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RINGS WITH QUASI-INJECTIVE IDEALS

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A short proof determining the structure of Q-rings with infinite identities is given. The structure of pQ-rings is determined and it is shown that essentially all wQ-rings with finite identities are pQ-rings.

Rings with an abundance of quasi-injective modules have been studied extensively since the late 1960's. Of particular interest are those rings whose (left) ideals are quasi-injective, the *(left) Q-rings*. It follows from the author's work in [3] that the study of Q-rings reduces to the study of the following 3 disjoint classes.

I. local Q-rings;
II. indecomposable Q-rings with more than one idempotent; and
III. Q-rings with an infinite set of orthogonal idempotents.

A ring's identity is *finite* if it is a sum of primitive orthogonal idempotents, otherwise it is *infinite*. It was further shown that the rings of class II have finite identities and are only the simple Artinian rings and the rings

$$H(m,D,V) = \begin{pmatrix} D & & & & V \\ V & D & & & \\ & V & \ddots & & \\ & & \ddots & D & \\ & & & V & D \end{pmatrix} \qquad \text{with } m \text{ rows, for } m \geqslant 2$$

where D is a skew field and V is a null D-algebra, one dimensional on both sides. The rings of class I are the local left self-injective left duo rings.

The structure of Q-rings of class III is determined in Section 1. It follows from that result that the conjecture in [4] is false. This structure theorem was obtained independently by Byrd [1] using different methods. The author's proof (which was obtained in 1976) appears to pre-date Byrd's and is very much shorter and simpler.

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It is included in this note, as its methods may be of use in studying other rings with infinite identities — usually an intractable task.

In Section 2 two generalisations of Q-rings are studied: *pQ-rings* — rings whose proper left ideals are quasi-injective; and *wQ-rings* — rings whose left ideals not isomorphic to the ring itself are quasi-injective. These were first studied in [5] and [6], respectively. It is shown that the only non-local pQ-rings which are not Q-rings are the rings $\begin{pmatrix} D & 0 \\ D & D \end{pmatrix}$ where D is a skew field. It is also shown that the only wQ-rings with finite identities which are not pQ-rings are those with a unique idempotent (the identity).

Throughout this note all rings have identity and all ideals and modules are left and unital. The letters e, f, with or without subscripts or superscripts, always denote idempotents and $E(M)$ denotes the injective hull of a module M. The methods used were developed in [3] so most of the arguments will be given only in outline.

1. Q-rings of Class III

Throughout this section R is a Q-ring of class III. The crucial result is the following generalisation of Lemma 3 of [3]. It does not require the assumption that R is of class III. This is essentially Theorem 5 of [1] which is the primary contribution of that paper. Its proof is both long and difficult.

Lemma 1.1. *Let $\{M_i \mid i \in I\}$ be a set of minimal ideals of R with injective hulls (in R) Re_i, $i \in I$, respectively. For only finitely many i is M_i an image of $R(1-e_i)$.*

Proof: Assume the converse, that I is infinite and that each M_i is an image of $R(1-e_i)$. By Lemma 3 of [3] only finitely many M_i are images of the Re_j so it may be assumed that none are. The M_i are mutually non-isomorphic since if two are isomorphic, say M_k and M_l, then by Lemma 2 of [3] they would be injective and hence images of Re_k $(= M_k)$, a contradiction to the last assumption. Hence if K_i is the kernel of an epimorphism $R \to M_i$ then the K_i are distinct maximal ideals. For an arbitrary pair i, $j \in I$ there is an idempotent $f_j \in K_i$ with the property that $f_j M_j \neq 0$ and $(1-f_j)M_i \neq 0$. One of f_j, $(1-f_j)$ satisfies the equation $xM_k \neq 0$ for infinitely many k. Denote this idempotent by f_1' and the other by f_1''. If $f_1'' = f_j$ then there is an idempotent $f_1 \in Rf_j \cap R(1-e_j)$ with the property that $f_1 M_j \neq 0$. If $f_1'' = 1 - f_j$ then there is an idempotent $f_1 \in Rf_1'' \cap R(1-e_i)$ with the property that $f_1 M_i \neq 0$. Note that in either case f_1 and f_1' are orthogonal and Rf_1 has a non-zero image in $R(1-f_1)$. Repeat the above procedure with M_i and M_j replaced by ideals M_l which are images of Rf_1 to obtain orthogonal idempotents f_2, $f_2' \in Rf_1$ with the properties that Rf_2 has a non-zero image in $R(1-f_2)$ and $f_2' M_l \neq 0$ for infinitely many l. In this fashion one can construct an infinite set $\{f_n\}$ of orthogonal idempotents with the property that each Rf_n has a non-zero image in $R(1-f_n)$. This contradicts Lemma 3 of [3] and proves the lemma. ∎

