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TRANSFORMATION AND REDUCTION FORMULAE FOR DOUBLE q-SERIES OF TYPE $\Phi^{2:1;\lambda}_{2:0;\mu}$

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Abstract. By applying the Sears non-terminating transformations, we establish four general transformation theorems for double basic hypergeometric series of type $\Phi^{2:1;\lambda}_{2:0;\mu}$. Moreover, several transformation, reduction and summation formulae on the double basic hypergeometric series $\Phi^{2:1;2}_{2:0;1}$, $\Phi^{2:1;3}_{2:0;2}$ and $\Phi^{2:1;4}_{2:0;3}$ are also derived through parameter specialisation.

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1. Introduction. For two indeterminates x and q, the shifted factorial is defined by

$$(x;q)_0 = 1 \quad \text{and} \quad (x;q)_n = \prod_{k=0}^{n-1}(1-q^k x) \quad \text{with} \quad n = 1, 2, \ldots .$$

When $|q| < 1$, we have the following well-defined infinite product expressions:

$$(x;q)_\infty = \prod_{k=0}^{\infty}(1-q^k x) \quad \text{and} \quad (x;q)_n = \frac{(x;q)_\infty}{(q^n x;q)_\infty} \quad \text{for} \quad n \in \mathbb{Z}.$$

For the sake of brevity, we also write the factorial product compactly as

$$[a, b, \ldots, c; q]_n := (a;q)_n (b;q)_n \ldots (c;q)_n.$$

Following Gasper and Rahman [**3**], the basic hypergeometric series is defined by

$$ {}_{1+r}\phi_s \left[\begin{matrix} a_0, & a_1, & \ldots, & a_r \\ & b_1, & \ldots, & b_s \end{matrix} \middle| q; z \right] = \sum_{n=0}^{\infty} \left\{(-1)^n q^{\binom{n}{2}}\right\}^{s-r} \frac{[a_0, a_1, \ldots, a_r; q]_n}{[q, b_1, \ldots, b_s; q]_n} z^n, \tag{1.1}$$

where the base q will be restricted to $|q| < 1$ for non-terminating q-series.

As the q-analogue of Kampé de Fériet function, Srivastava and Karlsson [**9**, p. 349] defined the generalised bivariate basic hypergeometric function by

$$\Phi^{\lambda:r;s}_{\mu:u;v}\left[\begin{matrix} \alpha_1, \ldots, \alpha_\lambda : & a_1, \ldots, a_r; & c_1, \ldots, c_s; & q : x, y \\ \beta_1, \ldots, \beta_\mu : & b_1, \ldots, b_u; & d_1, \ldots, d_v; & i, j, k \end{matrix}\right] \tag{1.2a}$$

$$= \sum_{m,n=0}^{\infty} \frac{[\alpha_1, \ldots, \alpha_\lambda; q]_{m+n}}{[\beta_1, \ldots, \beta_\mu; q]_{m+n}} \frac{[a_1, \ldots, a_r; q]_m [c_1, \ldots, c_s; q]_n}{[b_1, \ldots, b_u; q]_m [d_1, \ldots, d_v; q]_n} \frac{x^m y^n q^{i\binom{m}{2}+j\binom{n}{2}+kmn}}{(q;q)_m (q;q)_n}. \tag{1.2b}$$

It is not hard to check that when $i, j, k \in \mathbb{N}_0$, the double series $\Phi^{\lambda:r;s}_{\mu:u;v}$ is convergent for $|x| < 1$, $|y| < 1$ and $|q| < 1$.

For double basic hypergeometric series, there are fewer instances available in the literature [**1**, **2**, **4**–**8**, **10**–**12**]. Recently, Chu and Jia [**1**] established eight transformations by using the Sears transformation formulae and obtained a number of transformation, reduction and summation formulae on Φ^{122}_{022}, Φ^{123}_{023}, Φ^{033}_{111} and Φ^{034}_{112} as special cases. As a continuation, we will further investigate four general transformations for double basic hypergeometric series of type $\Phi^{2:1;\lambda}_{2:0;\mu}$, again by means of non-terminating Sears transformation formulae. Several new transformation formulae on Φ^{212}_{201}, Φ^{213}_{202}, Φ^{214}_{203} are also obtained as consequences.

