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# STRUCTURE OF RINGS WITH INVOLUTION APPLIED TO GENERALIZED POLYNOMIAL IDENTITIES

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**Introduction.** In [**14**, § 4], some theorems were obtained about generalized polynomial identities in rings with involution, but the statements had to be weakened somewhat because a structure theory of rings with involution had not yet been developed sufficiently to permit proofs to utilize enough properties of rings with involution. In this paper, such a theory is developed. The key concept is that of the central closure of a ring with involution, given in § 1, shown to have properties analogous to the central closure of a ring without involution. In § 2, the theory of primitive rings with involution, first set forth by Baxter-Martindale [**5**], is pushed forward, to enable a setting of generalized identities in rings with involution which can parallel the non-involutory situation.

**1. Prime and semiprime rings with involution.** All rings are associative with 1. Let $(R, *)$ denote a *ring with involution*, i.e., the ring $R$ has an anti-automorphism $(*)$ of degree $\leqq 2$. Clearly $(*)$ induces an automorphism of degree $\leqq 2$ on cent $R$. If this automorphism is the identity then $(*)$ is of the *first kind* on $(R, *)$; otherwise $(*)$ is of the *second kind* on $(R, *)$. $\text{Cent}(R, *) = \{c \in \text{cent } R | c^* = c\}$. An *ideal* $(B, *)$ of $(R, *)$ is an ideal $B$ of $R$ stable under $(*)$. $(R, *)$ is *prime* if the product of any two nonzero ideals (of $(R, *)$) is nonzero; $(R, *)$ is *semiprime* if $(B, *)^2 \neq 0$ for each nonzero ideal $(B, *)$ of $(R, *)$. (Much of this terminology is due to Jacobson.) Clearly, if $R$ is semiprime then $(R, *)$ is semiprime; the converse, due to Martindale [**9**] (who explored these objects under the terminology $(*)$-prime and $(*)$-semiprime) can be seen easily (cf. [**14**, § 4]).

Given a subset $A$ of $R$, let

$$\text{Ann } A \equiv \{r \in R | ar = 0, \text{ all } a \text{ in } A\},$$

and let

$$\text{Ann}' A \equiv \{r \in R | ra = 0, \text{ all } a \text{ in } A\}.$$

Suppose $(R, *)$ is semiprime. If $A$ is an ideal of $R$ then $\text{Ann}' A = \text{Ann } A$, as is well known. Moreover, if $(A, *)$ is an ideal of $(R, *)$ then $(\text{Ann } A)^* \subseteq \text{Ann } A$ (indeed, $(A(\text{Ann } A)^*)^* = (\text{Ann } A)A^* \subseteq (\text{Ann } A)A = 0$, so $A(\text{Ann } A)^* = 0$); hence $(\text{Ann } A, *)$ is an ideal of $(R, *)$.

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LEMMA 1. *The following conditions are equivalent.*

(1) $(R, *)$ *is prime;*

(2) *For each nonzero ideal* $(B, *)$ of $(R, *)$, $\operatorname{Ann} B = 0$;

(3) *If* $r_1, r_2 \in R$, $r_1 \neq 0$, *and if there exists an ideal* $(B, *) \neq 0$ *of* $(R, *)$ *such that* $r_1Br_2 = r_1{}^*Br_2 = 0$, *then* $r_2 = 0$*;*

(4) *If* $r_1, r_2 \in R$, $r_1 \neq 0$, *and if* $r_1Rr_2 = r_1{}^*Rr_2 = 0$, *then* $r_2 = 0$.

*Proof.* (1) $\Leftrightarrow$ (2) is trivial.

(1), (2) $\Rightarrow$ (3): $r_2 \in \operatorname{Ann}(Br_1B + Br_1{}^*B)$, so we are done unless $Br_1B = Br_1{}^*B = 0$. $R$ is semiprime by (1), so $Rr_1R + Rr_1{}^*R \subseteq \operatorname{Ann} B$; hence $r_1 = 0$.

(3) $\Rightarrow$ (4): This is immediate.

(4) $\Rightarrow$ (1): If $(A, *)$ and $(B, *)$ are ideals of $(R, *)$ with $A \neq 0$ and $AB = 0$, then for any $b$ in $B$, any nonzero $a$ in $A$, we have $aRb = a^*Rb = 0$, so $b = 0$ by (4).

LEMMA 2. *The following conditions are equivalent:*

(1) $(R, *)$ *is semiprime;*

(2) *If* $r \in R$ *and there exists an ideal* $(J, *)$ *such that* $\operatorname{Ann} J = 0$ *and* $rJr = r^*Jr = 0$, *then* $r = 0$*;*

(3) *If* $rRr = r^*Rr = 0$ *then* $r = 0$.

*Proof.* This parallels the proof of Lemma 1.

Now assume for the remainder of this paper that $(R, *)$ is semiprime. An ideal $(J, *)$ of $(R, *)$ is *essential* if $J \cap B \neq 0$ for each nonzero ideal $(B, *)$ of $(R, *)$. Clearly $(J, *)$ is essential $\Leftrightarrow \operatorname{Ann} J = 0 \Leftrightarrow J$ is essential in $R$, and we can apply Amitsur's construction in [**4**] to obtain a ring of quotients for $R$: Let

$$\mathscr{J} = \{\text{essential ideals of } (R, *)\}$$

and consider

$$\mathscr{T} = \{(f, J) \mid J \in \mathscr{J} \text{ and } f : J \to R \text{ is a right module homomorphism (disregarding the involution)}\}.$$

