TRANSFORMATION AND REDUCTION FORMULAE
FOR DOUBLE $q$-SERIES OF TYPE $\Phi_{2:1;2;0:1,2}^{2:1;3;0:2,3}$

CANGZHI JIA and XIANGDE ZHANG
Department of Mathematics, Northeast University, Shenyang 110005, PR China
e-mail: cangzhijia@yahoo.com.cn, zhangxdneu@yahoo.com.cn

(Received 2 December 2008; accepted 1 June 2009)

Abstract. By applying the Sears non-terminating transformations, we establish four general transformation theorems for double basic hypergeometric series of type $\Phi_{2:1;2;0:1,2}^{2:1;3;0:2,3}$. Moreover, several transformation, reduction and summation formulae on the double basic hypergeometric series $\Phi_{2:1;2;0:1,2}^{2:1;3;0:2,3}$ and $\Phi_{2:1;4}^{2:1;5}2:0:3$ are also derived through parameter specialisation.

2000 Mathematics Subject Classification. Primary, 33D15; secondary, 05A30.

1. Introduction. For two indeterminates $x$ and $q$, the shifted factorial is defined by

$$(x; q)_0 = 1 \quad \text{and} \quad (x; q)_n = \prod_{k=0}^{n-1}(1 - q^k x) \quad \text{with} \quad n = 1, 2, \ldots.$$  

When $|q| < 1$, we have the following well-defined infinite product expressions:

$$(x; q)_\infty = \prod_{k=0}^{\infty}(1 - q^k x) \quad \text{and} \quad (x; q)_n = \frac{(x; q)_\infty}{(q^n x; q)_\infty} \quad \text{for} \quad n \in \mathbb{Z}.$$  

For the sake of brevity, we also write the factorial product compactly as $[a, b, \ldots, c; q]_n := (a; q)_n(b; q)_n \ldots (c; q)_n$.

Following Gasper and Rahman [3], the basic hypergeometric series is defined by

$$1+r\Phi_s \left[ \begin{array}{l} a_0, a_1, \ldots, a_r; \\ b_1, \ldots, b_s; \end{array} q; z \right] = \sum_{m,n=0}^{\infty} \frac{(-1)^m q^{\binom{m}{2}}}{[q, b_1, \ldots, b_s; q]_m} \frac{[a_0, a_1, \ldots, a_r; q]_m}{[q, b_1, \ldots, b_s; q]_m} z^n,$$  

where the base $q$ will be restricted to $|q| < 1$ for non-terminating $q$-series.

As the $q$-analogue of Kampé de Fériet function, Srivastava and Karlsson [9, p. 349] defined the generalised bivariate basic hypergeometric function by

$$\Phi_{\lambda, \mu, c}^{\alpha, \beta, \gamma} \left[ \begin{array}{l} \alpha_1, \ldots, \alpha_{\lambda}; \\ \beta_1, \ldots, \beta_{\mu}; \\ c_1, \ldots, c_s; \\ q; x, y \end{array} q; m+n \right] = \sum_{m,n=0}^{\infty} \frac{[\alpha_1, \ldots, \alpha_{\lambda}; q]_{m+n}}{[\beta_1, \ldots, \beta_{\mu}; q]_{m+n}} \frac{[a_1, \ldots, a_r; q]_m}{[b_1, \ldots, b_s; q]_m} \frac{[c_1, \ldots, c_s; q]_m}{[d_1, \ldots, d_r; q]_m} \frac{x^m y^n q^{inom{m}{2} + \binom{n}{2} + kmn}}{(q; q)_m(q; q)_n}.$$  

\[1.2a\]  

\[1.2b\]
It is not hard to check that when \( i, j, k \in \mathbb{N}_0 \), the double series \( \Phi_{\lambda,t,s}^{\mu,k,u} \) is convergent for \( |x| < 1 \), \( |y| < 1 \) and \( |q| < 1 \).