2. Transformations between $\Phi^{2:1;\lambda}_{2:0;\mu}$ and $\Phi^{0:3;u}_{1:1;v}$.

THEOREM 2.1 (Transformation formula). *For an arbitrary complex sequence* $\{\Omega(j)\}$, *the transformation*

$$\sum_{i,j=0}^{\infty} \frac{(a;q)_{i+j}(c;q)_{i+j}(e;q)_i}{(b;q)_{i+j}(d;q)_{i+j}(q;q)_i(q;q)_j} \left(\frac{bd}{ace}\right)^i \Omega(j) \tag{2.1a}$$

$$= \frac{[d/e, bd/ac;q]_\infty}{[d, bd/ace;q]_\infty} \sum_{i,j=0}^{\infty} q^{ij} \left(\frac{d}{e}\right)^i \frac{[e, b/a, b/c;q]_i[a, c;q]_j}{(b;q)_{i+j}[q, bd/ac;q]_i[q, d/e;q]_j} \Omega(j) \tag{2.1b}$$

holds, provided that two double series displayed above are absolutely convergent.

Proof. Recalling the q-analogue of the Kummer–Thomae–Whipple transformation [**3**, p. 359, Appendix III.9],

$${}_3\phi_2\left[\begin{matrix} a, & c, & e \\ & b, & d\end{matrix} \middle| q; \frac{bd}{ace}\right] = \frac{[d/a, bd/ce;q]_\infty}{[d, bd/ace;q]_\infty} {}_3\phi_2\left[\begin{matrix} a, & b/c, & b/e \\ & b, & bd/ce\end{matrix} \middle| q; \frac{d}{a}\right], \tag{2.2}$$

we can reformulate the double sum in (2.1a) as follows:

$$\begin{aligned}
&\sum_{j=0}^{\infty} \frac{(a;q)_j(c;q)_j}{(q;q)_j(b;q)_j(d;q)_j} \Omega(j) {}_3\phi_2\left[\begin{matrix} e, q^ja, q^jc \\ q^jb, \ q^jd\end{matrix} \middle| q; \frac{bd}{ace}\right] \\
&= \sum_{j=0}^{\infty} \frac{[a, c;q]_j}{[q, b, d;q]_j} \Omega(j) \frac{[q^jd/e, bd/ac;q]_\infty}{[q^jd, bd/ace;q]_\infty} {}_3\phi_2\left[\begin{matrix} e, b/a, b/c \\ q^jb, \ bd/ac\end{matrix} \middle| q; \frac{q^jd}{e}\right] \\
&= \frac{[d/e, bd/ac;q]_\infty}{[d, bd/ace;q]_\infty} \sum_{j=0}^{\infty} \frac{[a, c;q]_j}{[q, b, d/e;q]_j} \Omega(j) {}_3\phi_2\left[\begin{matrix} e, b/a, b/c \\ q^jb, \ bd/ac\end{matrix} \middle| q; \frac{q^jd}{e}\right].
\end{aligned}$$

Writing the last double sum explicitly, we see that it coincides with (2.1b). □

When the Ω-sequence is specified by

$$\Omega(j) = \frac{[u_1, u_2, \ldots, u_\lambda;q]_j}{[v_1, v_2, \ldots, v_\mu;q]_j} w^j \tag{2.3}$$

the last theorem gives us a very general transformation between two non-terminating double series $\Phi^{2:1;\lambda}_{2:0;\mu}$ and $\Phi^{0:3;\lambda+2}_{1:1;\mu+1}$.

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In the proof of the last theorem, if we apply, instead of (2.2), the Hall transformation [**3**, p. 359, Appendix III.10]

$$ {}_3\phi_2\left[\begin{matrix} a, & c, & e \\ & b, & d \end{matrix}\middle| q; \frac{bd}{ace}\right] = \frac{[c, bd/ac, bd/ce; q]_\infty}{[b, d, bd/ace; q]_\infty}\, {}_3\phi_2\left[\begin{matrix} b/c, d/c, bd/ace \\ bd/ac, bd/ce \end{matrix}\middle| q; c\right], \quad (2.4) $$

then we can establish another transformation formula.

THEOREM 2.2 (Transformation formula). *For an arbitrary complex sequence* $\{\Omega(j)\}$, *the transformation*

$$ \sum_{i,j=0}^{\infty} \frac{(a;q)_{i+j}(c;q)_{i+j}(e;q)_i}{(b;q)_{i+j}(d;q)_{i+j}(q;q)_i(q;q)_j}\left(\frac{bd}{ace}\right)^i \Omega(j) \quad (2.5a) $$

$$ = \frac{[c, bd/ac, bd/ce; q]_\infty}{[b, d, bd/ace; q]_\infty} \sum_{i,j=0}^{\infty} q^{ij} c^i \frac{[b/c, d/c, bd/ace; q]_i (a;q)_j}{(bd/ce;q)_{i+j}[q, bd/ac; q]_i (q;q)_j}\Omega(j) \quad (2.5b) $$

holds, provided that two double series displayed above are absolutely convergent.