If $J' \in \mathscr{J}$ and $J' \subseteq J$, we let $(f, J')$ denote the restriction from $f$ to $J'$. There is an equivalence $\sim$ defined by: $(f_1, J_1) \sim (f_2, J_2)$ if, for some $J' \subseteq J_1 \cap J_2$, $(f_1, J') = (f_2, J')$. Let $Q_0(R) = \mathscr{T}/\sim$, and let $[f, J]$ denote the equivalence class of $(f, J)$; then $Q_0(R)$ has a canonical ring structure given by $[f_1, J_1] + [f_2, J_2] = [f_1 + f_2, J_1 \cap J_2]$ and $[f_1, J_1][f_2, J_2] = [f_1 \circ f_2, J_2J_1]$. Moreover, there is a canonical injection $R \hookrightarrow Q_0(R)$ given by $r \mapsto [f_r, R]$, where $f_r$ denotes left multiplication by $r$. Let $C = \operatorname{cent} Q_0(R)$. It has been shown in [**4**] that $[f, J] \in C$ if and only if $f : J \to R$ is a bimodule homomorphism. (Indeed, ($\Leftarrow$) is very easy; conversely, assume $[f, J] \in C$. For any $r$ in $R$, $(ff_r - f_rf, J_r) = 0$ for suitable $J_r$ in $\mathscr{J}$, so for all $x$ in $J$, $xJ_r \subseteq J_r$ and $(f(rx) - rf(x)) \in \operatorname{Ann} J_r = 0$.) Hence $(*)$ induces an automorphism on $C$ by $[f, J]^* =$

$[f^*, J]$ where $f^*(x) \equiv (f(x^*))^*$, for all $x$ in $J$. Let $\hat{R} \equiv RC \subseteq Q_0(R)$. $\hat{R}$ has a well-defined involution given by $(\sum r_i c_i)^* = \sum r_i^* c_i^*$, $r_i$ in $R$, $c_i$ in $C$. (Indeed, suppose $\sum r_i c_i = 0$. Let $c_i = [f_i, J_i]$; choosing suitably small $J$ in $\mathscr{J}$ we may assume $\sum r_i f_i(x) = 0$, for all $x$ in $J$. Then for all $x'$ in $J$,

$$0 = (\sum r_i f_i(x))^* x' = (\sum f_i(r_i x))^* x' = \sum f_i^*(x^* r_i^*) x' = x^*(\sum r_i^* f_i^*(x')).$$

Thus $\sum r_i^* f_i^*(x') \in \operatorname{Ann} J = 0$, all $x'$ in $J$, so $(\sum r_i^* f_i^*, J) = 0$.) $(\hat{R}, *) \equiv (RC, *)$ is called the *central* closure of $(R, *)$ and $\hat{C} \equiv \operatorname{cent}(RC, *)$ is called the *extended centroid* of $(R, *)$. (Note that when $R$ is prime, $RC$ is merely the central closure of $R$.)

THEOREM 1. *If $(R, *)$ is prime then its extended centroid $\hat{C}$ is a field and its central closure $(\hat{R}, *)(=(RC, *))$ is prime.*

*Proof.* If $[f, J] \neq 0$ in $\hat{C}$ we claim $f(J) \in \mathscr{J}$ and $f : J \to f(J)$ is an isomorphism of ideals of $(R, *)$. Indeed, for $f(x)$ in $f(J)$, $(f(x))^* = f^*(x^*) = f(x^*) \in f(J)$, so $(f(J), *)$ is an ideal of $(R, *)$. Also $(\ker f, *)$ is an ideal of $(R, *)$ and $(\ker f)f(J) = f((\ker f)J) = f(\ker f)J = 0$, implying $\ker f = 0$ since $(R, *)$ is prime. Hence the claim is proved, and $[f, J]^{-1} = [f^{-1}, f(J)] \in \hat{C}$, so $\hat{C}$ is a field.

To prove $(RC, *)$ is prime, we use criterion (4) of Lemma 1 and assume that

$$\sum r_i c_i \neq 0, \sum r_j' c_j' \in RC,$$

and

$$(\sum r_i c_i)\hat{R}(\sum r_j' c_j') = (\sum r_i^* c_i^*)\hat{R}(\sum r_j' c_j') = 0.$$

Then $(\sum r_i c_i)R(\sum r_j' c_j') = (\sum r_i^* c_i^*)R(\sum r_j' c_j') = 0$. Let

$$c_i = [f_i, J_i], c_j' = [f_j', J_j'],$$

for all $i, j$. Choosing $J$ in $\mathscr{J}$ suitably contained in the intersection of a suitable finite number of elements of $\mathscr{J}$, we may assume $c_i = [f_i, J]$, $c_j' = [f_j', J]$, and $0 = (\sum_i r_i f_i)(R(\sum_j r_j' f_j')x) = (\sum_i r_i^* f_i^*)(R(\sum_j r_j' f_j')x)$ for all $x$ in $J$. Since $J^2 \subseteq R$, we obtain

$$\left(\sum_i r_i f_i J\right) J \left(\sum_j r_j' f_j'\right) x = 0 \quad \text{and} \quad \left(\sum_i r_i^* f_i^* J\right) J \left(\sum_j r_j' f_j'\right) x = 0,$$

for all $x$ in $J$. Let $y = (\sum r_j' f_j')x$, and choose $x_1$ in $J$ such that $\sum_i r_i f_i(x_1) \neq 0$. Then $(\sum_i r_i f_i(x_1))Jy = 0$ and, for all $x'$ in $J$,

$$0 = x_1^*\left(\sum_i r_i^* f_i^*(x')\right) Jy = \left(\sum_i f_i^*(x_1^* r_i^* x')\right) Jy$$
$$= \left(\sum_i f_i^*(x_1^*) r_i^* x'\right) Jy = \sum_i (f_i(x_1)^* r_i^*) x' Jy = \sum_i (r_i f_i(x_1))^* x' Jy.$$

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Setting $a = \sum_i r_i f_i(x_1)$, we have $aJ^2y = 0$ and $a^*J^2y = 0$; hence $y = 0$ by Lemma 1(3), i.e., $(\sum r_j' f_j')x = 0$ for all $x$ in $J$, so $\sum r_j' c_j' = 0$. Therefore $(RC, *)$ is prime.

An analogous situation holds in general:

THEOREM 2. *If $(R, *)$ is semiprime then its central closure $(\hat{R}, *)$ is semiprime and its extended centroid $\hat{C}$ is von Neumann regular.*

*Proof.* The proof that $(\hat{R}, *)$ is semiprime follows the lines of the proof of Theorem 1, using Lemma 2(2). The fact that $\hat{C}$ is von Neumann regular is obtained analogously to Amitsur's proof in [**4**] that $C$ is von Neumann regular. A sketch: Let $[f, J] \in \hat{C}$. Then there exist ideals $(B, *)$, $(B', *)$ contained in $(J, *)$ such that

$$B \cap B' = 0, \quad B \oplus B' \in \mathscr{J}, \quad f(B') = 0, \quad \text{and} \quad f : (B, *) \to (f(B), *)$$

is an isomorphism. Choose $(B'', *)$ maximal with respect to $f(B) \cap B'' = 0$, let $J' = f(B) \oplus B'' \in \mathscr{J}$, and define $f' : J' \to R$ by $f'(f(b) + b'') = b$ for $b$ in $B$, $b''$ in $B''$. $[f', J'] \in \hat{C}$ and $[f, J][f', J'][f, J] = [f, J]$, so $\hat{C}$ is von Neumann regular.

