INDECOMPOSABLE REPRESENTATIONS OF THE EUCLIDEAN ALGEBRA \( \mathfrak{e}(3) \) FROM IRREDUCIBLE REPRESENTATIONS OF \( \mathfrak{sl}(4, \mathbb{C}) \)

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(Received 26 July 2010)

Abstract

The Euclidean group \( E(3) \) is the noncompact, semidirect product group \( E(3) \cong \mathbb{R}^3 \rtimes \text{SO}(3) \). It is the Lie group of orientation-preserving isometries of three-dimensional Euclidean space. The Euclidean algebra \( \mathfrak{e}(3) \) is the complexification of the Lie algebra of \( E(3) \). We embed the Euclidean algebra \( \mathfrak{e}(3) \) into the simple Lie algebra \( \mathfrak{sl}(4, \mathbb{C}) \) and show that the irreducible representations \( V(m, 0, 0) \) and \( V(0, 0, m) \) of \( \mathfrak{sl}(4, \mathbb{C}) \) are \( \mathfrak{e}(3) \)-indecomposable, thus creating a new class of indecomposable \( \mathfrak{e}(3) \)-modules. We then show that \( V(0, m, 0) \) may decompose.


Keywords and phrases: Euclidean algebra, indecomposable representations, Lie algebra embedding.

1. Introduction

The Euclidean group \( E(3) \) is the noncompact, semidirect product group \( E(3) \cong \mathbb{R}^3 \rtimes \text{SO}(3) \). It is the Lie group of orientation-preserving isometries of three-dimensional Euclidean space. The Euclidean algebra \( \mathfrak{e}(3) \) is the complexification of the Lie algebra of \( E(3) \). Its finite-dimensional irreducible representations are not very interesting, but classifying its indecomposable representations remains a significant challenge. We remind the reader that a representation is \textit{irreducible} if it has no proper subrepresentations. It is \textit{indecomposable} if it is not isomorphic to a direct sum of two nonzero subrepresentations.

Although a full classification of \( \mathfrak{e}(3) \)-indecomposable representations remains elusive, constructing large classes of indecomposable representations that may be classified is a viable option. Towards this end, in the current paper we embed the Euclidean algebra \( \mathfrak{e}(3) \) into the simple Lie algebra \( \mathfrak{sl}(4, \mathbb{C}) \) and examine certain irreducible representations of \( \mathfrak{sl}(4, \mathbb{C}) \) to determine whether or not they remain indecomposable upon restriction to \( \mathfrak{e}(3) \) under this embedding.

The work of A.D. is partially supported by the Professional Staff Congress/City University of New York (PSC/CUNY). The work of J.R. is partially supported by the Natural Sciences and Engineering Research Council (NSERC).

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This direction of research has been pursued, for instance, by Douglas and Premat [3], who showed that irreducible \(\text{sl}(3, \mathbb{C})\)-modules remain \(\varepsilon(2)\)-indecomposable, and later by Casati et al. [2], who established that irreducible \(\text{sl}(3, \mathbb{C})\)- and \(\text{so}(5, \mathbb{C})\)-modules remain indecomposable modules of the Diamond Lie algebra under appropriate embeddings. The Diamond Lie algebra is a central extension of the Poincaré Lie algebra in two dimensions.

In the current paper, we show that the irreducible representations \(V(m, 0, 0)\) and \(V(0, 0, m)\) of \(\text{sl}(4, \mathbb{C})\) remain \(\varepsilon(3)\)-indecomposable for all nonnegative integers \(m\), thus creating a new class of \(\varepsilon(3)\)-indecomposable modules. We then present examples in low dimension, based upon which we will conjecture that \(V(0, m, 0)\) is not indecomposable for any nonnegative integer \(m\).

The paper is organized as follows. In Section 2 we describe the basis and commutation relations of \(\varepsilon(3)\). Section 3 records information about the simple Lie algebra \(\text{sl}(4, \mathbb{C})\), and its irreducible representations that will be employed in the following section. In Section 4 we embed \(\varepsilon(3)\) into \(\text{sl}(4, \mathbb{C})\), and show that the \(\text{sl}(4, \mathbb{C})\) irreducible representations \(V(m, 0, 0)\) and \(V(0, 0, m)\) remain \(\varepsilon(3)\)-indecomposable under this embedding. The final section includes the presentation of examples illustrating the decomposition of \(V(0, m, 0)\).