For double basic hypergeometric series, there are fewer instances available in the literature [1, 2, 4–8, 10–12]. Recently, Chu and Jia [1] established eight transformations by using the Sears transformation formulae and obtained a number of transformation, reduction and summation formulae on \( \Phi_{12}^{22} \), \( \Phi_{023}^{123} \), \( \Phi_{111}^{033} \) and \( \Phi_{112}^{034} \) as special cases. As a continuation, we will further investigate four general transformations for double basic hypergeometric series of type \( \Phi_{2,0;0}^{21,\lambda} \), again by means of non-terminating Sears transformation formulae. Several new transformation formulae on \( \Phi_{201}^{212} \), \( \Phi_{202}^{213} \), \( \Phi_{203}^{214} \) are also obtained as consequences.

2. Transformations between \( \Phi_{2,0;0}^{21,\lambda} \) and \( \Phi_{1;1;\mu}^{0,3,\mu} \)

**Theorem 2.1** (Transformation formula). For an arbitrary complex sequence \( \{\Omega(j)\} \), the transformation

\[
\sum_{i,j=0}^{\infty} \frac{(a;q)_{i+j}(c;q)_{i+j}(e;q)_i}{(b;q)_{i+j}(d;q)_{i+j}(q;q)_i} \left( \frac{bd}{ace} \right)^i \Omega(j) = \frac{[d/e, bd/ace; q]_\infty}{[d, bd/ace; q]_\infty} \sum_{i,j=0}^{\infty} q^j \left( \frac{d}{e} \right)^i \frac{[e, b/a, b/c; q][a, c; q]_i}{[b, q]_{i+j}[q, bd/ace; q][q, d/e; q]} \Omega(j) \tag{2.1b}
\]

holds, provided that two double series displayed above are absolutely convergent.

**Proof.** Recalling the \( q \)-analogue of the Kummer–Thomae–Whipple transformation [3, p. 359, Appendix III.9],

\[
_{3}\Phi_{2} \left[ \begin{array}{c} a, c, e \\ b, d \end{array} \right| q; \frac{bd}{ace} = \frac{[d/a, bd/ce; q]_\infty}{[d, bd/ace; q]_\infty} \frac{1}{e} \frac{1}{c} \frac{1}{b} \frac{1}{d} \frac{1}{a} \tag{2.2}
\]

we can reformulate the double sum in (2.1a) as follows:

\[
\sum_{j=0}^{\infty} \frac{(a;q)_j(c;q)_j}{(b;q)_j(d;q)_j} \frac{\Omega(j)}{\Omega(j)} _3 \Phi_2 \left[ \begin{array}{c} e, q'a, q'c \\ q'b, q'd \end{array} \right| q; \frac{bd}{ace} \\
\sum_{j=0}^{\infty} \frac{[a, c; q]_j}{[b, d; q]_j} \frac{[q'd/e, bd/ace; q]_\infty}{[q'd, bd/ace; q]_\infty} \frac{1}{e} \frac{1}{c} \frac{1}{b} \frac{1}{d} \frac{1}{a} \tag{2.2}
\]

Writing the last double sum explicitly, we see that it coincides with (2.1b). \( \square \)

When the \( \Omega \)-sequence is specified by

\[
\Omega(j) = \frac{[u_1, u_2, \ldots, u_\mu; q]}{[v_1, v_2, \ldots, v_\mu; q]} w^j \tag{2.3}
\]

the last theorem gives us a very general transformation between two non-terminating double series \( \Phi_{2,0;0}^{21,\lambda} \) and \( \Phi_{1;1;\mu}^{0,3,\mu} \).
In the proof of the last theorem, if we apply, instead of (2.2), the Hall transformation [3, p. 359, Appendix III.10]

$$\phi^2_3\left[\begin{array}{c} a, \ c, \ e \\ b, \ d \end{array} \right| q^j; \frac{bd}{ace} \right] = \left[ b/c, d/c, bd/ace \right] \frac{[b, d, bd/ace; q]}{[b, d, bd/ace; q]}$$ (2.4)

then we can establish another transformation formula.