LEMMA 1.2. *Let $\{e_i\}$ be an infinite set of orthogonal central idempotents of R and let Re be an injective hull in R of $\oplus Re_i$. Then $Re = \prod e_i Re_i$.*

PROOF: As each Re_i is injective $\prod Re_i$ is injective and as each e_i is central, $\oplus Re_i$ is an essential submodule of $\prod Re_i$. Therefore Re is isomorphic to $\prod Re_i$. Since $(1-e)Re = 0$, $Re = eRe \cong \operatorname{End}(Re) \cong \operatorname{End}(\prod Re_i)$. As the e_i are orthogonal and central $\operatorname{End}(\prod Re_i) \cong \prod e_i Re_i$ and $e_i Re_i = Re_i$, thus $Re = eRe = \prod e_i Re_i$ as required. ∎

To simplify the statement of the main result we need the following definition.

DEFINITION: Let C be a Q-ring with infinite identity and all of whose idempotents are central, U a simple left C-module, D the endomorphism ring of U, acting on the right, so that U is a (C,D)-bimodule, and let V be a null D-algebra, one dimensional on both sides. Then we define $H(m,C,U,D,V)$, for an integer $m \geqslant 3$, to be the following matrix ring.

$$H(m,C,U,D,V) = \begin{pmatrix} D & & & & \\ V & \ddots & & & \\ & \ddots & D & & \\ & & V & D & \\ & & & U & C \end{pmatrix} \quad \text{with } m \text{ rows and columns.}$$

For $m = 2$ we define $H(m,C,U,D,V)$ to be the ring $H(2,C,U,D) = \begin{pmatrix} D & 0 \\ U & C \end{pmatrix}$. For $m = 1$, $H(m,C,U,D,V)$ is simply the ring C.

We can now state and prove the main theorem.

THEOREM 1.3. *A ring R is a Q-ring of class III with no summands of class II if, and only if, it is isomorphic to a product of a finite number of rings $H(m,C,U,D,V)$ each of which satisfies the following conditions:*

(i) $eU = 0$ for every primitive idempotent $e \in C$,

(ii) $C = C_0 \times C_1$ where C_0 is an infinite or empty product of local rings and C_1 has no primitive idempotents.

PROOF: Let $\mathcal{M} = \{M_i \mid i \in I\}$ be the set of minimal ideals with the property that $(1-e_i)M_i \neq 0$, where Re_i is the injective hull in R of M_i. By Lemma 1.1 the set I is finite and so Rf is the injective hull in R of $\bigoplus_{i\in I} M_i$, where $f = \sum_I e_i$. Hence by Lemma 2 of [3], $fR(1-f) = 0$ and all idempotents in $(1-f)R(1-f)$ are central in $(1-f)R(1-f)$.

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Let $\{f_j \mid j \in J\}$ be the set of primitive idempotents in $C = (1-f)R(1-f)$. Being central, these idempotents are mutually orthogonal. If Re is the injective hull of $\oplus Rf_i$ then, by Lemma 1.2, $Re = \prod Rf_i = \prod f_i Rf_i$. It follows that $C = C_0 \times C_1$ where $C_0 = \prod f_i Rf_i$ and $C_1 = R(1-f-e) = (1-f-e)R(1-f-e)$ has no primitive idempotents.

Let $\mathcal{M}_0 = \{M_i \mid i \in J\}$ be the subset of $\mathcal{M}$ consisting of those minimal ideals which are images of C. We now show that $C = \oplus C_i$ where the C_i are subrings with the property that $C_i M_i = M_i$ and $C_j M_i = 0$, for $i \neq j$. If $F = C/J(C)$ then since each M_i is annihilated by a maximal, ideal each M_i is canonically an F-module. Let K_i be the annihilator of M_i in F and let $a \in K_2 \backslash K_1$. Then $aM_1 = M_1$ and $aM_2 = 0$. Since F is a regular ring (Theorem 5.1 of [**2**]) there is an idempotent $e \in F$ such that $Fe = Fa$. Therefore M_1 is an image of Fe and M_2 is an image of $F(1-e)$. But e is central in F so F is a direct sum of the subrings Fe and $F(1-e)$. In this way we can obtain a decomposition $F = \oplus F_i$ where the F_i are subrings with the property that $F_i M_i = M_i$ and $F_i M_j = 0$, $i \neq j$. As idempotents lift modulo $J(C)$ (Theorem 5.6 of [**2**]) the required decomposition of C follows.