*Remark.* If $J \in \mathscr{J}$ then $(\hat{R}J\hat{R}, *)$ is essential in $(\hat{R}, *)$. Indeed, if $\sum r_i c_i \in \operatorname{Ann} \hat{R}J\hat{R}$, $r_i$ in $R$, $c_i$ in $C$, we have $(\sum r_i c_i)x = 0$, all $x$ in $J$. Let $c_i = [f_i, J_i]$ and let $J' = J \cap (\bigcap_i J_i)$. For all $x$ in $J'$, $\sum r_i f_i(x) = 0$, so $(\sum r_i f_i, J') = 0$, implying $\sum r_i c_i = 0$.

Using this remark, we can see that any bimodule homomorphism $f : J \to R$ can be extended to $f : \hat{R}J\hat{R} \to \hat{R}$ by

$$f\left(\sum_i \hat{r}_i x \hat{r}_i'\right) = \sum_i \hat{r}_i f(x) \hat{r}_i',$$

which is well-defined, and in this way one shows that $\hat{C}$ is also the extended centroid of $(\hat{R}, *)$.

(The theory becomes very easy when $(R, *)$ is a $PI$-algebra with involution, in view of [**13**]; applying the reasoning of [**13**, § 3], one can decompose $(\hat{R}, *)$ into a finite direct sum of Azumaya algebras of finite rank (with involution). Moreover, if $(R, *)$ is prime with a proper polynomial identity then $(\hat{R}, *)$ is its simple algebra with involution of central quotients (cf. [**12**]).) We shall be more interested here when $(R, *)$ is *not* a $PI$-algebra with involution. The point of departure is

PROPOSITION 1. *Suppose $(R, *)$ is prime. If $a, b \in R$ and $axb = bxa$, $axb^* = bxa^*$, all $x$ in $R$, then either $a = 0$ or $b = ca$ (in $(\hat{R}, *)$) for some $c$ in $\hat{C}$.*

*Proof* (as in Martindale [**10**]). Assume $a \neq 0$ and define a map $f : (RaR + Ra^*R) \to R$ by

$$f\left(\sum_i x_i a y_i + \sum_j x_j a^* y_j\right) = \sum_i x_i b y_i + \sum_j x_j b^* y_j.$$

To check that $f$ is well-defined, suppose $\sum x_i a y_i + \sum x_j a^* y_j = 0$. Then for all $r$ in $R$,

$$0 = br(\sum x_i a y_i + \sum x_j a^* y_j) = \sum b r x_i a y_i + \sum b r x_j a^* y_j$$
$$= \sum a r x_i b y_i + \sum a r x_j b^* y_j = ar(\sum x_i b y_i + \sum x_j b^* y_j);$$

likewise, for all $r$ in $R$,

$$0 = b^* r(\sum x_i a y_i + \sum x_j a^* y_j) = a^* r(\sum x_i b y_i + \sum x_j b^* y_j)$$

(noting that for all $x$ in $R$, $b^* x a^* = (a x^* b)^* = (b x^* a)^* = a^* x b^*$). Hence, by Lemma 1(3), $\sum x_i b y_i + \sum x_j b^* y_j = 0$, so $f$ is well-defined. But $f$ is a bimodule homomorphism and

$$f\left(\left(\sum_i x_i a y_i + \sum_j x_j a^* y_j\right)^*\right) = f\left(\sum_j y_j^* a x_j^* + \sum_i y_i^* a^* x_i^*\right)$$
$$= \sum y_j^* b x_j^* + \sum y_i^* b^* x_i^* = \left(\sum x_i b y_i + \sum x_j b^* y_j\right)^*,$$

so $[f, RaR + Ra^*R] \in \hat{C}$. Clearly $f(a) = b$, so we are done.

THEOREM 3. *If* $(R, *)$ *and* $(R', *)$ *are prime and* $(\hat{R}, *) \subseteq (R', *)$ *(as rings with involution), then for any ring* $H$ *such that* $\hat{C} \subseteq H \subseteq \mathrm{cent}(R', *)$,

$$(RH, *) \approx \left(\hat{R} \otimes_{\hat{C}} H, *\right)$$

*with the involution given by* $(\sum y_i \otimes h_i)^* = \sum y_i^* \otimes h_i$, $y_i$ *in* $\hat{R}$ *and* $h_i$ *in* $H$.

*Proof.* Viewing $(RH, *) \subseteq (R', *)$, we see by the definition of tensor product that there is a canonical homomorphism $\varphi : (\hat{R} \otimes_{\hat{C}} H, *) \to (RH, *)$ given by $\varphi(\sum y_i \otimes h_i) = \sum y_i h_i$, $y_i$ in $\hat{R}$, $h_i$ in $H$, and $(\ker \varphi, *)$ is an ideal of $(\hat{R} \otimes_{\hat{C}} H, *)$. We claim $\varphi$ is an isomorphism. Otherwise, there is a nonzero element $\sum_{i=1}^{u} y_i \otimes h_i$ in $\ker \varphi$ with $u$ minimal; note that $\{h_i | 1 \leqq i \leqq u\}$ are then $\hat{C}$-independent. For each $x$ in $\hat{R}$,

$$\sum_{i=1}^{u-1} (y_u x y_i - y_i x y_u) \otimes h_i =$$
$$(y_u x \otimes 1)\left(\sum_{i=1}^{u} y_i \otimes h_i\right) - \left(\sum_{i=1}^{u} y_i \otimes h_i\right)(x y_u \otimes 1) \in \ker \varphi;$$

by induction on $u$ we conclude $y_u x y_i - y_i x y_u = 0$, for each $i$, each $x$ in $R$. Likewise,

$$\sum_{i=1}^{u-1} (y_u x y_i^* - y_i x y_u^*) \otimes h_i =$$
$$(y_u x \otimes 1)(\sum y_i \otimes h_i)^* - (\sum y_i \otimes h_i)(x y_u^* \otimes 1) \in \ker \varphi,$$

so $y_u x y_i^* - y_i x y_u^* = 0$, for all $i$, all $x$ in $\hat{R}$. If $y_u = 0$ then we are done by induction on $u$; otherwise, by Proposition 1, there exist $c_i$ in $\hat{C}$ such that

