuth Semifields of Types I-IV.* Let $(\mathscr{G}, +) = (GF(q^2), +)$ and $t \in \mathscr{G} - (GF(q) = F)$. Define $t \cdot \alpha = t\alpha$ for $\alpha \in F$,

$$(t\alpha + \delta) \cdot (t\beta + \gamma) = (t\alpha) \cdot (t\beta) + t(\delta^\sigma\beta + \alpha\gamma) + \delta\gamma$$

where σ is an automorphism of F. If $f \neq 0$, σ, g are chosen so that $y^{\sigma+1}$

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$+ fy - g = 0$ has no solutions in F, then each of the multiplications I through IV gives rise to a semi-field $(\mathscr{G}, +, \cdot)$:

I. $(t\alpha) \cdot (t\beta) = t\alpha^\sigma \beta^{\sigma-1} f + \alpha^\sigma \beta^{\sigma-2} g$,

II. $(t\alpha) \cdot (t\beta) = t\alpha^\sigma \beta f + \alpha^\sigma \beta g$,

III. $(t\alpha) \cdot (t\beta) = t\alpha\beta^{\sigma-1} f + \alpha^{\sigma-1}\beta^{\sigma-2} g$,

IV. $(t\alpha) \cdot (t\beta) = t\alpha\beta f + \alpha^{\sigma-1}\beta g$.

(3.4) *Notes* (see [**8**]). (1) Any semifield $(\mathscr{G}, +. \cdot)$ such that

$$(t\alpha + \delta) \cdot (t\beta + \gamma) = (t\alpha) \cdot (t\beta) + t(\delta^\sigma \beta + \alpha\gamma) + \delta\gamma$$

for $\alpha, \beta, \delta, \gamma$ in F is of type II if and only if the right (nucleus) associator $=$ middle (nucleus) associator $= F$.

Proof. ($\Leftarrow$) That is,

$$\begin{aligned}(t\alpha) \cdot (t\beta) = ((t\alpha) \cdot t)\beta = (\alpha^{\sigma-1}t) \cdot (t\beta) &= \alpha^{\sigma-1}(t \cdot (t\beta)) \\ &= \alpha^{\sigma-1}((t \cdot t)\beta)\ t\alpha\beta f + \alpha^{\sigma-1}\beta g.\end{aligned}$$

(2) Also, [**8**] shows that if $(\mathscr{G}, +, \cdot)$ is a semifield which is 2-dimensional over F such that $a(bc) = (ab)c$ whenever any pair of elements of a, b, c are in F then there is a $t \in \mathscr{G} - F$ such that $\alpha t = t\alpha^\sigma$ for some automorphism σ of $\mathscr{G}$. So if F is the right and middle nucleus any such semifield 2-dimensional over F is of type II.

(3) Let P be the fixed field of σ and $(\mathscr{G}, +, \cdot)$ a semifield of type II. Then $tP + P = \{t\rho + \theta | \rho, \theta \in P\} =$ kernel $=$ left (nucleus) associator of $(\mathscr{G}, +, \cdot)$.

(3.5) *The Derived Knuth Semifields of Type I-IV* (see [**5**] (3.4)). Each of the following multiplications of $*$ of type i (with field addition) defines a (right) quasifield which coordinatizes a translation plane derived from a semifield plane of Knuth type i:

$$\begin{aligned}\text{I. } (t\alpha + \delta) * (t\beta + \gamma) = {}& t(\delta - \alpha^\sigma\beta^{-1}f - \alpha\beta^{-1}\gamma)^{\sigma-1}\beta \\ &+ (\delta - \alpha^\sigma\beta^{-1}f - \alpha\beta^{-1}\gamma)^{\sigma-1}\gamma + \alpha^\sigma\beta^{-\sigma-1}g,\ \sigma \neq 1, f \neq 0.\end{aligned}$$

$$\begin{aligned}\text{II. } (t\alpha + \delta) * (t\beta + \gamma) = {}& t(\delta - \alpha^\sigma\beta^{-\sigma}f - \alpha\beta^{-1}\gamma)^{\sigma-1}\beta \\ &+ (\delta - \alpha^\sigma\beta^{-\sigma}f - \alpha\beta^{-1}\gamma)^{\sigma-1}\gamma + \alpha^\sigma\beta^{-\sigma}g,\ \sigma \neq 1, f \neq 0.\end{aligned}$$