Let $M_{j0} \in \mathcal{M}_0$ be the image of C_j and let M_{j1} be the image of $Re_{j0} = E(M_{j0})$; let M_{j2} be the image of $Re_{j1} = E(M_{j1})$ and so on. This sequence is finite. As $\mathcal{M}_0$ is finite it is sufficient to show that each M_{ji} can appear only once. If not then one of these minimal ideals is an image of two Re_{ji} or of C_j and an Re_{ji}. In both cases R would have two isomorphic indecomposable injectives and thus would contain a simple Artinian ring as a summand (Lemma 2 and Theorem 1 of [**3**]) — a contradiction to the assumption that R has no summands of class II. If $Rf_j = C_j + Re_{j0} + Re_{j1} + \dots$ then Rf_j is a two sided ideal and a ring summand of R. For by construction Rf_j has no images in $R(1-f_j)$ and $R(1-f_j)$ can have no images in the Re_{ji}, by the above argument, and no images in any C_i, $i \neq j$, since all idempotents in C are central. The minimal ideals in $\mathcal{M}$ must be exhausted by the Rf_j's as otherwise R would have a summand of class II. Hence R is a direct product of the rings $Rf_j = R_j$. From now on we only consider these rings.

The matrix representation of R_j is obtained, in the usual manner, by considering it as its own ring of endomorphisms. The proof that the injective hull of each M_{ji} has a unique submodule (namely M_{ji}) is the same as the second paragraph of the proof of Lemma 4 of [**3**]. The rest of the details of the matrix representation of R_j are similar to the proof of Theorem 3 of [**3**]. We now show that condition (i) is satisfied.

Assume it is not. Then there is a primitive idempotent $e \in C$ such that some M_i, say M_1, is an image of Re. Let M_2 be an M_i which is an image of $Re_1 = E(M_1)$, M_3 an M_i which is an image of $Re_3 = E(M_2)$, ... and so on. As the number of the M_i is finite this process terminates at step n, say. That is, Re_n has no images outside itself.

Therefore $B = Re \oplus Re_1 \oplus \cdots \oplus Re_n$ is a two-sided ideal of R. But B has at least two idempotents and is indecomposable so it is a Q-ring of class II: a contradiction to the hypothesis that R has no summands of class II.

The proof of the converse is similar to that for Theorem 3 of [3]. ∎

COROLLARY 1.4. *A left Q-ring need not be right injective.*

PROOF: The right ideal $\begin{pmatrix} D & \\ & \end{pmatrix}$ in the ring $\begin{pmatrix} D & \\ U & C \end{pmatrix}$ is not injective. ∎

The following Corollary is the correct statement of the Remark at the end of [3].

COROLLARY 1.5. *A ring is a Q-ring if, and only if, it is a sum of (a finite number) of rings of class II and rings of type $H(m, C, U, D, V)$ which satisfy the conditions of Theorem 1.3.*

PROOF: By Lemma 3 of [3] a Q-ring can have only a finite number of central idempotents each of which generates a ring of class II. Hence a Q-ring is a finite sum of rings of class II and the rings $H(m, C, U, D, V)$. ∎

Note that products of rings of class I are contained in the rings $H(m, C, U, D, V)$ as part of the subrings C.

COROLLARY 1.6. *A ring is a left and right Q-ring if, and only if, it is the sum of rings of class II and a ring $Q = Q_1 \times Q_2$ all of whose idempotents are central and which has the following properties. Q_1 is a product of local left and right Q-rings and Q_2 is a left and right Q-ring with no primitive idempotents.*

PROOF: The rings of class II are left and right Q-rings and the rings $H(m, C, U, D, V)$ are not right injective if $m \geqslant 2$. So the only rings of the latter type that can appear are the rings C. ∎

2. pQ-RINGS AND wQ-RINGS

The study of pQ-rings was initiated in [5]. The authors showed that prime pQ-rings are Q-rings and that commutative Noetherian pQ-rings are Q-rings except for a, somewhat trivial, case. It was shown in [6] that nonsingular wQ-rings satisfying certain finiteness conditions are either Q-rings or the matrix rings in Theorem 2.3. Those finiteness conditions imply that the rings have finite identities. In this section we determine the structures of arbitrary pQ-rings and of wQ-rings with finite identities.