2. The Euclidean algebra \(\varepsilon(3)\)

The Euclidean algebra \(\varepsilon(3)\) is the complexification of the Lie algebra of the Euclidean Lie group \(E(3)\). For a more detailed discussion of \(E(3)\), and the calculation of its Lie algebra we refer the reader to [5]. The Euclidean algebra \(\varepsilon(3)\) has basis \(E, H, F, P_0, P_{\pm}\), and nonzero commutation relations,

\[
\begin{align*}
[H, P_{\pm}] &= \pm 2P_{\pm}, & [E, P_0] &= -P_+, & [F, P_0] &= -P_-, \\
[F, P_+] &= -2P_0, & [E, P_-] &= -2P_0.
\end{align*}
\]

One can easily see that \(\langle E, H, F \rangle \cong \text{sl}(2, \mathbb{C})\), and that \(\langle P_0, P_{\pm} \rangle\) is an abelian ideal of \(\varepsilon(3)\).

3. The simple Lie algebra \(\text{sl}(4, \mathbb{C})\) and its irreducible representations

The special linear algebra \(\text{sl}(4, \mathbb{C})\) is the 15-dimensional Lie algebra of traceless \(4 \times 4\) matrices with complex entries. It is the simple Lie algebra of type \(A_3\). Let \(\{x_i, y_j, h_j, 1 \leq i \leq 6, 1 \leq j \leq 3\}\) be the Chevalley basis of \(\text{sl}(4, \mathbb{C})\) defined by

\[
\begin{align*}
&ah_1 + bh_2 + ch_3 + dx_1 + ex_2 + fx_3 + gx_4 + hx_5 + ix_6 \\
&+ d'y_1 + e'y_2 + f'y_3 + g'y_4 + h'y_5 + i'y_6 \\
\end{align*}
\]

\[
= \begin{pmatrix}
a & d & -g & i \\
d' & b - a & e & -h \\
-g' & e' & c - b & f \\
i' & -h' & f' & -c
\end{pmatrix}.
\]
For \( i = 1, 2, \) or \( 3, \) define \( \Lambda_i \in \h^* \) by \( \Lambda_i(h_j) = \delta_{ij}. \) For each \( \lambda = m_1 \Lambda_1 + m_2 \Lambda_2 + m_3 \Lambda_3 \in \h^* \) with nonnegative integers \( m_1, m_2, m_3 \) there exists a finite-dimensional irreducible \( \mathfrak{sl}(4, \mathbb{C}) \)-module \( V(m_1, m_2, m_3) \) which can be realized as the quotient of the universal enveloping algebra \( \mathcal{U}(\mathfrak{sl}(4, \mathbb{C})) \) by the left ideal \( J_\lambda, \) generated by \( x_i, h_i - \lambda(h_i), y_i^{1+\lambda(h_i)}, 1 \leq i \leq 3 \) (here the action of \( \mathcal{U}(\mathfrak{sl}(4, \mathbb{C})) \) on itself and on \( V(m_1, m_2, m_3) \) is given by left multiplication). We will denote the element \( 1 + J_\lambda \) of \( V(m_1, m_2, m_3) \) by \( v_\lambda, \) or simply \( v \) if there is no ambiguity.

We describe here a basis of irreducible \( \mathfrak{sl}(4, \mathbb{C}) \) representations due to Littelman \[7\] (as reported in \[1\] in a more general setting).

**Theorem 3.1** \[7\]. For nonnegative integers \( m_1, m_2, m_3, \) let \( V(m_1, m_2, m_3) \) be the finite-dimensional irreducible representation of \( \mathfrak{sl}(4, \mathbb{C}) \) with highest weight \( \lambda = m_1 \lambda_1 + m_2 \lambda_2 + m_3 \lambda_3. \) Then the following is a basis of \( V(m_1, m_2, m_3): \)

\[
\mathcal{B}_\lambda = \left\{ y_1^{a_1} y_2^{a_2} y_3^{a_3} y_1^{a_4} v \right\}, \text{ where } y_i^{a_i} = \frac{y_i^a}{a!},
\]

subject to the following constraints:

\[
0 \leq a_1^3 \leq m_1, \quad a_1^3 \leq a_2^3 \leq m_2 + a_1^3, \quad a_2^3 \leq a_3^3 \leq m_3 + a_2^3, \quad 0 \leq a_2^3 \leq m_1 - 2a_3^3 + a_2^3, \quad a_2^3 \leq a_2^3 \leq m_2 + a_1^3 + a_2^3 - 2a_3^3 + a_2^3, \quad 0 \leq a_1^3 \leq m_1 - 2(a_3^3 + a_2^3) + a_3^3 + a_2^3.
\]