$$\begin{aligned}\text{III. } (t\alpha + \delta) * (t\beta + \gamma) = {}& t(\delta - \alpha\beta^{-1}f - \alpha\beta^{-1}\gamma)^{\sigma-1}\beta \\ &+ (\delta - \alpha\beta^{-1}f - \alpha\beta^{-1}\gamma)^{\sigma-1}\gamma + \alpha^{\sigma-1}\beta^{\sigma-1}g,\ \sigma \neq 1, f \neq 0.\end{aligned}$$

$$\begin{aligned}\text{IV. } (t\alpha + \delta) * (t\beta + \gamma) = {}& t(-\alpha\beta^{-\sigma}f - \alpha\beta^{-1}\gamma)^{\sigma-1}\beta \\ &+ (\delta - \alpha\beta^{-\sigma}f - \alpha\beta^{-1}\gamma)^{\sigma-1}\gamma + \alpha^{\sigma-1}\beta^{-\sigma}g,\ \sigma \neq 1, f \neq 0 \\ &\text{for } \alpha, \beta, \delta, \gamma \in F \cong GF(q).\end{aligned}$$

Also, $(t\alpha + \delta) * \gamma = t\alpha\gamma + \delta\gamma$ where σ is an automorphism of F and $y^{\sigma+1} + fy - y \neq 0$ for all y in F.

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(3.6) *Definition.* Let $(\mathcal{G}, +, \cdot)$ be a Knuth semifield of order q^2 and type II. We shall say that $(\mathcal{G}, +, \cdot)$ is *maximal* if and only if $tP + P \cong GF(q)$. That is, $P \cong GF(\sqrt{q})$ (see (3.4) (3)).

(Note that if $|tP + P| > q$ then $(\mathcal{G}, +, \cdot)$ is a field and $\sigma = 1$).

So $(\mathcal{G}, +, \cdot)$ is maximal precisely when the kernel is maximal (in terms of possible order).

(3.7) LEMMA. *Let $(\mathcal{G}, +, \cdot)$ be a maximal Knuth semifield of order q^2 and type II. Then $y^{\sigma+1} + fy - g \neq 0$ for all y in F (see (3.5)) if and only if $x^2 + fx - g$ is irreducible over $P \subseteq F$, $P \cong GF(\sqrt{q})$.*

Proof. If $P \cong GF(\sqrt{q})$ then $\sigma = \sqrt{q}$ so $y^{\sigma+1} \in P$. Hence $(y^{\sigma+1} + g)f^{-1}$ is in P. If $y^{\sigma+1} + fy - g = 0$ then $y \in P$. In this case, $y^\sigma = y$ and $y^{\sigma+1} = y^2$.

(3.8) THEOREM. *Let π be a derived Knuth plane of maximal type II. That is, π may be coordinatized by a derived semifield of maximal type. If the order of π is q^2 and $x^2 + fy - g$ is irreducible over $P \cong GF(\sqrt{q})$ then π is a translation plane of type $(h, \mathcal{G}, K)$ where*

$$h = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -f & -1 \\ g & 0 & -f & 0 \\ -fg & -g & 0 & -f \end{bmatrix}, \bar{K} = \left\{ \begin{bmatrix} \alpha & \beta \\ \beta g & \alpha + \beta f \end{bmatrix} \middle| \alpha, \beta \in P \right\},$$

$$K = \left\{ \begin{bmatrix} \delta & 0 \\ 0 & \delta \end{bmatrix} \middle| \delta \in \bar{K} \right\},$$

$$\mathcal{G} = \left\{ \begin{bmatrix} I & \delta \\ \mathcal{O} & I \end{bmatrix} \middle| \delta \in \bar{K} \right\} \cdot \left\{ \begin{bmatrix} \delta & \mathcal{O} \\ \mathcal{O} & \delta^{-1} \end{bmatrix} \middle| \delta \in K - \mathcal{O} \right\}.$$

(3.9) COROLLARY. *The Dempwolff planes π correspond to derived Knuth planes of maximal type II where $f = g = 1$. Thus $x^2 + x + 1$ is irreducible over P and $P \cong GF(2^r)$, r odd. That is,*

$$(t\alpha + \delta) * (t\beta + \gamma) = t(\delta + \alpha^{\sqrt{q}} + \alpha^{\sqrt{q}}\beta^{-\sqrt{q}} + \alpha\beta^{-1}\gamma)^{(\sqrt{q}-1)}\beta$$
$$+ (\delta + \alpha^{\sqrt{q}}\beta^{-\sqrt{q}} + \alpha\beta^{-1}\gamma)^{(\sqrt{q}-1)}\gamma + \alpha^{\sqrt{q}}\beta^{-\sqrt{q}}$$

for $\alpha, \beta, \delta, \gamma \in GF(\sqrt{q} = 2^r$, r odd) defines the quasifields coordinatizing the Dempwolff planes.