Under specification (2.3), this theorem yields a transformation between two non-terminating double series $\Phi^{2:1;\lambda}_{2:0;\mu}$ and $\Phi^{0:3;\lambda+1}_{1:1;\mu}$.

We remark that the relation $\Phi^{0:3;\lambda}_{1:1;\mu}$ and $\Phi^{2:1;\lambda}_{2:0;\mu+1}$ has first been discovered in [**1**].

2.1 Non-terminating reduction formula for $\Phi^{2:1;3}_{2:0;2}$. Specifying in Theorem 2.1 with

$$ \Omega(j) = \frac{[d/e, \alpha, \beta; q]_j}{[a, c; q]_j}\left(\frac{b}{\alpha\beta}\right)^j $$

and then evaluating the sum with respect to j displayed in (2.1b) by means of the q-Gauss summation theorem [**3**, p. 354, Appendix II.8]

$$ {}_2\phi_1\left[\begin{matrix} a, & b \\ & c \end{matrix}\middle| q; \frac{c}{ab}\right] = \frac{[c/a, c/b; q]_\infty}{[c, c/ab; q]_\infty}, \quad (2.6) $$

we find after some trivial simplification the following reduction formula.

PROPOSITION 2.3 (Reduction formula).

$$ \Phi^{2:1;3}_{2:0;2}\left[\begin{matrix} a, c: & e; & d/e, & \alpha, & \beta; & q: bd/ace, b/\alpha\beta \\ b, d: & -; & a, & c; & & 0, \quad 0, \quad 0 \end{matrix}\right] $$
$$ = \frac{[d/e, bd/ac, b/\alpha, b/\beta; q]_\infty}{[b, d, bd/ace, b/\alpha\beta; q]_\infty}\, {}_4\phi_3\left[\begin{matrix} e, b/a, b/c, b/\alpha\beta \\ bd/ac, b/\alpha, b/\beta \end{matrix}\middle| q; \frac{d}{e}\right]. $$

Note that when $a \to b$, then the ${}_4\phi_3$-series just displayed reduces to one. We can directly get the following summation formula.

COROLLARY 2.4 (Summation formula).

$$ \Phi^{1:1;3}_{1:0;2}\left[\begin{matrix} c: & e; & d/e, & \alpha, & \beta; & q: d/ce, b/\alpha\beta \\ d: & -; & b, & c; & & 0, \quad 0, \quad 0 \end{matrix}\right] = \frac{[d/c, d/e, b/\alpha, b/\beta; q]_\infty}{[d, d/ce, b/\alpha\beta; q]_\infty}. $$

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This summation formula can also be derived from (2.1a) by using twice the q-Gauss summation theorem (2.6).

Similar cases occur in other propositions. But for the space limitations, we have not listed all of the examples.

2.2 Non-terminating reduction formula for $\Phi^{2:1;2}_{2:0;1}$. Letting in Theorem 2.1

$$\Omega(j) = \frac{[\beta, \gamma; q]_j}{(c; q)_j}\left(\frac{bd}{ae\beta\gamma}\right)^j$$

and then reformulating the corresponding (2.1b) by using [**1**, Proposition 2.3]

$$\Phi^{0:3;3}_{1:1;1}\left[\begin{matrix} - : & a,\ b,\ c; & d/a,\ \beta,\ \gamma; & q: de/abc,\ de/bc\beta\gamma \\ d : & e; & de/abc; & 0,\quad 0,\quad 1 \end{matrix}\right]$$
$$= \frac{[d/a, de/bc\beta, de/bc\gamma; q]_\infty}{[d, de/abc, de/bc\beta\gamma; q]_\infty}\,{}_4\phi_3\left[\begin{matrix} a, e/b, e/c, de/bc\beta\gamma \\ e, de/bc\beta, de/bc\gamma \end{matrix}\,\middle|\, q; \frac{d}{a}\right],$$

we can derive the following reduction formula.

PROPOSITION 2.5 (Reduction formula).

$$\Phi^{2:1;2}_{2:0;1}\left[\begin{matrix} a, c : & e; & \beta,\ \gamma; & q : bd/ace, bd/ae\beta\gamma \\ b, d : & -; & c; & 0,\quad 0,\quad 0 \end{matrix}\right]$$
$$= \frac{[a, bd/ac, bd/ae\beta, bd/ae\gamma; q]_\infty}{[b, d, bd/ace, bd/ae\beta\gamma; q]_\infty}\,{}_4\phi_3\left[\begin{matrix} b/a, d/a, bd/ace, bd/ae\beta\gamma \\ bd/ac, bd/ae\beta, bd/ae\gamma \end{matrix}\,\middle|\, q; a\right].$$