The \( \mathfrak{sl}(4, \mathbb{C}) \) irreducible representations \( V(m, 0, 0), \) and \( V(0, 0, m) \) are the focus of the current paper. Note that \( V(m, 0, 0) \cong V(0, 0, m)^* \). In the special case \( V(m, 0, 0), \) the basis relations of Equation \( (3.3) \) for \( \mathcal{B}_{(m,0,0)} \) reduce to

\[
0 \leq a_1^3 = a_2^3 = a_3^3 \leq m, \quad 0 \leq a_2^3 = a_2^3 \leq m - a_1^3, \quad 0 \leq a_1^3 \leq m - a_3^3 - a_1^3.
\]

The following lemma will be used below.

**Lemma 3.2.** Suppose that \( 0 \leq a + b + c \leq m. \) Then the element \( y_1^a y_4^b y_6^c v \in V(m, 0, 0) \) is a nonzero scalar multiple of the element \( y_1^{(a)} y_2^{(b)} y_3^{(c)} y_2^{(c)} y_1^{(c)} v \in \mathcal{B}_{(m,0,0)} \).

**Proof.** From Equation \( (3.4) \), we can see that the element \( y_1^{(a)} y_2^{(b)} y_3^{(c)} y_2^{(c)} y_1^{(c)} v \) with \( 0 \leq a + b + c \leq m \) is a member of \( \mathcal{B}_{(m,0,0)} \).

We first show that \( y_2^b y_1^a y_6^c v \) is a nonzero scalar multiple of \( y_2^b y_6^c v. \) Since \([y_1, y_6] = [y_2, y_6] = [y_4, y_6] = 0, \) it suffices to show that \( y_2^b y_1^a v \) is a nonzero scalar multiple of \( y_4^a v. \) Let \( b = 1; \) using the fact that \( y_2 v = 0, \) we obtain

\[
y_2 y_1 v = -y_4 v.
\]
Assume that \( y^i_2 y^i_1 v = -\alpha_i y^b_i v \) for all \( i \) such that \( 1 \leq i \leq b - 1 < m \), with \( \alpha_i \) a nonzero scalar. Then, using \([y_2, y_4] = 0\) and \([y_1, y_2] = y_4\),
\[
y^b_2 y^b_1 v = y^b_2 y_1 y^b_2 y_1^{-1} v - \alpha_{b-1} y^b_4 v
\]
\[
= y^b_2 y_1 y^b_2 v - 2\alpha_{b-1} y^b_4 v + \alpha_{b-1} y^b_4 v
\]
\[
= y_2 y_1 y^b_2 v - (b - 1)\alpha_{b-1} y^b_4 v
\]
\[
= -\alpha_{b-1} y^b_4 v - (b - 1)\alpha_{b-1} y^b_4 v
\]
\[
= -b\alpha_{b-1} y^b_4 v.
\]

We now show that \( y^c_3 y^c_2 y^c_1 v \) is a nonzero scalar multiple of \( y^c_6 v \), proceeding by induction on \( c \). If \( c = 1 \), using the fact that \( y_2 v = y_3 v = 0 \),
\[
y_3 y_2 y_1 v = -y_3 y_4 v = -y_6 v.
\]

Assume that \( y^i_3 y^i_2 y^i_1 v = \beta_i y^i_6 v \) for all \( i \) such that \( 1 \leq i < c < m \), where \( \beta_i \) is an nonzero scalar. We show that it holds for \( i = c \). Note that from the above work we have \( y^c_2 y^c_1 v = \alpha y^c_6 v \) for a nonzero scalar \( \alpha \), and that \([y_3, y_6] = 0\), so
\[
\frac{1}{\alpha} y^c_3 y^c_2 y^c_1 v = y^c_3 y^c_2 y^c_1 v
\]
\[
= y^c_3^{-1} y_4 y_3 y^c_4^{-1} v - \beta_{c-1} y^c_6 v
\]
\[
= y^c_3^{-2} y_4 y^c_3 y^c_4^{-1} v - 2\beta_{c-1} y^c_6 v
\]
\[
= y^c_3 y_4 y^c_3^{-1} y^c_4^{-1} v - (c - 1)\beta_{c-1} y^c_6 v
\]
\[
= -c\beta_{c-1} y^c_6 v.
\]

We have shown that \( y^c_3 y^c_2 y^c_1 v \) is a nonzero scalar multiple of \( y^c_6 v \), and that \( y^b_2 y^b_1 y^c_1 v \) is a nonzero scalar multiple of \( y^c_4 y^c_6 v \), from which the result follows.