Proof. Let $(\mathcal{G}, +, \cdot)$ be the derived semifield of maximal Knuth type II as listed in (3.5) II. Let $P \cong GF(\sqrt{q})$, $P \subseteq F \cong GF(q)$ and $\mathcal{G}$ of order q^2. Thus, $\sigma = \sqrt{q}$.

By (3.7), $x^2 + fx - g$ is irreducible over P and by (3.2),

$$\left\{ \begin{bmatrix} \alpha & \beta \\ \beta g & \alpha + \beta f \end{bmatrix} \middle| \alpha, \beta \in P \right\}$$

is a field of order $(\sqrt{q})^2 = q$. We allow that $F = P[m]$, m a root of $x^2 + fx - g$. Also note that

$$(-(m+f))^2 = m^2 + 2mf + f^2 = -mf + g + 2mf + f^2$$
$$= mf + g + f^2 = (m+f)f + g = -(-(m+f)f + g.$$

That is, $-(m+f) = m^{\sqrt{q}} = m^{\sigma}$. So

$$(m\beta_1 + \beta_2)^{\sigma} = -(m+f)\beta_1 + \beta_2 = -m\beta_1 - f\beta_1 + \beta_2$$

for $\beta_i \in P$. By (3.5) II,

$$(t\alpha + \delta) * t = t(\delta^{\sigma^{-1}} - \alpha f^{\sigma^{-1}}) + \alpha^{\sigma} g = t(\delta^{\sigma} - \alpha g) + \alpha^{\sigma} g$$

($f \in P$ and $\sigma^{-1} = \sigma$). Let $\alpha, \delta \in F = m\alpha_1 + \alpha_2, m\delta_1 + \delta_2, \alpha_i, \delta_i \in P$, respectively. So

$$(t\alpha + \delta) * t = t(\delta^{\sigma} - \alpha f) + \alpha^{\sigma} g = (-m\alpha_1 - f\alpha_1 + \alpha_2)g$$
$$+ t(-m\delta_1 - f\delta_1 + \delta_2) - f(m\alpha_1 + \alpha_2)).$$

Now define $h \in GL(4, \sqrt{q})$ by $xh = x * t$ (where $x \in \mathcal{G}$) in terms of the basis $\{1, m, t, tm\}$ for $\mathcal{G}$ over P. If $x = t\alpha + \delta$ then

$$x * t = (\delta_2, \delta_1, \alpha_2, \alpha_1)h = (\delta_2, \delta_1, \alpha_2, \alpha_1)\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -f & -1 \\ g & 0 & -f & 0 \\ -fg & -g & 0 & -f \end{bmatrix}$$

in terms of $\{1, m, t, tm\}$.

$\{y = x * (t\alpha + \beta)$ for $\alpha \neq 0\}$ is an orbit under the group in the semifield plane

$$\bar{\pi} = \{(x, y) \to (x, y\beta) | \alpha, \beta \in F - \{\mathcal{O}\}\} \cdot \{(x, y) \to (x, x\alpha + y) | \alpha \in F\}.$$

And,

$$(t\alpha + \beta)*(mb + a) = t(m\alpha_1 + \alpha_2)(mb + a) + (m\beta_1 + \beta_2)(mb + a)$$
$$= t(m^2\alpha_1 b + m(b\alpha_2 + \alpha_1 a) + \alpha_2 a) + m^2(\beta_1 b) + m(b\beta_2 + \beta_1 a) + \beta_2 a$$
$$= t((-mf + y)\alpha_1 + b + m(b\alpha_2 + \alpha_1 a) + \alpha_2 a)$$
$$+ (-mf + y)\beta_1 b + m(b\beta_2 + \beta_1 a) + \beta_2 a$$
$$= (\beta_2, \beta_1, \alpha_2, \alpha_1)\begin{bmatrix} a & b & 0 & 0 \\ bg & a + bf & & 0 \\ & & a & b \\ & & bg & a + bf \end{bmatrix}.$$