2.3 Terminating reduction formula for $\Phi^{2:1;3}_{2:0;2}$. Setting in Theorem 2.1

$$a = q^{-n} \quad \text{and} \quad \Omega(j) = \frac{[d/e, \beta, \gamma; q]_j}{[c, q^{1-n}\beta\gamma/b; q]_j}q^j$$

and then rewriting the corresponding (2.1b) by [**1**, Proposition 2.6]

$$\Phi^{0:3;3}_{1:1;1}\left[\begin{matrix} - : & q^{-n},\ b,\ c; & q^n d,\ \beta,\ \gamma; & q: q,\ q^{-n}\alpha/\beta\gamma \\ d : & q^{1-n}bc/d; & \alpha; & 0,\quad 0,\quad 1 \end{matrix}\right]$$
$$= \frac{[d/b, d/c; q]_n}{[d, d/bc; q]_n}\frac{[\alpha/\beta, \alpha/\gamma; q]_\infty}{[\alpha, \alpha/\beta\gamma; q]_\infty}\,{}_4\phi_3\left[\begin{matrix} q^{-n},\ \beta,\ \gamma,\ d/bc \\ d/b,\ d/c,\ q\beta\gamma/\alpha \end{matrix}\,\middle|\, q; q\right],$$

we find the following reduction formula.

PROPOSITION 2.6 (Reduction formula).

$$\Phi^{2:1;3}_{2:0;2}\left[\begin{matrix} q^{-n}, c : & e; & d/e,\ \beta,\ \gamma; & q : q^n bd/ce, q \\ b, d : & -; & c,\ q^{1-n}\beta\gamma/b; & 0,\quad 0,\quad 0 \end{matrix}\right]$$
$$= \frac{[d/e, b/\beta, b/\gamma; q]_n}{[b,\ d,\ b/\beta\gamma; q]_n}\,{}_4\phi_3\left[\begin{matrix} q^{-n}, e, b/c, b/\beta\gamma \\ b/\beta, b/\gamma, q^{1-n}e/d \end{matrix}\,\middle|\, q; q\right].$$

Under replacements $\beta \to b/e$ and $\gamma \to c$, the last ${}_4\phi_3$-series reduces to a ${}_2\phi_1$-series. Evaluating it by the q-Chu–Vandermonde convolution formula [**3**, p. 354, Appendix II.6]

$$ {}_2\phi_1\left[\begin{matrix} q^{-n}, & a \\ & c \end{matrix}\middle| q; q\right] = \frac{(c/a;q)_n}{(c;q)_n} a^n, \tag{2.9}$$

we obtain the following closed formula.

Corollary 2.7 (Summation formula).

$$\Phi^{2:1;2}_{2:0;1}\left[\begin{matrix} q^{-n}, c: & e; & b/e, d/e; & q: q^n bd/ce, q \\ b, d: & -; & q^{1-n}c/e; & 0, \quad 0, \quad 0 \end{matrix}\right] = \frac{[b/c, d/c, e; q]_n}{[b, d, e/c; q]_n}.$$

2.4 Terminating reduction formula for $\Phi^{2:1;2}_{2:0;1}$. Putting in Theorem 2.1

$$a = q^{-n} \quad \text{and} \quad \Omega(j) = \frac{(\alpha;q)_j(\beta;q)_j}{(q^{1-n}ce\alpha\beta/bd;q)_j} q^j$$

and then transforming the corresponding (2.1b) by [**1**, Proposition 2.9]

$$\begin{aligned}\Phi^{0:3;4}_{1:1;2}&\left[\begin{matrix} -: & a, b, q^n d; & q^{-n}, & d/a, & \alpha, \beta; & q: q^{-n}e/ab, q \\ d: & e; & q^{-n}e/ab, & qb\alpha\beta/e; & 0, \quad 0, \quad 1 \end{matrix}\right] \\ &= b^n \frac{[a, qd/e; q]_n}{[d, qab/e; q]_n} \frac{[e/a, e/b; q]_\infty}{[e, e/ab; q]_\infty} {}_4\phi_3\left[\begin{matrix} q^{-n}, d/a, qb\alpha/e, qb\beta/e \\ q^{1-n}/a, qd/e, qb\alpha\beta/e \end{matrix}\middle| q; \frac{q}{b}\right],\end{aligned}$$

we deduce the following reduction formula.