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$y_i = c_i y_u$, $1 \leqq i \leqq u - 1$. Then

$$\sum_{i=1}^{u} y_i \otimes h_i = \sum_{i=1}^{u} y_u \otimes c_i h_i = y_u \otimes \sum_{i=1}^{u} c_i h_i,$$

and $y_u \sum c_i h_i = 0$ implies $\sum c_i h_i = 0$ since $(R', *)$ is prime; hence $y_u \otimes \sum_{i=1} c_i h_i = 0$ after all, so $\ker \varphi = 0$.

COROLLARY 1. *Any collection of $\hat{C}$-independent elements of $(R, *)$ are* $\operatorname{cent}(R', *)$*-independent in $(R', *)$, notation as in Theorem* 3.

COROLLARY 2. *If $\sum_{i=1}^{u} y_i h_i = 0$, $y_i$ in $\hat{R}$, $\{h_i | 1 \leqq i \leqq u\}$ $\hat{C}$-independent in* $\operatorname{cent}(R', *)$, *then $y_i = 0$, $1 \leqq i \leqq u$.*

**2. Primitive rings with involution.** Let an irreducible (left) module $M$ of $R$ be *faithful* for $(R, *)$ if $rM \neq 0$ or $r^*M \neq 0$ for each nonzero $r$ in $R$. Following Baxter-Martindale [**5**], we call $(R, *)$ *primitive* if $(R, *)$ has a faithful irreducible module.

LEMMA 3. *$(R, *)$ is primitive if and only if $R$ has a maximal left ideal which contains no nonzero ideals of $(R, *)$.*

*Proof.* Jacobson [**7**, p. 6] has shown a left $R$-module $M$ is irreducible if and only if there is a maximal left ideal $J$ of $R$ such that $M \approx R/J$ (as left $R$-modules). Let $B = \{r \in R | rM = 0 \text{ and } r^*M = 0\}$. Clearly $(B, *)$ is an ideal of $(R, *)$ and $B \subseteq J$; $(B, *) = 0$ if and only if $M$ is faithful for $(R, *)$.

Now suppose a primitive ring $R'$ has a minimal left ideal. In this case all minimal left ideals are isomorphic (as left $R'$-modules), and each faithful irreducible (left) module is isomorphic to any given minimal left ideal (by [**7**, Proposition 2, p. 45]). The sum of the minimal left ideals is the *socle*, which is also the sum of all minimal right ideals of $R'$ and is contained in each nonzero ideal of $R'$ by [**7**, Theorem 1, p. 65]. In view of this fact, we define $\operatorname{soc}(R, *)$ to be the intersection of the nonzero ideals of $(R, *)$.

Given a ring $E$, the *opposite ring* $E^o$ is defined as follows: The elements of $E^o$ are $\{x^o | x \in E\}$ with addition given by $x_1^o + x_2^o = (x_1 + x_2)^o$ and multiplication given by $(x_1^o x_2^o) = (x_2 x_1)^o$. Thus, $x \to x^o$ is the canonical anti-isomorphism from $E$ to $E^o$. If $E$ has an involution $(*)$, then $(*)$ can be thought of as an isomorphism from $E$ to $E^o$ given by $x \to (x^*)^o$. On the other hand, the map $x \to x^o$ induces a canonical involution on $E \oplus E^o$ given by $(x_1, x_2^o) \to (x_2, x_1^o)$, called the *exchange involution.*

Consider $D = \operatorname{End}_R M$. By Schur's lemma, $D$ is a division ring; we shall view $M$ as $R - D$ bimodule. Also $M$ is a left $D^o$-module with action $d^o y \equiv yd$, all $d$ in $D$, $y$ in $M$, called the *opposite action*. Note that $(\operatorname{End} M_D)^o \approx \operatorname{End}_{D^o} M$. The structure of primitive rings with involution has the following neat characterization:

THEOREM 4. *Let* $(R, *)$ *be primitive with faithful irreducible module* $M$, *and let* $D = \operatorname{End}_R M$.

(1) *If* $R$ *is primitive then* $\operatorname{soc}(R, *) = \operatorname{soc} R$.

(2) *Suppose* $R$ *is not primitive. Let* $U = \{r \in R | rM = 0\}$. $R$ *is a subdirect product of* $R_1$ *and* $R_2$, *where* $R_1 = R/U$ *has faithful irreducible left module* $M$ (with action $(r + U)y = ry$, $r$ in $R$ and $y$ in $M$) *and* $R_2 = R/U^*$ *has faithful irreducible right module* $M$ (with action $y(r + U^*) = r^*y$, $r$ in $R$ and $y$ in $M$). *Let* $D_1 = \operatorname{End}_{R_1} M$, $D_2 = \operatorname{End} M_{R_2}$, *and let* $E = \operatorname{End} M_D$. $D_1 \approx D$, $D_2 \approx D^o$, $\operatorname{End} M_{D_1} \approx E$, $\operatorname{End}_{D_2} M \approx E^o$, *and, under these identifications,* $R_1$ *is a dense subring of* $E$ *and* $R_2$ *is a dense subring of* $E^o$. *Let* $(o)$ *be the exchange involution on* $E \oplus E^o$, *and define* $\varphi : (R, *) \rightarrow (E \oplus E^o, o)$ *by* $\varphi(r) = (r + U, r + U^*)$. $\varphi$ *is an injection in the category of rings with involution, and* $\varphi(R)$ *is a dense subring of* $E \oplus E^o$ (i.e. for any $t$, given $y_1, \ldots, y_t$ $D$-linearly independent in $M$, and given arbitrary $y_1', \ldots, y_t', y_1'', \ldots, y_t''$ in $M$, there exists $r$ in $R$ such that $ry_j = y_j'$, $r^*y_j = y_j''$, $1 \leqq j \leqq t$). *Finally,* $\varphi(\operatorname{soc}(R, *)) = \operatorname{soc} R_1 \oplus \operatorname{soc} R_2$ *and* $\operatorname{soc} R_1 = (\operatorname{soc} R_2)^o$.