4. Representations of \( \mathfrak{e}(3) \) from irreducible representations of \( \mathfrak{sl}(4, \mathbb{C}) \)

We may embed \( \mathfrak{e}(3) \) into \( \mathfrak{sl}(4, \mathbb{C}) \) as follows:
\[
\phi : \mathfrak{e}(3) \hookrightarrow \mathfrak{sl}(4, \mathbb{C})
\]
\[
E \leftrightarrow x_2 + x_6
\]
\[
H \leftrightarrow h_1 + 2h_2 + h_3
\]
\[
F \leftrightarrow y_2 + y_6
\]
\[
P_+ \leftrightarrow -2x_4
\]
\[
P_0 \leftrightarrow x_1 - y_3
\]
\[
P_- \leftrightarrow 2y_5.
\]
In this section we will show that $V(m, 0, 0)$, and $V(0, 0, m)$ are $\epsilon(3)$-indecomposable under the embedding $\phi$. Since $V(0, 0, m)^* \cong V(m, 0, 0)$, the following proposition reduces this to showing that $V(m, 0, 0)$ is $\epsilon(3)$-indecomposable.

**Proposition 4.1.** Suppose that $V$ is a finite-dimensional representation of $\epsilon(3)$. Then $V$ is indecomposable if and only if its dual (that is, contragredient) $V^*$ is indecomposable.

**Proof.** Suppose that the representation $V$ decomposes: $V = V_1 \oplus V_2$. Then it is easy to see that the natural decomposition $V^* = V_1^* \oplus V_2^*$ of vector spaces is in fact a decomposition of representations. The converse follows from the fact that $V^{**} \cong V$. □

The following lemmas will be used to establish the indecomposability of $V(m, 0, 0)$ and $V(0, 0, m)$ in Theorem 4.6 below.

**Lemma 4.2.** Let $v$ be the maximal vector of $V(m, 0, 0)$, and $0 \leq i + j + k \leq m$; then

\[
H \cdot y_1^i y_4^j y_6^k v = (m - 2(j + k))y_1^i y_4^j y_6^k v, \tag{4.2}
\]

\[
E \cdot y_1^i y_4^j y_6^k v = \eta_1(i, j, k)y_1^i y_4^j y_6^k v + \eta_2(i, j, k)y_1^{i+1} y_4^{j-1} y_6^k v, \tag{4.3}
\]

\[
P_+ \cdot y_1^i y_4^j y_6^k v = \eta_+(i, j, k)y_1^i y_4^{j-1} y_6^k v, \tag{4.4}
\]

\[
P_0^i \cdot y_1^i v = \Pi_{t=1}^t (m - t + 1), \tag{4.5}
\]

where

\[
\eta_1(i, j, k) = k(m - i - j - k + 1), \quad \eta_2(i, j, k) = -j, \tag{4.6}
\]

\[
\eta_+(i, j, k) = -2j(m - i - j - k + 1). \tag{4.7}
\]

**Proof.** We prove only Equations (4.2) and (4.3). The other equations are proved in a similar fashion. Since $[H, y_1] = 0$, $[H, y_4] = -2y_4$, and $[H, y_6] = -2y_6$. Equation (4.2) follows from a simple count of weights:

\[
H \cdot y_1^i y_4^j y_6^k v = (m - 2(j + k))y_1^i y_4^j y_6^k v. \tag{4.8}
\]