Proposition 2.8 (Reduction formula).

$$\begin{aligned}\Phi^{2:1;2}_{2:0;1}&\left[\begin{matrix} q^{-n}, c: & e; & \alpha, \beta; & q: q^n bd/ce, q \\ b, d: & -; & q^{1-n}ce\alpha\beta/bd; & 0, \quad 0, \quad 0 \end{matrix}\right] \\ &= \frac{[c, bd/ce\alpha, bd/ce\beta; q]_n}{[b, d, bd/ce\alpha\beta; q]_n} {}_4\phi_3\left[\begin{matrix} q^{-n}, b/c, d/c, bd/ce\alpha\beta \\ q^{1-n}/c, bd/ce\alpha, bd/ce\beta \end{matrix}\middle| q; q\right].\end{aligned}$$

When $\alpha \to b/e$ and $\beta \to q^{n-1}d$, the last ${}_4\phi_3$-series reduces to a ${}_2\phi_1$-series. Evaluating it by the q-Chu–Vandermonde convolution formula (2.9) again, we obtain the following summation formula.

Corollary 2.9 (Summation formula).

$$\Phi^{2:1;2}_{2:0;1}\left[\begin{matrix} q^{-n}, c: & e; & b/e, q^{n-1}d; & q: q^n bd/ce, q \\ b, d: & -; & c; & 0, \quad 0, \quad 0 \end{matrix}\right] = \frac{[e, d/c; q]_n}{[b, d; q]_n}\left(\frac{b}{e}\right)^n.$$

2.5 Terminating reduction formula for $\Phi^{2:1;3}_{2:0;2}$. Taking in Theorem 2.1

$$a = q^{-n} \quad \text{and} \quad \Omega(j) = \frac{[b/e, d/e, \beta; q]_j}{[c, \gamma; q]_j}\left(\frac{q^n e\gamma}{\beta}\right)^j$$

and then rewriting the corresponding (2.1b) by [**1**, Proposition 2.10]

$$\Phi^{0:3;3}_{1:1;1}\left[\begin{matrix} - : & q^n d,\ b,\ c; & q^{-n},\ d/b,\ \beta; & q: q^{-n}e/bc,\ q^n b\gamma/\beta \\ d : & e; & \gamma; & 0, \quad 0, \quad 1 \end{matrix}\right]$$
$$= c^n \frac{[b, qb/e; q]_n}{[d, qbc/e; q]_n} \frac{[e/b, e/c; q]_\infty}{[e, e/bc; q]_\infty} {}_4\phi_3 \left[\begin{matrix} q^{-n}, d/b, \gamma/\beta, q^{-n}e/bc \\ \gamma, q^{1-n}/b, q^{-n}e/b \end{matrix} \middle| q; q\right],$$

we have the following reduction formula.

PROPOSITION 2.10 (Reduction formula).

$$\Phi^{2:1;3}_{2:0;2}\left[\begin{matrix} q^{-n}, c : & e; & b/e, d/e, \beta; & q : q^n bd/ce, q^n e\gamma/\beta \\ b, d : & -; & c, \gamma; & 0, \quad 0, \quad 0 \end{matrix}\right]$$
$$= \frac{[e, bd/ce; q]_n}{[b,\ d; q]_n} {}_4\phi_3 \left[\begin{matrix} q^{-n}, & b/e, & d/e, & \gamma/\beta \\ & q^{1-n}/e, & bd/ce, & \gamma \end{matrix} \middle| q; q\right].$$

2.6 Terminating reduction formula for $\Phi^{2:1;2}_{2:0;1}$. Setting in Theorem 2.1

$$a = q^{-n} \quad \text{and} \quad \Omega(j) = \frac{[b/e, \beta; q]_j}{(\gamma; q)_j} \left(\frac{q^n d\gamma}{c\beta}\right)^j$$

and then reformulating the corresponding (2.1b) by [**1**, Proposition 2.11]

$$\Phi^{0:3;4}_{1:1;2}\left[\begin{matrix} - : & a,\ b,\ q^n d; & q^{-n},\ d/a,\ d/b,\ \beta; & q: q^{-n}e/ab,\ e\gamma/d\beta \\ d : & e; & q^{-n}e/ab,\ \gamma; & 0, \quad 0, \quad 1 \end{matrix}\right]$$
$$= \frac{(qd/e; q)_n}{(qab/e; q)_n} \frac{[e/a, e/b; q]_\infty}{[e, e/ab; q]_\infty} \left(\frac{ab}{d}\right)^n {}_4\phi_3 \left[\begin{matrix} q^{-n}, & d/a, d/b, \gamma/\beta \\ & qd/e,\ d,\ \gamma \end{matrix} \middle| q; q\right],$$

we deduce the following reduction formula.