**Theorem 2.2 (Transformation formula).** For an arbitrary complex sequence \(\Omega(j)\), the transformation

$$\sum_{i,j=0}^{\infty} \frac{(a; q)_{i+j}(c; q)_{i+j}(e; q)_i}{(b; q)_{i+j}(d; q)_i} \left( \frac{bd}{ace} \right)^i \Omega(j)$$ (2.5a)

$$= \left[ b/c, d/c, bd/ace \right] \frac{[b, d, bd/ace; q]}{[b, d, bd/ace; q]} \sum_{i,j=0}^{\infty} q^i c^j \left( \frac{bd/ce}{ace} \right)^i \left( \frac{bd/ace}{q} \right)^j \Omega(j)$$ (2.5b)

holds, provided that two double series displayed above are absolutely convergent.

Under specification (2.3), this theorem yields a transformation between two non-terminating double series \(\phi_{2,0,\mu}^{2;1,3}\) and \(\phi_{1;1;\mu}^{0;3,\lambda+1}\).

We remark that the relation \(\phi_{2,0,\mu}^{2;1,3}\) and \(\phi_{1;1;\mu}^{0;3,\lambda+1}\) has first been discovered in [1].

### 2.1 Non-terminating reduction formula for \(\phi_{2,0;2}^{2;1;3}\)

Specifying in Theorem 2.1 with

$$\Omega(j) = \left[ \frac{[d/e, \alpha, \beta; q]}{[a, c; q]} \right] \left( \frac{b}{\alpha \beta} \right)^j$$

and then evaluating the sum with respect to \(j\) displayed in (2.1b) by means of the \(q\)-Gauss summation theorem [3, p. 354, Appendix II.8]

$$\phi_1^{2;0;2}\left[\begin{array}{c} a, \ b \\ c, \ ab \end{array} \right| q^j; \frac{c/a}{c/ab} \right] = \left[ c/a, c/b; q \right] \frac{[c, c/ab; q]}{[c, c/ab; q]}$$ (2.6)

we find after some trivial simplification the following reduction formula.

**Proposition 2.3 (Reduction formula).**

$$\phi_{2,0;2}^{2;1;3}\left[\begin{array}{c} a, \ c; e, \ d/e, \alpha, \beta \\ b, \ d; e, \alpha, \beta \end{array} \right| q; bd/ace, b/\alpha \beta] = \left[ b/d/e, b/\alpha, b/\beta; q \right]$$ (2.7)

$$\phi_3^{1;0;2}\left[\begin{array}{c} e, \ b/a, b/c, b/\alpha \beta \\ d; e, \alpha, \beta ; q \end{array} \right| q; d/e, b/\alpha \beta, c/ab \right] = \left[ d/c, d/e, b/\alpha, b/\beta; q \right]$$ (2.8)

Note that when \(a \to b\), then the \(4\phi_3\)-series just displayed reduces to one. We can directly get the following summation formula.

**Corollary 2.4 (Summation formula).**

$$\phi_{1;0;2}^{1;1;3}\left[\begin{array}{c} e; d/e, \alpha, \beta \\ d; e, \alpha, \beta ; q \end{array} \right| q; \alpha \beta \right] = \left[ d/c, d/e, b/\alpha, b/\beta; q \right]$$ (2.9)
This summation formula can also be derived from (2.1a) by using twice the \( q \)-Gauss summation theorem (2.6).

Similar cases occur in other propositions. But for the space limitations, we have not listed all of the examples.

2.2 Non-terminating reduction formula for \( \Phi_{2:0;1}^{2:1,2} \). Letting in Theorem 2.1

\[
\Omega(j) = \frac{\beta, \gamma; q^j}{(c; q)_j} \left( \frac{bd}{ae\beta\gamma} \right)^j
\]

and then reformulating the corresponding (2.1b) by using [1, Proposition 2.3]

\[
\Phi_{1:1:1}^{0:3:3} \left[ - : \ a, b, c; \ d/a, \beta, \gamma; \ q: de/abc, de/bc\beta\gamma \ \\
\quad d : \ e; \ de/abc; \ 0, \ 0, \ 1 \right] = \frac{[d/a, de/bc\beta, de/bc\gamma; q]_\infty}{[d, de/abc, de/bc\beta\gamma; q]_\infty} 4\Phi_3 \left[ a, e/b, e/c, de/bc\beta\gamma \mid q; \frac{d}{a}, e, de/bc\beta, de/bc\gamma \right].
\]

we can derive the following reduction formula.