Since

\[
[E, y_1] = x_5, \quad [E, y_4] = x_3 - y_1, \quad [E, y_6] = h_1 + h_2 + h_3,
\]

\[
[x_5, y_1] = [x_5, y_4] = 0, \quad [x_5, y_6] = -y_1 \quad \text{and} \quad [x_3, y_6] = y_4,
\]

we have

\[
E \cdot y_1^i y_4^j y_6^k v = y_1^i E y_4^j y_6^k v + i y_1^{i-1} y_4^j x_5 y_6^k v
\]

\[
= y_1^i y_4^j y_6^k v + j y_1^{i+1} y_4^{j-1} x_3 y_6^k v - j y_1^{i+1} y_4^{j-1} y_6^k v - i k y_1^i y_4^j y_6^{k-1} v
\]

\[
= \sum_{l=0}^{k-1} y_1^i y_4^j y_6^l h_1 y_6^{k-1-l} v + \sum_{l=0}^{k-1} y_1^i y_4^j y_6^l h_3 y_6^{k-1-l} v
\]

\[
- j k y_1^i y_4^j y_6^{k-1} v - j y_1^{i+1} y_4^{j-1} y_6^k v - i k y_1^i y_4^j y_6^{k-1} v
\]
\[
\begin{align*}
&= (km - k^2 + \frac{k(k + 1)}{2})y_1^j y_4^j y_6^{k-1}v \\
&\quad + (-k^2 + \frac{k(k + 1)}{2})y_1^j y_4^j y_6^{k-1}v \\
&\quad - jky_1^i y_4^j y_6^{k-1}v - jy_1^i y_4^j y_6^{k-1}v - iky_1^i y_4^j y_6^{k-1}v \\
&= (km - k^2 + k - jk - ik)y_1^i y_4^j y_6^{k-1}v - jy_1^i y_4^j y_6^{k-1}v - jy_1^i y_4^j y_6^{k-1}v \\
&= k(m - k + 1 - j - i)y_1^i y_4^j y_6^{k-1}v - jy_1^i y_4^j y_6^{k-1}v \\
&= \eta_1(i, j, k)y_1^i y_4^j y_6^{k-1}v + \eta_2(i, j, k)y_1^i y_4^j y_6^{k-1}v.
\end{align*}
\]

This concludes the proof. \(\square\)

**Lemma 4.3.**
\[
\dim(V(m, 0, 0)) = \sum_{i=0}^{\lfloor m/2 \rfloor} (m - 2i + 1)^2, \tag{4.10}
\]

where \(\lfloor m/2 \rfloor\) is the largest integer less than or equal to \(m/2\).

**Proof.** Recall Weyl’s character formula for the dimension of \(V(\lambda)\) [6]:
\[
\dim(V(\lambda)) = \frac{\prod_{\alpha > 0} \langle \lambda + \delta, \alpha \rangle}{\prod_{\alpha > 0} \langle \delta, \alpha \rangle}. \tag{4.11}
\]

The positive roots of \(A_3\) are \(\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3,\) and \(\alpha_1 + \alpha_2 + \alpha_3\). Accordingly, for \(\lambda = m\lambda_1\), the denominator is \(1 \cdot 1 \cdot 1 \cdot 2 \cdot 2 \cdot 3 = 12\), while the numerator is \((m + 1) \cdot 1 \cdot 1 \cdot (m + 2) \cdot 2 \cdot (m + 3)\). Thus,
\[
\dim(V(m, 0, 0)) = \frac{(m + 1)(m + 2)(m + 3)}{6}. \tag{4.12}
\]

It is then not difficult to show that
\[
\frac{(m + 1)(m + 2)(m + 3)}{6} = \sum_{i=0}^{\lfloor m/2 \rfloor} (m - 2i + 1)^2. \tag{4.13}
\]

For odd \(m\), this follows easily from the familiar formula
\[
\sum_{k=1}^{N} k^2 = \frac{N(N + 1)(2N + 1)}{6}.
\]

Subtracting this result from the whole sum then recovers the result for even \(m\). \(\square\)

**Lemma 4.4.** The \(H\)-maximal vectors that occur in \(V(m, 0, 0)\) have weights \(m - 2M\), where \(0 \leq M \leq \lfloor m/2 \rfloor\). A basis for the \(H\)-highest weight vectors of \(H\)-weight \(m - 2M\)
is given by
\[
\begin{aligned}
w(M, i) &= \sum_{j=0}^{M} \alpha_j(M, i) y_1^{i+j} y_4^{M-j} y_6^{j} v, \\
\end{aligned}
\]
for \(0 \leq i \leq m - 2M\), where the nonzero scalars \(\alpha_j(M, i)\) are defined recursively by
\[
\begin{aligned}
\alpha_M(M, i) &= 1, \\
\alpha_j(M, i) &= \frac{-\alpha_{j+1}(M, i) \eta_1(i + j + 1, M - j - 1, j + 1)}{\eta_2(i + j, M - j, j)}, \quad 0 \leq j < M.
\end{aligned}
\]