PROPOSITION 2.11 (Reduction formula).

$$\Phi^{2:1;2}_{2:0;1}\left[\begin{matrix} q^{-n}, c : & e; & b/e,\ \beta; & q : q^n bd/ce, q^n d\gamma/c\beta \\ b, d : & -; & \gamma; & 0, \quad 0, \quad 0 \end{matrix}\right]$$
$$= \frac{(d/c; q)_n}{(d; q)_n} {}_4\phi_3 \left[\begin{matrix} q^{-n}, & c, & b/e, & \gamma/\beta \\ & q^{1-n}c/d, & b, & \gamma \end{matrix} \middle| q; q\right].$$

2.7 Non-terminating reduction formulae for $\Phi^{2:1;3}_{2:0;2}$ and $\Phi^{2:1;4}_{2:0;3}$. Specialising in Theorem 2.2 with

$$\Omega(j) = \frac{[b/e, \alpha, \beta; q]_j}{[c, b\alpha\beta/c; q]_j} \left(\frac{bd}{ac}\right)^j$$

and then evaluating the corresponding (2.5b) by means of [**1**, Proposition 2.5]

$$\Phi^{0:3;4}_{1:1;2}\left[\begin{matrix} - : & a,\ b,\ c; & d/a, & d/b, & \alpha, & \beta; & q: de/abc, e \\ d : & e; & & de/abc, & c\alpha\beta; & & 0,\ 0,\ 1 \end{matrix}\right] \tag{2.13a}$$
$$= \frac{[e/c, de/ab; q]_\infty}{[e, de/abc; q]_\infty} {}_4\phi_3 \left[\begin{matrix} d/a, & d/b, & c\alpha, & c\beta \\ & d, & de/ab, & c\alpha\beta \end{matrix} \middle| q; e/c\right], \tag{2.13b}$$

we find the following reduction formula.

PROPOSITION 2.12 (Reduction formula).

$$\Phi^{2:1;3}_{2:0;2}\left[\begin{matrix} a, c: & e; & b/e, \ \alpha, \ \beta; & q: bd/ace, bd/ac \\ b, d: & -; & c, \ b\alpha\beta/c; & 0, \quad 0, \quad 0 \end{matrix}\right] = \frac{[d/a, bd/ce; q]_\infty}{[d, bd/ace; q]_\infty} {}_4\phi_3\left[\begin{matrix} a, b/e, b\alpha/c, b\beta/c \\ b, bd/ce, b\alpha\beta/c \end{matrix}\middle| q; d/a\right].$$

Similarly, letting in Theorem 2.2

$$\Omega(j) = \frac{[b/e, d/e, \alpha, \beta; q]_j}{[a, c, bd\alpha\beta/ace; q]_j}\left(\frac{bd}{ac}\right)^j$$

and then evaluating the corresponding (2.5b) by (2.13a)–(2.13b) again, we get the following reduction formula.

PROPOSITION 2.13 (Reduction formula).

$$\Phi^{2:1;4}_{2:0;3}\left[\begin{matrix} a, c: & e; & b/e, \ d/e, \ \alpha, \ \beta; & q: bd/ace, bd/ac \\ b, d: & -; & a, \ c, \ bd\alpha\beta/ace; & 0, \quad 0, \quad 0 \end{matrix}\right] = \frac{[e, bd/ae, bd/ce; q]_\infty}{[b, \ d, \ bd/ace; q]_\infty} {}_4\phi_3\left[\begin{matrix} b/e, d/e, bd\alpha/ace, bd\beta/ace \\ bd/ae, \ bd/ce, \ bd\alpha\beta/ace \end{matrix}\middle| q; e\right].$$

3. Transformations between $\Phi^{2:1;\lambda}_{2:0;\mu}$ and $\Phi^{2:1;r}_{2:0;s}$.

THEOREM 3.1 (Transformation formula). *For an arbitrary complex sequence $\{\Omega(j)\}$, the transformation*

$$\sum_{i,j=0}^{\infty} \frac{(a;q)_{i+j}(c;q)_{i+j}}{(b;q)_{i+j}(d;q)_{i+j}} \frac{(e;q)_i}{(q;q)_i} \frac{(b/e;q)_j}{(q;q)_j(c;q)_j}\left(\frac{bd}{ace}\right)^i \Omega(j) \tag{3.1a}$$

$$= \frac{[d/a, bd/ce; q]_\infty}{[d, bd/ace; q]_\infty} \sum_{i,j=0}^{\infty}\left(\frac{d}{a}\right)^i \frac{[a, b/e; q]_{i+j}(b/c;q)_i}{[b, bd/ce; q]_{i+j}(q;q)_i(q;q)_j}\Omega(j) \tag{3.1b}$$

holds, provided that two double series displayed above are absolutely convergent.

Under specification (2.3), this theorem becomes a transformation between two non-terminating double series $\Phi^{2:1;\lambda+1}_{2:0;\mu+1}$ and $\Phi^{2:1;\lambda}_{2:0;\mu}$.