There are trivial examples of pQ-rings and wQ-rings with unique idempotents which are not Q-rings. The available techniques seem to be unable to shed much light on the structure of these rings or, in fact, on local Q-rings. So the major remaining open questions are to determine the structures of these rings and of wQ-rings with infinite identities. Some progress is made on the last question in [7] where the authors obtain some results on arbitrary nonsingular wQ-rings, but even their structure remains unknown.

We first study pQ-rings and show that, apart from some simple exceptions, they are Q-rings.

LEMMA 2.1. *If R is a pQ-ring with a unique idempotent (the identity) then R is a local ring and every element in its radical is a zero divisor.*

PROOF: Let $a \in R$ be an element which is not a zero divisor. Then $Ra \cong R$ so if $Ra \neq R$ then Ra, and hence R, is quasi-injective. This implies that R is a local ring (Proposition 5.8 of [**2**]) and every element in its radical is a zero divisor (Theorem 5.1 of [**2**]): a contradiction. Hence every regular element of R is a unit. Therefore, R is a local ring and its radical consists of zero divisors. ∎

LEMMA 2.2. *If R is a pQ-ring with at least three orthogonal idempotents then it is a Q-ring.*

PROOF: Let e_1, e_2, e_3 be three orthogonal idempotents whose sum is the identity of R and let L be a left ideal of R. To show that R is self-injective it is sufficient to show that any homomorphism ϕ from L to R can be extended to an endomorphism of R. If L is a direct summand of R then that can be done trivially. If L is not a direct summand then it has a complement K and $L \oplus K$ is essential and proper in R. Clearly ϕ can be extended to a homomorphism from $L \oplus K$ to R. Hence we can assume that L is essential in R. As L is quasi-injective it is invariant under endomorphisms of its injective hull and so it is closed under right multiplication by elements of R. If $Re_i \cap L$ is denoted by L_i then $L = L_1 \oplus L_2 \oplus L_3$ and

$$\phi = \begin{pmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{21} & \phi_{22} & \phi_{23} \\ \phi_{31} & \phi_{32} & \phi_{33} \end{pmatrix}$$

where $\phi_{ij} : L_i \to Re_j$ is the appropriate restriction of ϕ. Clearly ϕ can be extended to be an endomorphism of R provided each ϕ_{ij} can be extended to a homomorphism $Re_i \to Re_j$. This can clearly be done as $Re_i \oplus Re_j$ is quasi-injective, being a proper ideal. Hence R is injective and therefore is a Q-ring. ∎

THEOREM 2.3. *A non-local ring is a pQ-ring if, and only if, it is either a Q-ring or the ring $\begin{pmatrix} D & 0 \\ D & D \end{pmatrix}$, for some skew field D.*

PROOF: By Lemmas 2.1 and 2.2 it may be assumed that R has exactly two orthogonal primitive idempotents e_1 and e_2. If R decomposes then it is a product of two Q-rings and so is itself a Q-ring. Hence it may be assumed that R is indecomposable. If $Re_1 \cong Re_2$ then, since Re_1 is quasi-injective, R is quasi-injective and hence is a Q-ring. So it will be assumed that $Re_1 \not\cong Re_2$. Since R is indecomposable one of e_1Re_2,

e_2Re_1 is non-zero, say $e_2Re_1 \neq 0$. If L is an essential proper submodule of Re_1 then $L \oplus Re_2$ is quasi-injective, hence invariant under all endomorphisms of $E(L \oplus Re_2) \supseteq R$ [**2**, Proposition 3.1] and thus $L \oplus Re_2$ is a right ideal of R. The proof of Lemma 2 of [**3**] now shows that $e_2Re_1 \subseteq S(Re_1) \neq 0$.

In fact, $S(Re_1)$ is simple and essential in Re_1. To prove this it is sufficient to show that an indecomposable quasi-injective module M must be uniform. If it is not then the map which kills the complement of a non-essential submodule K of M and is the identity on K can be extended to an endomorphism of M which, by Theorem 5.1 and Proposition 5.8 of [**2**], must be an automorphism — a contradiction. Therefore $S(Re_1)$ is simple and essential in Re_1.