Proposition 2.5 (Reduction formula).

\[
\Phi_{2:0;1}^{2:1,2} \left[ a, c : \ e; \ \beta, \gamma; \ q: bd/ace, bd/ae\beta\gamma \ \\
\quad b, d : \ - ; \ c; \ 0, \ 0, \ 0 \right] = \frac{[a, bd/ace, bd/ae\beta, bd/ae\gamma; q]_\infty}{[b, d, bd/ace, bd/ae\beta\gamma; q]_\infty} 4\Phi_3 \left[ b/a, d/a, bd/ace, bd/ae\beta\gamma \ \\
\quad bd/ace, bd/ae\beta, bd/ae\gamma \mid q; a \right].
\]

2.3 Terminating reduction formula for \( \Phi_{2:0;2}^{2:1,3} \). Setting in Theorem 2.1

\[
a = q^{-n} \quad \text{and} \quad \Omega(j) = \frac{[d/e, \beta, \gamma; q_j]}{(c.q^{-n}\beta\gamma/b; q)_j} q^j
\]

and then rewriting the corresponding (2.1b) by [1, Proposition 2.6]

\[
\Phi_{1:1:1}^{0:3:3} \left[ - : \ q^{-n}, b, c; \ q^n d, \beta, \gamma; \ q, q, q^{-n}\alpha/\beta\gamma \ \\
\quad d : \ q^{1-n}bc/d; \ \alpha; \ 0, \ 0, \ 1 \right] = \frac{[d/b, d/c; q]_n \ [\alpha/\beta, \alpha/\beta\gamma; q]_\infty}{[d, d/bc; \ q]_n \ [\alpha, \alpha/\beta\gamma; q]_\infty} 4\Phi_3 \left[ q^{-n}, \beta, \gamma, d/bc \mid q, q \right].
\]

we find the following reduction formula.

Proposition 2.6 (Reduction formula).

\[
\Phi_{2:0;2}^{2:1,3} \left[ q^{-n}, c : \ e; \ d/e, \beta, \gamma; \ q : q^nbd/ce, q \ \\
\quad b, d : \ - ; \ c, q^{1-n}\beta\gamma/b; \ 0, \ 0, \ 0 \right] = \frac{[d/e, b/\beta, b/\gamma; q]_n}{[b, d, b/\beta\gamma; q]_n} 4\Phi_3 \left[ q^{-n}, e, b/c, b/\beta\gamma \ \\
\quad b/\beta, b/\gamma, q^{1-n}e/d \mid q, q \right].
\]
Under replacements \( \beta \rightarrow b/e \) and \( \gamma \rightarrow c \), the last \( \phi_3 \)-series reduces to a \( \phi_1 \)-series. Evaluating it by the \( q \)-Chu–Vandermonde convolution formula [3, p. 354, Appendix II.6]

\[
\begin{align*}
\Phi_2^{2:1;2}_{0:1;1} & \left[ q^{-n}, c ; e ; b/e, d/e ; q : q^\alpha bd/ce, q \right] = \frac{(c/ae; q)_n}{(ce; q)_n} d^n,
\end{align*}
\]

we obtain the following closed formula.

**Corollary 2.7 (Summation formula).**

\[
\Phi_2^{2:1;2}_{0:1;1} \left[ q^{-n}, c ; e ; b/e, d/e ; q : q^\alpha bd/ce, q \right] \Phi_1 \left[ q^a, q/c ; q_n \right] = \frac{[b/c, d/c, e; q)_n}{[b, d, e/c; q)_n}.
\]