*Proof.* (1) $rR$ is a minimal right ideal if and only if $Rr^* = (rR)^*$ is a minimal left ideal, so clearly $(\operatorname{soc} R)^* = \operatorname{soc} R$, implying $(\operatorname{soc} R, *) = \operatorname{soc}(R, *)$.

(2) Baxter-Martindale [**5**] have shown in a straightforward argument that, with the given actions, $M$ is a faithful, irreducible left $R_1$-module and right $R_2$-module. For all $r$ in $R$, $y$ in $M$, $d$ in $D$, $((r + U)y)d = (ry)d = r(yd) = (r + U)(yd)$, so we can view $d$ in $D_1$; conversely, for all $d$ in $D_1$, $(ry)d = ((r + U)y)d = (r + U)(yd) = r(yd)$, so $D \approx D_1$. Likewise, for $d$ in $D^o$, $d(y(r + U^*)) = d(r^*y) = r^*(dy) = (dy)(r + U^*)$, so we can view $d$ in $D_2$, and, as before, we get $D^o \approx D_2$. Now $\operatorname{End} M_{D_1} \approx \operatorname{End} M_D \approx (\operatorname{End}_{D^o} M)^o \approx (\operatorname{End}_{D_2} M)^o$, and, by the density theorem, $R_1$ is dense in $E$ and $R_2$ is dense in $E^o$. Now $\varphi : R \rightarrow R_1 \oplus R_2 \subseteq E \oplus E^o$ is clearly an injection of rings. Moreover, $\varphi(r^*) = (r^* + U, r^* + U^*)$, and one can check as before that $(r + U)^o = r^* + U^*$ and $(r + U^*)^o = r^* + U$, implying $\varphi$ is an injection of rings with involution.

Now $\varphi(U + U^*) = ((U + U^*)/U) \oplus (U + U^*)/U^*$; since a nonzero ideal of a primitive ring is dense, this implies $\varphi(U + U^*)$, and hence $\varphi(R)$, is dense in $E \oplus E^o$. Moreover, $\operatorname{soc}(R, *) \subseteq U + U^*$, an ideal of $(R, *)$, implying $\operatorname{soc}(R, *) = \bigcup\{\text{ideals of } (R, *) \text{ contained in } (U + U^*)\}$. Likewise, $\operatorname{soc} R_1 = \bigcup\{\text{ideals of } R_1 \text{ contained in } (U + U^*)/U\}$ and $\operatorname{soc} R_2 = \bigcup\{\text{ideals of } R_2 \text{ contained in } (U + U^*)/U^*\}$. Hence, $\varphi(\operatorname{soc}(R, *)) = \operatorname{soc} R_1 \oplus \operatorname{soc} R_2$. Moreover, for each minimal ideal $(B + U)/U$ of $R_1$, $((B + U)/U)^* = (B^* + U^*)/U^*$ is a minimal ideal of $R_2$ and vice versa, so $\operatorname{soc} R_2 = (\operatorname{soc} R_1)^o$.

Note for $c$ in $\operatorname{cent}(R, *)$, that there is an element $\hat{c}$ in $\operatorname{End}_R M$ given by $\hat{c}(y) = cy$, $y$ in $M$ (notation as in Theorem 4), yielding an injection $\psi : \operatorname{cent}(R, *) \hookrightarrow \operatorname{cent} D$ given by $\psi(c) = \hat{c}$, all $c$ in $\operatorname{cent}(R, *)$. Also note that, in the notation of Theorem 4, $U = \{r \in R | rM = 0\}$ is a primitive ideal of $R$ such that $U \cap U^* = 0$, so any primitive ring with involution is quasi-primitive in the sense of [**14**].

**3. Generalized identities in rings with involution.** Let $W$ and $R$ be rings. In [**14**, § 1], $R$ is called a *W-algie* if $R$ is a $W - W$ bimodule such that the canonical map $\varphi : W \rightarrow R$ given by $w \rightarrow w \cdot 1$ is actually a ring homomorphism with $\varphi(\text{cent } W) \subseteq \text{cent } R$. It was shown in [**14**, § 1] that

$$W\{X\} \equiv W\{X_1, X_2, \ldots\},$$

the free product of $W$ with the free algebra $C\{X\}$ (where $C = \text{cent } W$), is a free object in the category of $W$-algies. An element $f(X_1, \ldots, X_m)$ in $W\{X\}$ which lies in the kernel of each algie homomorphism $\psi : W\{X\} \rightarrow R$ is a *GI of R*. $f$ is *multilinear* if each indeterminate occurring in $f$ occurs exactly once in each monomial of $f$; explicitly a multilinear $GI$ can be written in the form

$$f(X_1, \ldots, X_m) = \sum_{i,\pi} w_{i1}X_{\pi 1}w_{i2}X_{\pi 2} \ldots w_{im}X_{\pi m}w_{i,m+1},$$

$\pi$ ranging over permutations of $(1, \ldots, m)$. The generalized monomial $f_\pi$ of $f$ is the sum of those monomials of $f$ for which $\pi$ is a fixed permutation; clearly $f$ is the sum of its generalized monomials $f_\pi$. $f$ is *R-proper* if at least one of its generalized monomials is not a $GI$ of $R$. It is shown in [**14**, § 1–§ 3] that proper $GI$'s are the fundamental concepts in the theory of algies with $GI$.

Analogously, $(R, *)$ is a *$(W, *)$-algie with involution* if $R$ is a $W$-algie such that the canonical map $\varphi$ is a homomorphism of rings with involution such that $\varphi(\text{cent}(W, *)) \subseteq \text{cent}(R, *)$. In this case we consider the free product (of rings) of $W$ with the free algebra of rings with involution. $W\{X\}$ is seen to have an involution $(*)$ induced by $(*)$ on $W$ and $X_{2i-1}{}^* = X_{2i}, X_{2i}{}^* = X_{2i-1}$, $1 \leqq i < \infty$. Any $f$ in $(W\{X\}, *)$ lying in the kernel of each homomorphism $\psi : (W\{X\}, *) \rightarrow (R, *)$ is a *GI of* $(R, *)$.