**Proof.** We first show the \(w(M, i)\) are linearly independent. First note that, by Lemma 3.2, each summand in \(w(M, i)\) is a nonzero basis vector (up to a nonzero scalar multiple) in \(B_{(m, 0, 0)}\). By Lemma 4.2, the weight of \(w(M, i)\) is \(m - 2M\). So it suffices to check that \(w(M, i)\) are linearly independent for fixed \(M\), where \(0 \leq M \leq \lfloor m/2 \rfloor\), and all \(i\) such that \(0 \leq i \leq m - 2M\). This, however, follows easily by noting that the leading term \(y_1^{i} y_4^{M} v\) of \(w(M, i)\) occurs as a summand in \(w(M, i')\) if and only if \(i = i'\).

We now check that \(E \cdot w(M, i) = 0\). Using Lemma 4.2,
\[
\begin{aligned}
E \cdot w(M, i) &= \sum_{j=0}^{M} \alpha_j(M, i) E \cdot y_1^{i+j} y_4^{M-j} y_6^{j} v \\
&= \sum_{j=1}^{M-1} \alpha_j(M, i) (\eta_1(i + j, M - j, j) y_1^{i+j} y_4^{M-j} y_6^{j} v \\
&\quad + \eta_2(i + j, M - j, j) y_1^{i+j+1} y_4^{M-j-1} y_6^{j} v) \\
&\quad + \alpha_0(M, i) \eta_2(i, M, 0) y_1^{i+1} y_4^{M-1} v \\
&\quad + \alpha_M(M, i) \eta_1(i + M, 0, M) y_1^{i+M} y_6^{M-1} v \\
&= (\alpha_0(M, i) \eta_2(i, M, 0) + \alpha_1(M, i) \eta_1(i + 1, M - 1, 1)) y_1^{i+1} y_4^{M-1} v \\
&\quad + \sum_{j=1}^{M-1} (\alpha_j(M, i) \eta_2(i + j, M - j, j) \\
&\quad + \alpha_{j+1}(M, i) \eta_1(i + j + 1, M - j - 1, j + 1)) y_1^{i+j+1} y_4^{M-j-1} y_6^{j} v \\
&\quad + (\alpha_{M-1}(M, i) \eta_2(i + M - 1, 1, M - 1) \\
&\quad + \alpha_M(M, i) \eta_1(i + M, 0, M) y_1^{i+M} y_6^{M-1} v \\
&= \left(\frac{-\alpha_1(M, i) \eta_1(i + 1, M - 1, 1)}{\eta_2(i, M, 0)} \eta_2(i, M, 0) \\
&\quad + \alpha_1(M, i) \eta_1(i + 1, M - 1, 1)\right) y_1^{i+1} y_4^{M-1} v
\end{aligned}
\]
\[+ \sum_{j=1}^{M-1} \left( -\alpha_{j+1}(M, i) \eta_1(i + j + 1, M - j - 1, j + 1) \right) \eta_2(i + j, M - j, j) \times \eta_2(i + j, M - j, j) + \alpha_{j+1}(M, i) \eta_1(i + j + 1, M - j - 1, j + 1) \right) y_1^{i+j+1} y_4^{M-j-1} y_6^j v + \left( -\frac{\eta_1(i + M, 0, M)}{\eta_2(i + M - 1, 1, M - 1)} \eta_2(i + M - 1, 1, M - 1) + \eta_1(i + M, 0, M) \right) y_1^{i+M} y_6^{M-1} v = 0.\]

We thus have
\[\langle w(M, i) \rangle \cong_{\text{sl}(2, \mathbb{C})} V(m - 2M), \quad (4.16)\]
for each \(i\) and \(M\) such that \(0 \leq M \leq [m/2]\) and \(0 \leq i \leq m - 2M\). By linear independence of the \(w(M, i)\), we have a direct sum \(\text{sl}(2, \mathbb{C})\)-subrepresentation of \(V(m, 0, 0)\):
\[\bigoplus_{i=0}^{[m/2]} (m - 2i + 1) \langle w(M, i) \rangle \cong_{\text{sl}(2, \mathbb{C})} \bigoplus_{i=0}^{[m/2]} (m - 2i + 1) V(m - 2i). \quad (4.17)\]
By dimension considerations, Lemma 4.3 and the fact that \(\dim(V(m - 2i)) = m - 2i + 1\),
\[V(m, 0, 0) \cong_{\text{sl}(2, \mathbb{C})} \bigoplus_{i=0}^{[m/2]} (m - 2i + 1) V(m - 2i). \quad (4.18)\]
This concludes the proof. \(\square\)