The group $(x, y) \to (x, y\beta)$ in $\bar{\pi}$ appears in the form

$$(tx_1 + y_1, tx_2 + y_2) \to (tx_1 + y_1\beta, tx_2 + y_2\beta)$$

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in π. We are representing F as the matrix field

$$\left\{\begin{bmatrix} a & b \\ bg & a+bf \end{bmatrix} \middle| a, b \in P\right\}$$

when considering multiplication $x * \alpha = x * (mb + a)$. So,

$$(tx_1 + y_1, tx_2 + y_2) \rightarrow (tx_1 + y_1\beta, tx_2 + y_2\beta)$$

in π appears in the form

$$\begin{bmatrix} \begin{matrix} I & \mathcal{O} \\ \mathcal{O} & \beta \end{matrix} & \mathcal{O} \\ \mathcal{O} & \begin{matrix} I & \mathcal{O} \\ \mathcal{O} & \beta \end{matrix} \end{bmatrix}$$

where

$$\beta = \begin{bmatrix} c & d \\ dg & a+df \end{bmatrix} \quad \text{for some } c, d \text{ in } P.$$

The elation group $(x, y) \rightarrow (x, x\alpha + y)$ in $\bar{\pi}$ is

$$\begin{bmatrix} \begin{matrix} I & \alpha \\ 0 & I \end{matrix} & \\ & \begin{matrix} I & \alpha \\ 0 & I \end{matrix} \end{bmatrix}$$

in π.

Thus, π is of type $(h, \mathcal{G}, K)$ where

$$h = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -f & -1 \\ g & 0 & -f & 0 \\ -fg & -f & 0 & -f \end{bmatrix},$$

$$K = \left\{\begin{bmatrix} a & b & 0 & 0 \\ bg & a+bf & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & bg & a+bg \end{bmatrix} \middle| a, b \in P\right\},$$

$$\mathcal{G} = \left\{\begin{bmatrix} I_2 & \mathcal{O}_2 \\ \mathcal{O}_2 & \begin{matrix} a & b \\ bg & a+bf \end{matrix} \end{bmatrix} \middle| a, b \in P\right\} \left\{\begin{bmatrix} I_2 & \begin{matrix} a & b \\ bg & a+bf \end{matrix} \\ \mathcal{O} & I_2 \end{bmatrix}\right\}.$$

Note that the group $\mathcal{G}$ we have used is not the full normalizer in $GL(2, \sqrt{q})$ of $S = \left\{\begin{bmatrix} I & \alpha \\ \mathcal{O} & I \end{bmatrix} \middle| \alpha \in F\right\}$. However, it is easy to check that the full normalizer $\mathcal{G}^*$ of S within $GL(2, \sqrt{q})$ leaves $h^{\mathcal{G}}$ invariant (by conjugation) so π is also of type $(h, \mathcal{G}^*, K)$.

In the Dempwolff plane,

$$h = \begin{bmatrix} \mathcal{O} & \begin{matrix} 1 & 0 \\ 1 & 1 \end{matrix} \\ \begin{matrix} 1 & 0 \\ 1 & 1 \end{matrix} & I_2 \end{bmatrix}.$$

Obviously in this case $f = g = 1$ and $P \cong GF(2^r)$ implies r is odd.

(3.10) COROLLARY. *Let V_4 be a 4-dimensional vector space over $P \cong GF(p^r)$. Let $x^2 + fx - g$ be irreducible over P and let*

$$K = \left\{ \begin{bmatrix} \begin{matrix} a & b \\ bg & a+bf \end{matrix} & \mathcal{O}_2 \\ \mathcal{O}_2 & \begin{matrix} a & b \\ bg & a+bf \end{matrix} \end{bmatrix} \middle| a, b \in P \right\}.$$

Let

$$\mathcal{G} = \left\{ \begin{bmatrix} I_2 & \mathcal{O}_2 \\ \mathcal{O}_2 & \beta \end{bmatrix} \middle| \beta \in K \right\} \left\{ \begin{bmatrix} I_2 & \alpha \\ \mathcal{O}_2 & I_2 \end{bmatrix} \middle| \alpha \in K \right\}.$$

Let

$$h = \begin{bmatrix} \mathcal{O}_2 & \begin{matrix} 1 & 0 \\ -f & -1 \end{matrix} \\ \begin{matrix} g & 0 \\ -fg & -g \end{matrix} & \begin{matrix} -f & 0 \\ 0 & -f \end{matrix} \end{bmatrix}.$$

Then $(h^{\mathcal{G}} \cup K)V_4$ is a sharply 2-transitive set on V_4.

In a similar manner, other derived Knuth type planes may be constructed. The possible planes are given in the following table. We leave the details for the reader.