Proof. Recalling transformation (2.2), we have the expression

$$\sum_{j=0}^{\infty} \frac{(a;q)_j(b/e;q)_j}{(q;q)_j(b;q)_j(d;q)_j}\Omega(j)\,{}_3\phi_2\left[\begin{matrix} q^j a, & q^j c, & e \\ & q^j b, & q^j d \end{matrix}\middle| q; \frac{bd}{ace}\right]$$

$$= \sum_{j=0}^{\infty} \frac{[a, b/e; q]_j}{[q, b, d; q]_j}\Omega(j) \frac{[d/a, q^j bd/ce; q]_\infty}{[q^j d, bd/ace; q]_\infty} {}_3\phi_2\left[\begin{matrix} q^j a, b/c, q^j b/e \\ q^j b, \ q^j bd/ce \end{matrix}\middle| q; \frac{d}{a}\right]$$

$$= \frac{[d/a, bd/ce; q]_\infty}{[d, bd/ace; q]_\infty} \sum_{j=0}^{\infty} \frac{[a, b/e; q]_j}{[q, b, bd/ce; q]_j}\Omega(j)\,{}_3\phi_2\left[\begin{matrix} q^j a, b/c, q^j b/e \\ q^j b, \ q^j bd/ce \end{matrix}\middle| q; \frac{d}{a}\right],$$

which is exactly (3.1b) when writing explicitly as a double sum. □

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Similarly, by applying the Hall transformation (2.4), we derive the following transformation.

THEOREM 3.2 (Transformation formula). *For an arbitrary complex sequence* $\{\Omega(j)\}$*, the transformation*

$$\sum_{i,j=0}^{\infty} \frac{[a,c;q]_{i+j}}{[b,d;q]_{i+j}} \frac{(e;q)_i}{(q;q)_i} \frac{[b/e,d/e;q]_j}{[q,a,c;q]_j} \left(\frac{bd}{ace}\right)^i \Omega(j) \tag{3.2a}$$

$$= \frac{[e,bd/ae,bd/ce;q]_\infty}{[b,d,bd/ace;q]_\infty} \sum_{i,j=0}^{\infty} e^i \frac{[b/e,d/e;q]_{i+j}(bd/ace;q)_i}{[bd/ae,bd/ce;q]_{i+j}(q;q)_i(q;q)_j} \Omega(j) \tag{3.2b}$$

holds, provided that two double series displayed above are absolutely convergent.

Under specification (2.3), this theorem reduces to a transformation between two non-terminating double series $\Phi^{2:1;\lambda+2}_{2:0;\mu+2}$ and $\Phi^{2:1;\lambda}_{2:0;\mu}$.

3.1 Semi-terminating reduction formula for $\Phi^{2:1;4}_{2:0;3}$. Setting in Theorem 3.1

$$e = q^n b \quad \text{and} \quad \Omega(j) = \frac{[q^{-n}d/b,\beta,\gamma;q]_j}{[a,q^{1-n}\beta\gamma/b;q]_j} q^j$$

and then rewriting the corresponding expression (3.1b) by Proposition 2.6, we find the following reduction formula.

PROPOSITION 3.3 (Reduction formula).

$$\Phi^{2:1;4}_{2:0;3}\left[\begin{matrix} a,c: & q^n b; & q^{-n}, & q^{-n}d/b, & \beta, & \gamma; & q: q^{-n}d/ac, q \\ b,d: & -; & & a, & c, & q^{1-n}\beta\gamma/b; & 0, \quad 0, \quad 0 \end{matrix}\right]$$
$$= \left(\frac{ac}{b}\right)^n \frac{[qb/d,b/\beta,b/\gamma;q]_n}{[b,qac/d,b/\beta\gamma;q]_n} \frac{[d/a,d/c;q]_\infty}{[d,d/ac;q]_\infty} {}_4\phi_3\left[\begin{matrix} q^{-n},b/a,b/c,b/\beta\gamma \\ b/\beta,b/\gamma,qb/d \end{matrix}\middle| q;q\right].$$

For the special case $\beta \to a$ and $\gamma \to c$, the last ${}_4\phi_3$-series reduces to a ${}_2\phi_1$-series. Evaluating it by (2.9), we obtain the following expression.

COROLLARY 3.4 (Summation formula).

$$\Phi^{2:1;2}_{2:0;1}\left[\begin{matrix} a,c: & q^n b; & q^{-n}, & q^{-n}d/b; & q: q^{-n}d/ac, q \\ b,d: & -; & & q^{1-n}ac/b; & 0, \quad 0, \quad 0 \end{matrix}\right] = \frac{[b/a,b/c;q]_n}{[b,b/ac;q]_n} \frac{[d/a,d/c;q]_\infty}{[d,d/ac;q]_\infty}.$$

The special case $a = q^{-m}$ of this corollary reduces to the same summation formula as the case $e = q^n b$ of Corollary 2.7.