Now write $Y_i = X_{2i-1}$, $Y_i^* = X_{2i}$, $1 \leqq i < \infty$, and call

$$f(Y_1, Y_1^*, \ldots, Y_m, Y_m^*)$$

*multilinear* if the sum of the degrees of $Y_i$ and $Y_i^*$ in each monomial of $f$ is 1, for each $i$. A multilinear element $f(Y_1, Y_1^*, \ldots, Y_m, Y_m^*)$ in $(W\{X\}, *)$ is *proper* if $f(X_1, \ldots, X_{2m})$ is proper. Clearly any proper $GI$ of $R$ is a (proper) $GI$ of $(R, *)$ because each homomorphism $\varphi : (W\{X\}, *) \rightarrow (R, *)$ induces a homomorphism $\varphi : W\{X\} \rightarrow R$. The major result of [**14**, § 4] is a partial converse, namely [**14**, Theorem 10]: if $(W, *)$ is prime with a proper $GI$ then $W$ has a proper $GI$. The proof there sacrifices categorial consistency for speed. Since the necessary concepts have been developed here, we give somewhat stronger results (which parallel what is known for $GI$ in rings without involution). The following results from [**1**] are quoted:

If $P$ is a primitive ring with faithful irreducible module $M$, let $D = \text{End}_P M$. Viewing $P \subseteq \text{End } M_D \subseteq \text{End } M_{\mathbf{Z}} \approx \text{End}_{\mathbf{Z}} M$ and $D \subseteq \text{End}_{\mathbf{Z}} M$, let $F$ be a maximal subfield of $D$ and let $P_F = PF$ in $\text{End}_{\mathbf{Z}} M$; if $W \subseteq P$ let $W_F = WF$ in $\text{End}_{\mathbf{Z}} M$. Note that $P_F$ is primitive with faithful irreducible module $M$ (which has centralizer $F$), and by Jacobson's structure theorem [**7**, p. 75],

$W_F \cap \mathrm{soc}(P_F) = \{$set of finite-ranked transformations of $\mathrm{End}_F M$ in $W_F\}$. Since any $D$-independent set is obviously $F$-independent, we observe that $\mathrm{soc}(P_F) \cap P \subseteq \mathrm{soc}\, P$, so $W \cap \mathrm{soc}(P_F) \subseteq W \cap \mathrm{soc}\, P$. So suppose $W \subseteq P$ and $P$ is a $W$-algie satisfying each $GI$ of $W$. From [**14**] we shall need

[**14**, Theorem 2]: If $W$ satisfies a proper $GI$ then $W_F \cap \mathrm{soc}(P_F) \neq 0$. If $W_F \cap \mathrm{soc}(P_F) \neq 0$ and if each $GI$ of $W_F$ is a $GI$ of $P_F$, then $W \cap \mathrm{soc}\, P_F \neq 0$ and the dimension of $D$ over its center is finite.

[**14**, Proposition 6]: If $(R, *)$ is primitive but $R$ is not primitive and if $f(X_1, X_1^*, \ldots, X_m, X_m^*)$ is a $GI$ of $(R, *)$ then $f(X_1, X_2, \ldots, X_{2m-1}, X_{2m})$ is a $GI$ of $R$.

[**14**, Proposition 4]: If $P$ is primitive, if $(P, *)$ is a primitive $(W, *)$-algie with involution satisfying each $GI$ of $(W, *)$, and if $(W, *)$ satisfies a proper $GI$, then $W_F \cap \mathrm{soc}(P_F) \neq 0$.

To proceed further we need a structure theorem inspired by Martindale [**11**]. (A *semiprimitive* ring with involution is a subdirect product of primitive rings with involution.)

THEOREM 5. *Suppose $(W, *)$ is prime and view $(W, *)$ as a $(W, *)$-algie with involution. $(W, *)$ can be embedded in a primitive $(W, *)$-algie with involution satisfying each multilinear GI of $(W, *)$.*

*Proof.* We give a sequence of $(W, *)$-algies with involution $(A_1, *)$, $(A_2, *)$, and $(A_3, *)$, each satisfying every multilinear $GI$ of $(W, *)$, and such that the canonical map $(W, *) \to (A_i, *)$, given by $w \to w \cdot 1$, is an embedding, $1 \leqq i \leqq 3$. It will turn out that $(A_1, *)$ has no nonzero nil ideals, $(A_2, *)$ is semiprimitive, and $(A_3, *)$ is primitive.

Let $(T, *)$ be the complete direct product of a countably infinite number of copies of $(A_1, *)$, and let $N$ = sum of the nil ideals of $T$. Clearly $N^* \subseteq N$; let $(W, *) = (T/N, *)$. By [**11**, Theorem 2.5], $A_1$ has no nil ideals. Let $A_2 = A_1[\lambda]$, $\lambda$ an indeterminate commuting with $A_1$, and define an involution $(*)$ on $A_2$ by $(\sum a_i\lambda^i)^* = \sum a_i^*\lambda^i$, $a_i$ in $A_1$ (well-known to be well-defined). $A_2$ is semiprimitive by [**7**, p. 10]; hence $(A_2, *)$ is semiprimitive by Baxter-Martindale [**5**]. Note that $(W, *) \subseteq (A_1, *) \subseteq (A_2, *)$ are $(W, *)$-algies with involution, satisfying each multilinear $GI$ of $(W, *)$.