By the second paragraph of the proof of Lemma 4 of [**3**], this minimal submodule is the unique proper submodule of Re_1 and is the set e_2Re_1. As Re_1 is quasi-injective e_2Re_1 is a one dimensional right vector space over e_1Re_1 and a simple left e_2Re_2-module. If $e_1Re_2 = 0$ then by the proof of Lemma 4 of [**3**], $e_2Re_2 = Re_2$ is a skew field and $R \cong \begin{pmatrix} D & 0 \\ D & D \end{pmatrix}$ where $D \cong e_2Re_2 \cong e_1Re_1$. If $e_1Re_2 \neq 0$ then, as for Re_1, it follows that Re_2 has a unique submodule which is the set e_1Re_2 and is a one dimensional right vector space over the skew field e_2Re_2. It can now be shown, as in [**3**], that $R \cong \begin{pmatrix} D & V \\ V & D \end{pmatrix}$ where $D \cong e_1Re_1 \cong e_2Re_2$ is a skew field and V is a null D-algebra, one dimensional on both sides. Thus R is a Q-ring, by Theorem 3 of [**3**]. This proves the theorem. □

We now turn to wQ-rings.

If a wQ-ring decomposes (as a ring) then it is a Q-ring as each summand is quasi-injective and thus a Q-ring. Therefore it is only necessary to consider indecomposable wQ-rings. Since principal ideal domains are trivially wQ-rings (that is, by default) the main interest is in wQ-rings with more than one idempotent.

THEOREM 2.4. *If R is a wQ-ring with finite identity and more than one idempotent then R is a pQ-ring.*

PROOF: Let $1 = e_1 + \cdots + e_n$, $n \geqslant 2$, be a decomposition of the identity of R into orthogonal primitive idempotents. We want to show that every proper left ideal is quasi-injective. Clearly each Re_i is quasi-injective so if all the Re_i are isomorphic R is itself quasi-injective and is therefore a Q-ring. So we may assume that all the Re_i are not mutually isomorphic. Assume that $e_1Re_2 \neq 0$ and let $K = Ra$ for some non-zero $a \in e_1Re_2$. If $K \oplus Re_1 \cong R$ then $n = 2$ and $K \cong Re_2$. It follows that $Re_2 \cong Re_1$ since by the projectivity of Re_2, the isomorphism from Re_2 to K can be factored through $Re_1 \to K$. But this case has been excluded by assumption. Therefore $K \oplus Re_1$ is quasi-injective. By Proposition 3.1 of [**2**], $K \oplus Re_1$ is invariant under endomorphisms of its injective hull and so is invariant under endomorphisms of $Re_1 \oplus Re_2$. That is, $K \oplus Re_1$ is closed under right multiplication by elements of $(e_1 + e_2)R(e_1 + e_2)$. Therefore $e_1 Re_2 = e_1^2 Re_2 \subseteq K$. As Re_2 is indecomposable and quasi-injective it is uniform, so K is simple. Consequently, if $i \neq j \neq k$ then $e_i Re_j . e_j Re_k = 0$ and $e_i Re_j . e_j Re_i = 0$.

Let $\phi : R \to L$ be an isomorphism to a left ideal L of R. Then for each i, $e_i \phi = a_i e_i + \sum_{j \neq i} b_{ij} e_j$ for some $a_i \in e_i Re_i$, $b_{ij} \in e_i Re_j$. Since each b_{ij} generates a simple or zero left ideal, $a_i \neq 0$. As $e_i Re_i$ is a local ring, a_i must be a unit in $e_i Re_i$ [2, Proposition 5.8]. If a_i^{-1} is the inverse of a_i in $e_i Re_i$, then $g_1 = b_{1i} a_i^{-1} e_i \phi = b_{1i} e_i + \sum_{j \neq i} b_{1i} a_i^{-1} b_{ij} e_j = b_{1i} e_i$, since $b_{1i} a_i^{-1} b_{ij} \in e_1 Re_i . e_i Re_j$, $i \neq j$. Therefore $e_1 \phi - g_1 - \cdots - g_n = a_1 e_1 \in L$. This means that $Re_1 \subseteq L$. Similarly it can be shown that each $Re_i \subseteq L$ and therefore $R = L$. That shows that R has no proper left ideals isomorphic to itself and therefore all proper left ideals are quasi-injective. That is, R is a pQ-ring. □

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