### 2.4 Terminating reduction formula for \( \Phi_2^{2:1;2}_{0:1;1} \).

Putting in Theorem 2.1

\[
a = q^{-n} \quad \text{and} \quad \Omega(j) = \frac{(j; q)_j (\beta; q)_j}{(q^{-n} c e \alpha \beta / bd ; q)_j} q^j/
\]

and then transforming the corresponding (2.1b) by [1, Proposition 2.9]

\[
\begin{align*}
\Phi_3^{0:3;4}_{1:1:2} & \left[ q^{-n}, c ; e ; b/a, q^n d, d/a, \alpha, \beta ; q : q^{-n} e/ab, q \right] \\
& = b^n [a, qd/e; q]_{\infty} [e/a, e/b; q]_n [e, e/ab; q]_n \Phi_3 \left[ q^{-n}, d/a, q\alpha \beta / e, q^{-n} e/ab, q \right] q^j / b, q \Phi_3 \left[ q^{-n}, d/a, q\alpha \beta / e, q^{-n} e/ab, q \right] q^j / b,
\end{align*}
\]

we deduce the following reduction formula.

**Proposition 2.8 (Reduction formula).**

\[
\begin{align*}
\Phi_2^{2:1;2}_{0:1;1} & \left[ q^{-n}, c ; e ; b/a, q^n d, d/a, \alpha, \beta ; q : q^{-n} e/ab, q \right] \\
& = \frac{[c, bd/ce \alpha \beta /bd ; q]_n}{[b, bd/ce \alpha \beta ; q]_n} \Phi_3 \left[ q^{-n}, b/c, d/c, bd/ce \alpha \beta ; q \right] q^{-n} / c, bd/ce \alpha \beta, bd/ce \beta / q^j / q.
\end{align*}
\]

When \( \alpha \rightarrow b/e \) and \( \beta \rightarrow q^{-n-1} d \), the last \( \phi_3 \)-series reduces to a \( \phi_1 \)-series. Evaluating it by the \( q \)-Chu–Vandermonde convolution formula (2.9) again, we obtain the following summation formula.

**Corollary 2.9 (Summation formula).**

\[
\Phi_2^{2:1;2}_{0:1;1} \left[ q^{-n}, c ; e ; b/a, q^{-n-1} d, d/a, \alpha, \beta ; q : q^{-n} e/ab, q \right] \Phi_3 \left[ q^{-n}, b/c, d/c, bd/ce \alpha \beta ; q \right] q^{-n} / c, bd/ce \alpha \beta, bd/ce \beta / q^j / q.
\]

### 2.5 Terminating reduction formula for \( \Phi_2^{2:1;3}_{0:1:2} \).

Taking in Theorem 2.1

\[
a = q^{-n} \quad \text{and} \quad \Omega(j) = \frac{[j, b/c, d/c ; q]_n (q^j e^j)}{[j, c, \gamma ; q]_n}.
\]
and then evaluating the corresponding (2.1b) by means of [1, Proposition 2.10]

\[
\Phi^{0;3}_{1;1;1} \left[ - : \begin{array}{cccc}
q^n d, & b, & c; & q^{-n},
d, & b, & \beta; & q, & q^{n} e / bc, & q^{n} y / b^\beta
\end{array} \gamma; \right] = e^n [b, q b / e; q^n b [e / b, e / c; q]_n [d, q b / c / e; q]_n [e, e / b c; q]_n 4\Phi_3 \left[ q^{-n}, d / b, \gamma / \beta, q^{n} e / b c \right] q; q \right],
\]

we have the following reduction formula.

**Proposition 2.10 (Reduction formula).**

\[
\Phi^{2;1;3}_{2;0;2} \left[ q^{-n}, c : e; \begin{array}{cccc}
b / e, & d / e, & \beta; & q, & q^{n} b d / c e, & q^{n} e y / b^\beta
\end{array} \gamma; \right] = [e, b d / c e; q]_n \left[ q^{-n}, d / e, \gamma / \beta, q^{n} e / b c \right] q; q \right]
\]

we deduce the following reduction formula.

**Proposition 2.11 (Reduction formula).**

\[
\Phi^{0;3;4}_{1;1;2} \left[ - : \begin{array}{cccc}
a, & b, & q^n d; & q^{-n},
d, & a, & d / b, & \beta; & q, & q^{n} e / a b, & q^{n} y / d b
\end{array} \gamma; \right] = \frac{(q d / e; q) [e / a, e / b; q]_n (ab / d)}{(q a b / e; q) [e, e / a b; q]_n} \left[ q^{-n}, d / a, \gamma / \beta, q^{n} e / a b \right] q; q \right]
\]

we deduce the following reduction formula.