**Lemma 4.5.** Suppose that \(0 \leq M \leq [m/2]\), \(0 \leq i \leq m - 2M\). Then
\[P_+^M \cdot w(M, i) = \alpha_0(M, i)(\Pi_{k=1}^M \eta_+(i, k, 0)) y_1^i v. \quad (4.19)\]

**Proof.** Equation \((4.4)\) of Lemma 4.2 implies that \(P_+^M \cdot y_1^i y_4^{i+j} y_6^{-j} v = 0\) for \(j > 0\) since in this case the exponent of \(y_4\) is less than \(M\). Further,
\[P_+^M \cdot y_1^i y_4^M v = (\Pi_{k=1}^M \eta_+(i, k, 0)) y_1^i v\]
follows from Lemma 4.2. The result follows. \(\square\)

**Theorem 4.6.** The \(\text{sl}(4, \mathbb{C})\)-modules \(V(m, 0, 0)\) and \(V(0, 0, m)\) are \(\epsilon(3)\)-indecomposable.

**Proof.** By Proposition 4.1, since \(V(m, 0, 0)^* \cong V(0, 0, m)\), it suffices to show that \(V(m, 0, 0)\) is \(\epsilon(3)\)-indecomposable. Suppose that \(V(m, 0, 0)\) decomposes as
Based on these examples we conjecture that the examples were calculated with the assistance of the GAP computer algebra system.

We have shown that the irreducible \( \mathfrak{sl}(4, \mathbb{C}) \)-modules \( V(m, 0, 0) \) and \( V(0, 0, m) \) are \( \varepsilon(3) \)-indecomposable under the embedding described above. However, not all \( \mathfrak{sl}(4, \mathbb{C}) \)-modules are \( \varepsilon(3) \)-indecomposable, as the following examples illustrate. All the examples were calculated with the assistance of the GAP computer algebra system [4].

The \( \mathfrak{sl}(4, \mathbb{C}) \) representations \( V(0, 1, 0) \) and \( V(0, 2, 0) \), of dimension 6 and 20 respectively, decompose over \( \varepsilon(3) \) as follows:

\[
V(0, 1, 0) \cong_{\varepsilon(3)} \langle y_2 v - y_6 v \rangle \oplus \langle v, y_1 y_2 v, y_5 v, y_2 y_6 v, y_2 y_6 v \rangle,
\]
\[
V(0, 2, 0) \cong_{\varepsilon(3)} \langle y_2 y_5 v - y_5 y_6 v + y_2^2 v \rangle \\
\oplus \langle y_2 v - y_6 v, y_2 y_2 v - y_4 y_6 v, y_2 y_5 v - y_5 y_6 v, y_2^2 y_6 v - y_2 y_2^2 v, y_2^2 y_6 v - y_2^2 y_6 v \rangle \\
\oplus \langle v, y_5 v, y_2 v - y_6 v, y_4 v, y_2^2 v, y_2 y_5 v + y_5 y_6 v, y_4 y_5 v, y_2^2 v + y_2 y_6 v + y_2^2 v, y_2^2 v, y_4 y_6 v + y_2 y_4 y_6 v, y_2 y_5 y_6 v, y_2^2 y_6 v + y_2 y_4 y_6 v, y_2 y_4 y_6 v, y_2^2 y_6 v \rangle.
\]  

5. Conclusions

We have shown that the irreducible \( \mathfrak{sl}(4, \mathbb{C}) \)-modules \( V(m, 0, 0) \) and \( V(0, 0, m) \) are \( \varepsilon(3) \)-indecomposable under the embedding described above. However, not all \( \mathfrak{sl}(4, \mathbb{C}) \)-modules are \( \varepsilon(3) \)-indecomposable, as the following examples illustrate. All the examples were calculated with the assistance of the GAP computer algebra system [4].