3.2 Semi-terminating reduction formula for $\Phi^{2:1;3}_{2:0;2}$. Letting in Theorem 3.1

$$e = q^n b \quad \text{and} \quad \Omega(j) = \frac{[\alpha,\beta;q]_j}{(qa\alpha\beta/d;q)_j} q^j$$

and then reformulating the corresponding (3.1b) by Proposition 2.8, we get the following reduction formula.

PROPOSITION 3.5 (Reduction formula).

$$\Phi^{2:1;3}_{2:0;2}\left[\begin{matrix} a, c: & q^n b; & q^{-n}, & \alpha, & \beta; & q: q^{-n}d/ac, q \\ b, d: & -; & c, & qa\alpha\beta/d; & 0, & 0, \quad 0 \end{matrix}\right] = a^n \frac{(b/a;q)_n}{(b;q)_n}\frac{[d/a, d/c;q]_\infty}{[d, d/ac;q]_\infty}\, {}_4\phi_3\left[\begin{matrix} q^{-n}, a, qa\alpha/d, qa\beta/d \\ q^{1-n}a/b, qac/d, qa\alpha\beta/d \end{matrix}\,\middle|\, q; \frac{qc}{b}\right].$$

Its special case $\alpha = d/aq$ reduces further to the following identity.

COROLLARY 3.6 (Summation formula).

$$\Phi^{2:1;2}_{2:0;1}\left[\begin{matrix} a, c: & q^n b; & q^{-n}, & d/aq; & q: q^{-n}d/ac, q \\ b, d: & -; & & c; & 0, \quad 0, \quad 0 \end{matrix}\right] = a^n \frac{(b/a;q)_n}{(b;q)_n}\frac{[d/a, d/c;q]_\infty}{[d, d/ac;q]_\infty}.$$

It should be pointed out that when $a = q^{-m}$ in Proposition 3.5 and Corollary 3.6, they reduce to, respectively, the same summation formulae as the case $e = q^n b$ of Proposition 2.6 and Corollary 2.9.

3.3 Semi-terminating reduction formulae for $\Phi^{2:1;3}_{2:0;2}$ and $\Phi^{2:1;2}_{2:0;1}$. Specialising in Theorem 3.1 with

$$e = q^n b \quad \text{and} \quad \Omega(j) = \begin{cases} \dfrac{[q^{-n}d/b, \beta;q]_j}{(\gamma;q)_j}\left(\dfrac{q^n b\gamma}{a\beta}\right)^j, \\ \dfrac{[c, \beta;q]_j}{(\gamma;q)_j}\left(\dfrac{d\gamma}{ac\beta}\right)^j \end{cases}$$

and then transforming the corresponding double sum (3.1b) through Proposition 2.11, we derive two further reduction formulae, respectively.

PROPOSITION 3.7 (Reduction formula).

$$\Phi^{2:1;3}_{2:0;2}\left[\begin{matrix} a, c: & q^n b; & q^{-n}, & q^{-n}d/b, & \beta; & q: q^{-n}d/ac, q^n b\gamma/a\beta \\ b, d: & -; & & c, & \gamma; & 0, \quad 0, \quad 0 \end{matrix}\right] = a^n \frac{[b/a, qc/d;q]_n}{[b, qac/d;q]_n}\frac{[d/a, d/c;q]_\infty}{[d, d/ac;q]_\infty}\, {}_4\phi_3\left[\begin{matrix} q^{-n}, a, q^{-n}d/b, \gamma/\beta \\ q^{1-n}a/b, q^{-n}d/c, \gamma \end{matrix}\,\middle|\, q;q\right].$$

PROPOSITION 3.8 (Reduction formula).

$$\Phi^{2:1;2}_{2:0;1}\left[\begin{matrix} a, c: & q^n b; & q^{-n}, & \beta; & q: q^{-n}d/ac, d\gamma/ac\beta \\ b, d: & -; & & \gamma; & 0, \quad 0, \quad 0 \end{matrix}\right] = \frac{[d/a, d/c;q]_\infty}{[d, d/ac;q]_\infty}\, {}_4\phi_3\left[\begin{matrix} q^{-n}, & a, & c, & \gamma/\beta \\ & b, & qac/d, & \gamma \end{matrix}\,\middle|\, q;q\right].$$

Similarly, setting $a = q^{-m}$ in Propositions 3.7 and 3.8, they reduce to, respectively, the same summation formulae as the case $e = q^n b$ in Propositions 2.10 and 2.11.

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