**Proposition 2.11 (Reduction formula).**

\[
\Phi^{2;1;2}_{2;0;1} \left[ q^{-n}, c : e; \begin{array}{cccc}
b / e, & b, & \beta; & q, & q^{n} b d / c e, & q^{n} e y / c b
\end{array} \gamma; \right] = \frac{(d / c; q) [e / a, e / b; q]_n (ab / d)}{(d / q) [e, e / a b; q]_n} \left[ q^{-n}, c / d, \gamma / \beta, q^{n} e / c b \right] q; q \right]
\]

we find the following reduction formula.

**2.6 Terminating reduction formulae for \( \Phi^{2;1;2}_{2;0;1} \).** Setting in Theorem 2.1

\[
a = q^{-n} \quad \text{and} \quad \Omega(j) = \frac{[b / e, \beta; q]_j \left( \frac{q^n d y}{c b} \right)^j}{(q^n d y / c b)}
\]

we specialise in Theorem 2.2 with

\[
\Omega(j) = \frac{[b / e, \alpha, \beta; q]_j \left( \frac{bd}{c b} \right)^j}{(b d / c b)}
\]

and then evaluating the corresponding (2.5b) by means of [1, Proposition 2.5]

\[
\Phi^{0;3;4}_{1;1;2} \left[ - : \begin{array}{cccc}
a, & b, & c; & q^n d, 
d, & a, & d / b, & \alpha, \beta; & q, & q^{n} e / a b c, & q^{n} e / c
\end{array} \gamma; \right] = \frac{[e / c, d e / a b c; q]_n [e, d e / a b c; q]_n (de / a b c)}{(e, c b)} \left[ d / a, d / b, \alpha / c b, q^{n} e / c \right] q; q \right]
\]
PROPOSITION 2.12 (Reduction formula).

\[
\Phi_{2;0:2}^{2;1:3} \left[ \begin{array}{c}
a, c : e; \\
b, d : -;
\end{array} \right]
\Phi_{2;0:2}^{2;1:3} \left[ \begin{array}{c}
b / e, \alpha, \beta; \\
c ; ba \beta / c; \
\end{array} \right] \\
\frac{[d / a, b / d / e ; q]_{\infty}}{[b / d / c e ; q]_{\infty}} 3 \Phi_{3} \left[ \begin{array}{c}
b, b / d / e, ba \beta / c ; \\
b, b / d / c e, ba \beta / c;
\end{array} \right] q; d / a.
\]

Similarly, letting in Theorem 2.2

\[
\Omega(j) = \frac{[b / e, d / e, \alpha, \beta ; q]_{l}}{[a, c, bd \alpha \beta / ace ; q]_{l}} \left( \frac{bd}{ace} \right)^{j}
\]

and then evaluating the corresponding (2.5b) by (2.13a)–(2.13b) again, we get the following reduction formula.

PROPOSITION 2.13 (Reduction formula).

\[
\Phi_{2;0:3}^{2;1:4} \left[ \begin{array}{c}
a, c : e; \\
b, d : -;
\end{array} \right]
\Phi_{2;0:3}^{2;1:4} \left[ \begin{array}{c}
b / e, d / e, \alpha, \beta; \\
a, c , bd \alpha \beta / ace;
\end{array} \right] \\
\frac{[e, b / d a e, b / d / c e ; q]_{\infty}}{[b, b / d / a e, b / d / c e, bd \alpha \beta / ace ; q]_{\infty}} 4 \Phi_{3} \left[ \begin{array}{c}
b / e, d / e, bd \alpha / ace, bd \beta / ace ; \\
b / d / a e, b / d / c e, bd \alpha \beta / ace,
\end{array} \right] q; e.
\]