Let $P \cong GF(p^r)$, $x^2 + xf - g$ irreducible over P. Let

$$K = \left\{ \begin{bmatrix} \begin{matrix} a & b \\ bg & a+bf \end{matrix} & \mathcal{O}_2 \\ \mathcal{O}_2 & \begin{matrix} a & b \\ bg & a+bf \end{matrix} \end{bmatrix} \middle| a, b \in P \right\}.$$

Note that $h_{\mathrm{IV}} = h_{\mathrm{III}}$ but we use a different group $\mathcal{G}$ to define $h^{\mathcal{G}}$. Also note that K is not a collineation group of a derived type III or IV plane.

4. Remarks on the collineation groups of the Dempwolff planes. Recall from (3.3),

$$K = \left\{ \begin{bmatrix} Z & 0 \\ 0 & Z \end{bmatrix} \middle| Z \in \bar{K} \right\}.$$

Fixed point free element	Group	Sharply 2-transitive	Translation Plane
$h_{II} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ g & 0 & -f & -1 \\ g & 0 & -f & 0 \\ -fg & -f & 0 & -f \end{bmatrix}$	$\mathscr{G} = S_0H_0$ $S_0 = \left\{\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \middle\| Z \in K\right\}$ $H_0 = \left\{\begin{bmatrix} Z & 0 \\ 0 & I \end{bmatrix} \middle\| Z \in K\right\}$ K is a collineation group of π	$(h_{II}{}^{\mathscr{G}} \cup K)V_4$	Derived Knuth type II plane
$h_{III} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & -f & -1 \\ g & 0 & -f & 0 \\ -fg & -f & f^2 & f \end{bmatrix}$	$\mathscr{G} = S_0H_0$ K is not a collineation group of π	$(h_{III}{}^{\mathscr{G}} \cup K)V_4$	Derived Knuth type III
$h_{IV} = h_{III}$	$\mathscr{G} = S_0H_1$ $H_1 = \left\{\begin{bmatrix} I & 0 \\ 0 & Z \end{bmatrix} \middle\| Z \in K\right\}$ K is not a collineation group of π	$(h_{IV}{}^{\mathscr{G}} \cup K)V_4$	Derived Knuth type IV

Let

$$S = \left\{\begin{bmatrix} I & \alpha \\ 0 & I \end{bmatrix} \middle| \alpha \in \bar{K}\right\}$$

$$Q = \left\{\begin{bmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{bmatrix} \middle| \alpha \in K\right\}.$$

Then the full normalizer of S with $GL(2, K)$ is KSQ. We may take SQ or KSQ as $\mathscr{G}$. The full normalizer of K within $GL(4, P)$ is $KSQ\langle T\rangle$ where $T = \begin{bmatrix} j & \mathscr{O} \\ \mathscr{O} & j \end{bmatrix}$, $j = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Recall $h = \begin{bmatrix} \mathscr{O} & j \\ j & I_2 \end{bmatrix}$.

Let $\mathscr{G}$ be KSQ. Dempwolff shows that the full linear translation complement is $\mathscr{G} \times \mathbf{Z}_{2^r+1}$ where $\mathbf{Z}_{2^r+1}$ is a cyclic group of order $q + 1$.

$$\begin{aligned} \mathscr{G} \times \mathbf{Z}_{2^r+1} &= KSQ\langle T\rangle \times \mathbf{Z}_{2^r+1} \\ &= S\left\{\begin{bmatrix} I & 0 \\ 0 & Z \end{bmatrix} \middle| Z \in \bar{K}\right\}\left\{\begin{bmatrix} Z & 0 \\ 0 & I \end{bmatrix}\right\}\langle T\rangle \times \mathbf{Z}_{2^r+1} \\ &= SH_0H_1\langle T\rangle \times \mathbf{Z}_{2^r+1}. \end{aligned}$$

By our analysis we may identify the groups as follows.

(4.1) *Notes.* S is an elation group in the semifield plane $\bar{\pi}$ with axis $X = \mathscr{O}$ (under appropriate choice of coordinates).

H_0 and H_1 are homology groups with axes $Y = \mathscr{O}$ and $X = \mathscr{O}$, respectively.

The kernel of $\bar{\pi} = tP + P$ induces the kern of π and the group $\mathbf{Z}_{2^r+1}$. (π is a vector space of dimension 8 over $GF(2^r)$.)

The group $\langle T \rangle$ is obtained from an automorphism of order 2 of the semifield that fixes $tP + P$ pointwise.