The \( \mathfrak{sl}(4, \mathbb{C}) \) representations \( V(0, 1, 0) \) and \( V(0, 2, 0) \), of dimension 6 and 20 respectively, decompose over \( \varepsilon(3) \) as follows:

\[
V(0, 1, 0) \cong_{\varepsilon(3)} \langle y_2 v - y_6 v \rangle \oplus \langle v, y_1 y_2 v, y_5 v, y_2 y_6 v, y_2 y_6 v \rangle,
\]
\[
V(0, 2, 0) \cong_{\varepsilon(3)} \langle y_2 y_5 v - y_5 y_6 v + y_2^2 v \rangle \\
\oplus \langle y_2 v - y_6 v, y_2 y_2 v - y_4 y_6 v, y_2 y_5 v - y_5 y_6 v, y_2^2 y_6 v - y_2 y_2^2 v, y_2^2 y_6 v - y_2^2 y_6 v \rangle \\
\oplus \langle v, y_5 v, y_2 v - y_6 v, y_4 v, y_2^2 v, y_2 y_5 v + y_5 y_6 v, y_4 y_5 v, y_2^2 v + y_2 y_6 v + y_2^2 v, y_2^2 v, y_4 y_6 v + y_2 y_4 y_6 v, y_2 y_5 y_6 v, y_2^2 y_6 v + y_2 y_4 y_6 v, y_2 y_4 y_6 v, y_2^2 y_6 v \rangle.
\]  

Based on these examples we conjecture that \( V(0, m, 0) \) decomposes for all \( m \). Indecomposability in the general case \( V(m_1, m_2, m_3) \) is less clear. However, it is clear
that a class larger than \( V(m, 0, 0) \), and \( V(0, 0, m) \) does remain \( \epsilon(3) \)-indecomposable; for instance, the modules \( V(1, 0, 1) \), and \( V(1, 1, 0) \) are \( \epsilon(3) \)-indecomposable.

It is also interesting to note that \( \epsilon(3) \) may be embedded into other simple Lie algebras. For instance, we may embed \( \epsilon(3) \) into \( \mathfrak{so}(5, \mathbb{C}) \), the simple Lie algebra of type \( B_2 \). We are currently investigating the irreducible \( \mathfrak{so}(5, \mathbb{C}) \) representations restricted to \( \epsilon(3) \). Embedding \( \epsilon(3) \) into \( \mathfrak{sl}(4, \mathbb{C}) \) was investigated in the present paper since this is a natural generalization of embedding \( \epsilon(2) \) into \( \mathfrak{sl}(3, \mathbb{C}) \) examined in [3].

Since \( \epsilon(3) \) may be embedded into \( \mathfrak{so}(5, \mathbb{C}) \), it naturally embeds into \( \mathfrak{so}(7, \mathbb{C}) \), the simple Lie algebra of type \( B_3 \). An embedding is given by

\[
\phi : \epsilon(3) \hookrightarrow \mathfrak{so}(7, \mathbb{C})
\]

\[
E \mapsto x_5 \\
H \mapsto 2h_2 + h_3 \\
F \mapsto y_5 \\
P_+ \mapsto x_9 \\
P_0 \mapsto \frac{1}{2}x_6 \\
P_- \mapsto x_1.
\]

(5.2)

However, irreducible representations of \( \mathfrak{so}(7, \mathbb{C}) \), even in small dimension, appear to \( \epsilon(3) \)-decompose as the following examples in dimensions 7, 21 and 8, respectively, illustrate:

\[
V_{\mathfrak{so}(7,\mathbb{C})}(1,0,0) \cong_{\epsilon(3)} \langle v, y_1v, y_6v, y_1y_9v \rangle \oplus \langle y_4v \rangle \oplus \langle y_8v \rangle,
\]

\[
V_{\mathfrak{so}(7,\mathbb{C})}(0,1,0) \cong_{\epsilon(3)} \langle y_4y_9v, y_4v, y_2, y_2y_6v, y_2y_9v \rangle \oplus \langle y_8y_9v, y_9v \rangle \oplus \langle v, y_5v, y_2y_7v, y_6v, -y_2y_8v - y_4y_7v + y_9v, y_5y_9v, y_4y_8v, y_2y_6v, y_5^2, y_2y_8v, y_4y_7v \rangle,
\]

\[
V_{\mathfrak{so}(7,\mathbb{C})}(0,0,1) \cong_{\epsilon(3)} \langle v, y_5v, y_6v, y_9v \rangle \oplus \langle y_3v, y_7, y_8, y_3y_9v \rangle.
\]

(5.3)

Acknowledgements

A.D. would like to thank the Department of Mathematics at the University of Toronto and J.R. for their hospitality and support during the preparation of this paper.

References


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