3. Transformations between \( \Phi_{2;0:3}^{2;1:4} \) and \( \Phi_{2;0:3}^{2;1:4} \).

THEOREM 3.1 (Transformation formula). For an arbitrary complex sequence \( \{ \Omega(j) \} \), the transformation

\[
\sum_{l=0}^{\infty} (a ; q)_{l+j} (c ; q)_{l+j} (e ; q)_{l+j} (b / e ; q)_{l+j} \left( \frac{b d}{ace} \right)^{i} \Omega(j) = \frac{[d / a, b / d / c e ; q]_{\infty}}{[b / d / a e, b / d / c e, bd \alpha \beta / ace ; q]_{\infty}} \sum_{l=0}^{\infty} \left( \frac{d}{a} \right)^{i} [a, b / e ; q]_{l+j} (b / c ; q)_{l+j} \Omega(j)
\]

holds, provided that two double series displayed above are absolutely convergent.

Under specification (2.3), this theorem becomes a transformation between two non-terminating double series \( \Phi_{2;0:3}^{2;1:4+1} \) and \( \Phi_{2;0:3}^{2;1:4} \).

Proof. Recalling transformation (2.2), we have the expression

\[
\sum_{j=0}^{\infty} \frac{(a ; q)_{j} (b / e ; q)_{j}}{(q ; q)_{j} (b ; q)_{j} (d ; q)_{j}} \Omega(j) 3 \Phi_{2} \left[ \begin{array}{c}
q^{a}, q^{c}, e \\
q^{b}, q^{d}
\end{array} \right] q ; b d / ace
\]

which is exactly (3.1b) when writing explicitly as a double sum.
Similarly, by applying the Hall transformation (2.4), we derive the following transformation.

**THEOREM 3.2 (Transformation formula).** For an arbitrary complex sequence \( \{\Omega(j)\} \), the transformation

\[
\sum_{i,j=0}^{\infty} \frac{[a, c; q]_{i+j} (e; q)_i [b/e, d/e; q]_i}{[b, d; q]_{i+j} (q; q)_i [a, c; q]_i} \left( \frac{bd}{ace} \right)^i \Omega(j)
\]

(3.2a)

\[
\frac{[e, bd/ae, bd/ce; q]_\infty}{[b, d, bd/ae, bd/ce; q]_\infty} \sum_{i,j=0}^{\infty} e^i [b/e, d/e; q]_{i+j} (bd/ace; q)_i - \Omega(j)
\]

(3.2b)

holds, provided that two double series displayed above are absolutely convergent.

Under specification (2.3), this theorem reduces to a transformation between two non-terminating double series \( \Phi_{2;0;\mu+2}^{2;1,\lambda} \) and \( \Phi_{2;0;\mu}^{2;1,\lambda} \).

### 3.1 Semi-terminating reduction formula for \( \Phi_{2;0;3}^{2;1,4} \)

Setting in Theorem 3.1

\[ e = q^n b \quad \text{and} \quad \Omega(j) = \frac{[q^{-n} d/b, \beta, \gamma; q]_j}{[a, q^{-n} \beta \gamma / b; q]_j} q^j \]

and then rewriting the corresponding expression (3.1b) by Proposition 2.6, we find the following reduction formula.

**PROPOSITION 3.3 (Reduction formula).**

\[
\Phi_{2;0;3}^{2;1,4} \left[ \begin{array}{c}
 a, c : q^n b; q^{-n}, q^{-n} d/b, \beta, \gamma; q^{-n} d/ae, q \\
 b, d : -; a, c, q^{-n} \beta \gamma / b; 0, 0, 0
\end{array} \right] = \left( \frac{ac}{b} \right)^n \left[ \begin{array}{c}
 [q^{-n} b/a, b/c, b/\beta \gamma; q]_n [d/a, d/c; q]_\infty \\
 [b, qac / d, b/\beta \gamma; q]_n [d, d/ac; q]_\infty \end{array} \right] 4 \Phi_{3}^{2;1} \left[ q^{-n}, b/\beta, b/\gamma, q^j b/d \right]_{q; q}.
\]

For the special case \( \beta \rightarrow a \) and \( \gamma \rightarrow c \), the last \( 4 \Phi_3 \)-series reduces to a \( 2 \Phi_1 \)-series. Evaluating it by (2.9), we obtain the following expression.