The next step uses ultraproducts in a manner introduced by Amitsur [**2**] (cf. also Herstein [**6**, pp. 97–99]). Let $\{(P_\lambda, *) | \lambda \in \Lambda\}$ be the set of primitive ideals of $(A_2, *)$, and write $(T_\lambda, *) = (A_2/(P_\lambda \cap P_\lambda^*), *)$, a primitive algie with involution for each $\lambda$ in $\Lambda$. A *filter* on $\Lambda$ is a collection $\mathscr{F}$ of subsets of $\Lambda$ such that (i) $\phi \notin \mathscr{F}$, (ii) if $\Lambda_\alpha \in \mathscr{F}$ and $A_\beta \supseteq A_\alpha$ then $\Lambda_\beta \in \mathscr{F}$; (iii) if $\Lambda_\alpha \in \mathscr{F}$ and $\Lambda_\beta \in \mathscr{F}$ then $\Lambda_\alpha \cap \Lambda_\beta \in \mathscr{F}$. It is well-known in logic (cf. Herstein [**6**, p. 98]) that any filter $\mathscr{F}$ can be embedded in an *ultrafilter* $\mathscr{F}'$ which has the additional property that for each $\Lambda_\alpha \subseteq \Lambda$, either $\Lambda_\alpha \in \mathscr{F}'$ or $(\Lambda - \Lambda_\alpha) \in \mathscr{F}'$. Given an ultrafilter $\mathscr{F}'$ one defines the *ultraproduct* of the $(T_\lambda, *)$ as follows: Let $(T', *) = \prod_{\lambda \in \Lambda} (T, *)$. Writing a typical element of $T'$ as $(x(\lambda))$, we define an equivalence $\sim$ on $T'$ by $(x_1(\lambda)) \sim (x_2(\lambda))$ if $\{\lambda \in \Lambda | x_1(\lambda) = x_2(\lambda)\} \in \mathscr{F}$.

Likewise, letting $M_\lambda$ be a faithful irreducible module of $T_\lambda$, we set $M' = \prod_{\lambda \in \Lambda} M_\lambda$ and define $\sim$ on $M'$ in the same way. Let $[(x(\lambda)]$ denote the equivalence class of $(x(\lambda))$ under $\sim$ in $T'/\sim$; let $[(y(\lambda))]$ denote the equivalence class of $(y(\lambda))$ in $M'/\sim$. We claim that $T'/\sim$ becomes a $(W, *)$-algie with involution, satisfying each $GI$ of $(W, *)$, when endowed with operations

$$w[(x(\lambda))] = [(wx(\lambda))],\ w \text{ in } W;$$
$$[(x_1(\lambda))] + [(x_2(\lambda))] = [((x_1 + x_2)(\lambda))];$$
$$[(x_1(\lambda))][(x_2(\lambda))] = [(x_1x_2(\lambda))];$$

$[(x(\lambda))]^* = [(x(\lambda)^*)]$, and $M'/\sim$ becomes a faithful irreducible $(T'/\sim, *)$-module when given the operation $[(x(\lambda))][(y(\lambda))] = [(xy(\lambda))]$, $[(x(\lambda))]$ in $(T'/\sim, *)$, $[(y(\lambda))]$ in $M'/\sim$. Indeed, this is true because all relevant sentences are elementary. For example, "$M_\lambda$ is a faithful irreducible $(T_\lambda, *)$-module" can be written as

$$((\forall y \in M_\lambda)(\forall y' \in M_\lambda)(\exists x \in (T_\lambda, *) : xy = y')) \wedge (\forall x \in (T_\lambda, *)$$
$$(x = 0 \vee (\exists y \in M_\lambda : xy \neq 0 \vee x^*y \neq 0))).$$

Furthermore, $\mathscr{F}$ can be chosen in the following manner (as in Amitsur [**2**, Theorem 3]): Embed $(W, *)$ in $(T', *)$ in the natural way $(w \to (\hat{w}(\lambda))$ where $\hat{w}(\lambda) = w$, all $\lambda$ in $\Lambda$), and let $\Lambda_w = \{\lambda \in \Lambda | w \notin P_\lambda \cap P_\lambda^*\}$ for each nonzero $w$ in $W$. Observe that $\Lambda_w = \Lambda_{w^*}$. Since $(W, *)$ is prime, given nonzero $w_1$, $w_2$ in $W$, there exists (by Lemma 1) $w$ in $W$ such that $w_1ww_2 \neq 0$ or $w_1^*ww_2 \neq 0$. Hence, $\mathscr{F} = \{$all subsets of $\Lambda$ containing finite intersections of the $\Lambda_w$, $w \neq 0$ in $W\}$ is a filter. Embedding $\mathscr{F}$ in an ultrafilter $\mathscr{F}'$, one has $(W, *) \subseteq (T'/\sim, *)$ in the given construction, so we let $(A_3, *) = (T'/\sim, *)$, which is primitive with faithful irreducible module $M'$.

We are finally ready to improve [**14**, Theorems 9 and 10].

THEOREM 6. *Let $(R, *)$ be a prime $(R, *)$-algie with involution satisfying a proper GI.*

(i) *The central closure $(\hat{R}, *)$ of $(R, *)$ is primitive.*

(ii) *Let $M$ be a faithful irreducible module of $(\hat{R}, *)$ and let $D = \mathrm{End}_{\hat{R}}M$. Then $D$ is finite dimensional over its center and $R \cap \mathrm{soc}(\hat{R}, *) \neq 0$.*

*Proof.* (i) Let us embed $(\hat{R}, *)$ in a primitive $(\hat{R}, *)$-algie with involution $(P, *)$ satisfying each multilinear $GI$ of $(\hat{R}, *)$, as in Theorem 5, and let $f(X_1, X_1^*, \ldots, X_m, X_m^*)$ be a proper $GI$ of $(R, *)$. Clearly $f$ is also a proper $GI$ of $(P, *)$.

*Case* I. $P$ is not primitive. Then, by Theorem 4, $(P, *)$ can be embedded as a dense subring of $(E \oplus E^o, o)$ where $E$ is a ring of endomorphisms of a vector space $M'$ over a division ring $D$; by the density, each $GI$ of $(P, *)$ can be seen to be a $GI$ of $(E \oplus E^o, o)$. Now let $F$ be a maximal subfield of $D$ and let $E_F$ be the $F$-subalgebra of $\mathrm{End}_{\mathbf{Z}}M'$ generated by $E$. Since $(E^o)_{F^o} \approx (E_F)^o$ and since $F \approx F^o$, we may replace $F$ by $\{(\alpha, \alpha)$ in $E_F \oplus (E_F)^o | \alpha \in F\}$, which

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we shall instead call $F$. Thus, $F = \text{cent}(E_F \oplus (E_F)^o, o)$ and, by Theorem 3, $(\hat{R}_F, o) \approx (\hat{R} \otimes_C F, *)$, where $\hat{C}$ is the extended centroid of $(R, *)$ (and thus of $(\hat{R}, *)$ also). Let $\pi_1$, $\pi_2$ denote the respective projections of $E_F \oplus (E_F)^o$ onto $E_F$, $(E_F)^o$.