That is, let $s \in GF(2^{2r}) - \{tP + P\}$ such that $s^2 = s + 1$. Since s^{2^r} is also a root of $x^2 + x + 1$ we have $s^{2^r} = s + 1 = s^2$. Write out the multiplication of the semifield in terms of the basis $\{1, s\}$ over $tP + P$. Letting $k(t\alpha + \beta) = t\alpha$ for $\alpha, \beta \in F$, it is not difficult to verify that

$$(\bar{\alpha}s + \bar{\beta})(\bar{\delta}s + \bar{\gamma}) = (\bar{\alpha}\bar{\delta} + \bar{\alpha}k(\bar{\delta}) + \bar{\beta}\bar{\delta} + \bar{\alpha}\bar{\gamma})s + (\bar{\alpha}\bar{\delta} + \bar{\alpha}k(\bar{\gamma}) + \bar{\beta}\bar{\gamma})$$

for all $\bar{\alpha}$, $\bar{\beta}$, $\bar{\delta}$, $\bar{\gamma}$ in $tP + P$. From here, it follows that the mapping $\hat{T}$ such that

$$\hat{T}(\bar{\alpha}s + \bar{\beta}) = \bar{\alpha}(s + 1) + \bar{\beta}$$

is the unique automorphism of order 2 which fixes $tP + P$ pointwise. Since $\hat{T}$ maps s to $s + 1$, $\hat{T}$ leaves $GF(2^{2r})$ invariant. $GF(2^{2r})$ corresponds to $\mathscr{G}_0$ in Dempwolff's construction, so $\hat{T}$ induces a collineation of π which fixes a Baer subplane pointwise and thus may be considered an element in $GL(4, 2^r)$ which normalizes $\mathscr{G}_0$.

Note T in the Dempwolff plane is $\begin{bmatrix} j & 0 \\ 0 & j \end{bmatrix}$ and as a collineation fixes the components $y = xg$ if and only if $jgj^{-1} = g$. So T fixes $y = xh$ and

$$y = x\begin{bmatrix} Z_{a,0} & \\ & Z_{a,0} \end{bmatrix} = x\begin{bmatrix} a & & & \\ & a & & \\ & & a & \\ & & & a \end{bmatrix}$$

for $a \in P$. T also fixes

$$y = xh\begin{bmatrix} a & & & \\ & a & & \\ & & a & \\ & & & a \end{bmatrix}.$$

Note

$$x * (ta) = xh\begin{bmatrix} a & & & \\ & a & & \\ & & a & \\ & & & a \end{bmatrix}$$

since $a \in P$.

5. Open questions and problems. (1) Let π be a translation plane of type $(h, \mathscr{G}, K)$ of order $\neq 16$. If K^+ is a field then we have, see (2.6), that if $\mathscr{G}$ centralizes K^+ and leaves π_{K^+} invariant then π_{K^+} is a Baer subplane and π is derivable. Characterize the derived planes.

(2) If π is of type $(h, \mathscr{G}, K)$, K^+ a field, $\mathscr{G}$ centralizes K^+ but does not leave π_{K^+} invariant. Is π a Hall plane?

(3) If π is of type $(h, \mathscr{G}, K)$, K^+ a field, is there a subgroup $\mathscr{G}$ of $\mathscr{G}$ such that $\mathscr{G}$ centralizes K and π is of type $(h, \overline{\mathscr{G}}, K)$?

(4) For planes of order $\neq 16$ the Hall planes are the only known $(h, \mathscr{G}, K)$ planes to admit a nonsolvable collineation group. If π is $(h, \mathscr{G}, K)$ and $\mathscr{G}$ is nonsolvable, is π Hall?

References

1. V. Dempwolff, *Eine Klasse Von Translationsebenen uber* $GF(2^{2k+1})$. (To appear.)
2. ——— *Einige Translationsebenen der Ordnung* 16 *und Ihre Kollineationen.* (To appear.)
3. D. A. Foulser, N. L. Johnson and T. G. Ostrom, *Characterization of the Desarguesian and Hall planes of order* q^2 *by* $SL(2, q)$. (To appear.)
4. V. Jha, *Finite semi-transitive planes.* (To appear.)
5. N. L. Johnson, *Translation planes constructed from semi-field planes*, Pacific J. Math. *36* (1971), 701–711.
6. ——— *Distortion and generalized Hall planes*, Geom. Ded. *4* (1975), 437–456.
7. ——— *The translation planes of order* 16 *that admit* $SL(2, 4)$. (To appear.)
8. D. E. Knuth, *Finite semifields and projective planes*, J. Algebra *2* (1965), 182–217.

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