**COROLLARY 3.4 (Summation formula).**

\[
\Phi_{2;0;1}^{2;1,2} \left[ \begin{array}{c}
 a, c : q^n b; q^{-n}, q^{-n} d/b, q^{-n} d/ac, q \\
 b, d : -; q^{1-n} ac / b; 0, 0, 0
\end{array} \right] = \left[ b/a, b/c, q]_n [d/a, d/c; q]_\infty \\
 [b, b/ac; q]_n [d, d/ac; q]_\infty
\]

The special case \( a = q^{-m} \) of this corollary reduces to the same summation formula as the case \( e = q^n b \) of Corollary 2.7.

### 3.2 Semi-terminating reduction formula for \( \Phi_{2;0;2}^{2;1,3} \)

Letting in Theorem 3.1

\[ e = q^n b \quad \text{and} \quad \Omega(j) = \frac{[\alpha, \beta; q]_j}{(qac \beta / d; q)_j} q^j \]
and then reformulating the corresponding (3.1b) by Proposition 2.8, we get the following reduction formula.

**Proposition 3.5 (Reduction formula).**

\[
\phi_{1}^{2.1.3,2.0.2}[a, c : q^n b; q^{-n}, \alpha, \beta; q : q^{-n} d/ac, q] = a^n(b/a; q) [d/a, d/c; q] \phi_3^{[q^{-n}, \alpha, qac/d, qac/b, qac'\beta/d]}(q, q'c/b). 
\]

Its special case \( \alpha = d/qn \) reduces further to the following identity.

**Corollary 3.6 (Summation formula).**

\[
\phi_{2}^{2.0.1,1.2}[a, c : q^n b; \alpha aq, \alpha aq, q^{-n} d/ac, q] = a^n(b/a; q) [d/a, d/c; q] \phi_3^{[q^{-n}, \alpha, qac/d, qac/b, qac'\beta/d]}(q, q'c/b). 
\]

It should be pointed out that when \( \alpha = q^{-n} \) in Proposition 3.5 and Corollary 3.6, they reduce to, respectively, the same summation formulae as the case \( e = q^n b \) of Proposition 2.6 and Corollary 2.9.

### 3.3 Semi-terminating reduction formulae for \( \phi_{2}^{2.0.1,1.2,2.0.2,1.2} \) and \( \phi_{2}^{2.1.3,2.0.2} \)

Specialising in Theorem 3.1 with

\[
e = q^n b \quad \text{and} \quad \Omega(j) = \left\{ \begin{array}{l}
[q^{-n}d/b, \beta; q] j \left( \frac{q^n b^j \gamma}{\alpha \beta} \right), \\
[c, \beta; q] j \left( \frac{d \gamma}{\alpha \epsilon} \right) \end{array} \right\}
\]

and then transforming the corresponding double sum (3.1b) through Proposition 2.11, we derive two further reduction formulae, respectively.

**Proposition 3.7 (Reduction formula).**

\[
\phi_{2}^{2.1.3,2.0.2}[a, c : q^n b; q^{-n}, q^{-n} d/b, \beta; q : q^{-n} d/ac, q^{n}b^j \gamma/a\beta] = \beta c, \gamma; 0, 0, 0 
\]

\[
= \phi_{2}^{2.0.2}[a, c : q^n b; q^{-n}, \beta; q : q^{-n} d/ac, q^{n}b^j \gamma/a\beta] 
\]

**Proposition 3.8 (Reduction formula).**

\[
\phi_{2}^{2.0.1,1.2}[a, c : q^n b; q^{-n}, \beta; q : q^{-n} d/ac, q^{n}b^j \gamma/a\beta] = \gamma c, \gamma; 0, 0, 0 
\]

Similarly, setting \( \alpha = q^{-n} \) in Propositions 3.7 and 3.8, they reduce to, respectively, the same summation formulae as the case \( e = q^n b \) in Propositions 2.10 and 2.11.
ACKNOWLEDGEMENTS. We are thankful to the referee for his many valuable suggestions and comments. This work is supported by the Chinese National Science Foundation (youth grant 10801026).

REFERENCES