Since $f(X_1, X_1^*, \ldots, X_m, X_m^*)$ is a $GI$ of $(E_F \oplus (E_F)^o, o)$, a trivial application of [**14**, Proposition 6] shows $f(X_1, X_2, \ldots, X_{2m-1}, X_{2m})$ is a proper $GI$ of $E_F \oplus (E_F)^o$. Clearly $f(X_1, \ldots, X_{2m})$ is proper either for $E_F$ or for $(E_F)^o$; without loss of generality we assume $f(X_1, \ldots, X_{2m})$ is proper for $E_F$. By [**14**, Theorem 2], there exists nonzero $\pi_1(w)$ in $\pi_1(\hat{R}_F) \cap \text{soc}\, E_F$. We claim that $\hat{R}_F \cap \text{soc}(E_F \oplus (E_F)^o, o) \neq 0$. Indeed, let $w = \sum_{i=1}^u r_i\alpha_i$, $r_i$ in $\hat{R}$, $\alpha_i$ in $F$. For each $r$ in $\hat{R}$, $wrw^* \in \hat{R}_F \cap \text{soc}(E_F \oplus (E_F)^o, o)$ so we are done unless $w\hat{R}w^* = 0$, i.e. $(\sum_i r_i\alpha_i)\hat{R}(\sum_j r_j^*\alpha_j) = 0$. Let $\{\alpha_t'\}$ be a $\hat{C}$-base for $F$, and let $\alpha_i\alpha_j = \sum_t \beta_{ijt}\alpha_t'$, $\beta_{ijt}$ in $\hat{C}$. Then, for each $r$ in $R$,

$$\sum_{i,j,t} (r_i r r_j^* \beta_{ijt})\alpha_t' = 0, \quad \text{implying} \sum_{i,j} r_i r r_j^* \beta_{ijt} = 0$$

since $\hat{R}_F \approx \hat{R} \otimes_C F$. So $\sum_t r_i X_1 r_j^* \beta_{ijt}$ is a $GI$ for $(\hat{R}, *)$, thus for $(E_F \oplus (E_F)^o, o)$, implying $(\sum_i r_i\alpha_i)x(\sum_j r_j^*\alpha_j) = 0$, for each $x$ in $(E_F \oplus (E_F)^o, o)$. Hence $\pi_1(w)E_F\pi_1(w^*) = 0$, so $\pi_1(w^*) = 0$ since $E_F$ is prime. But this means that $\pi_2(w) = 0$, so $0 \neq w \in \hat{R}_F \cap \text{soc}(E_F \oplus (E_F)^o, o)$, as claimed. (Incidentally, a similar argument shows $(\hat{R}_F, *)$ is prime, but we do not need this fact.)

Now we choose $w = \sum_{i=1}^u r_i\alpha_i$, $r_i$ in $\hat{R}$, $\alpha_i$ in $F$, with $u$ minimal such that $0 \neq w \in \hat{R}_F \cap \text{soc}(E_F \oplus (E_F)^o, o)$. (In particular, $r_u \neq 0$.) We claim $u = 1$. Otherwise, for each $r$ in $\hat{R}$,

$$\sum_{i=1}^{u-1} (r_i r r_u - r_u r r_i)\alpha_i = \left(\sum_{i=1}^{u} r_i\alpha_i\right) r r_u - r_u r\left(\sum_{i=1}^{u} r_i\alpha_i\right) \in \hat{R}_F \cap \text{soc}(E_F \oplus (E_F)^o, o).$$

Hence, by induction (in view of Theorem 3, Corollary 2), $r_i r r_u = r_u r r_i$, all $i$. Moreover,

$$\sum_{i=1}^{u-1} (r_i^* r r_u - r_u^* r r_i)\alpha_i = \left(\sum_{i=1}^{u} r_i\alpha_i\right)^* r r_u - r_u^* r\left(\sum_{i=1}^{u} r_i\alpha_i\right) \in \hat{R}_F \cap \text{soc}(E_F \oplus (E_F)^o, o),$$

so $r_i^* r r_u = r_u^* r r_i$, all $i$. By Proposition 1, $r_i = c_i r_u$, for suitable $c_i$ in $\hat{C}$, so $w = r_u(\sum c_i\alpha_i)$. Hence $u = t$, as claimed, so $w = r_1\alpha_1$. But

$$\alpha_1^{-1}w \in R \cap \text{soc}(E_F \oplus (E_F)^o, o),$$

which is therefore nonzero. A proof analogous to [**11**, Theorem 2.10] shows $(\hat{R}, *)$ is primitive.

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*Case* II. $P$ is primitive. By [**14**, Proposition 4], $\hat{R}_F \cap \operatorname{soc} P_F \neq 0$; hence $\hat{R}$ is primitive by [**11**, Theorems 2.9 and 2.10], so certainly $(\hat{R}, *)$ is primitive.

(ii) By part (i), we may assume $P = \hat{R}$. If $\hat{R}$ is primitive then $\operatorname{soc} \hat{R}_F \neq 0$ by [**14**, Proposition 4], so by [**14**, Theorem 2], $R \cap \operatorname{soc} \hat{R}_F \neq 0$ and $D$ is finite dimensional over its center. It follows easily from Theorem 4 that $R \cap \operatorname{soc}(\hat{R}, *) \neq 0$.

Hence we are done unless $\hat{R}$ is *not* primitive, i.e. case I of part (i), with $P = \hat{R}$ and $M' = M$. Since $E$ satisfies a proper $GI$, $D$ is finite dimensional over its center, by [**14**, Theorem 2]. Moreover, obviously $R_F = \hat{R}_F$, and the identical argument as in part (i) case I, shows $R \cap \operatorname{soc}(E_F \oplus (E_F)^o, o) \neq 0$. Hence $0 \neq R \cap (\hat{R} \cap \operatorname{soc}(E_F \oplus (E_F)^o, o) \subseteq R \cap \operatorname{soc}(\hat{R}, *)$ (in light of Theorem 4).

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