# A CLASS OF MULTIPLIERS

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## 1. The problem and notation

Throughout this paper $G$ denotes an infinite compact connected Hausdorff Abelian group with character group $X$. Given a map $\alpha$ of $X$ into itself, we are concerned with the set of $a \in G$ such that the function $\varphi_a \in l^\infty(X)$ defined by

$$\varphi_a(\chi) = (a, \alpha(\chi)) \text{ for } \chi \in X$$

is a multiplier of type $(p, p)$, where it can be assumed without loss of generality that $1 \leqq p < \infty$.

In approaching this question we shall use the following notation. $L^p(G)$ denotes the usual Lebesgue space built over $G$, $T(G)$ the space of trigonometric polynomials on $G$, $A(G)$ the space of continuous functions $f$ on $G$ such that the Fourier transform $\hat{f} \in l^1(X)$, and $P(G)$ the space of pseudomeasures on $G$. For $f \in T(G)$ and $a \in G$,

$$U_a f = \sum_{\chi \in X} \varphi_a(\chi)\hat{f}(\chi)\chi, \tag{1.1}$$

itself a trigonometric polynomial on $G$, and

$$N_p(a) = \text{Sup}\,\{\|U_a f\|_p : f \in T(G), \|f\|_p \leqq 1\} \leqq \infty. \tag{1.2}$$

It is almost evident that $U_{a+b} = U_a U_b$, $U_0 = I$ (the identity operator), that $N_p$ is a lower semicontinuous function on $G$, and that for $a, b \in G$

$$N_p(a+b) \leqq N_p(a)N_p(b), \quad N_p(-a) = N_p(a), \tag{1.3}$$

provided one defines $\infty \cdot \infty = \infty$, $k \cdot \infty = \infty \cdot k = \infty$ if $k$ is real and positive, and $0 \cdot \infty = \infty \cdot 0 = 0$, and agrees that $k < \infty$ for any real $k$.

It is also simple to verify that the following three statements are equivalent:

(a) $\varphi_a \in (p, p)$ (the set of multiplier functions on $X$ of type $(p, p)$);

(b) there exists a number $B = B(p, a) < \infty$ such that $\|U_a f\|_p \leqq B\|f\|_p$ for $f \in T(G)$;

(c) $N_p(a) < \infty$;

in this connection it is to be noticed that $T(G)$ is dense in $L^p(G)$ whenever $1 \leqq p < \infty$.

Write

$$A_p = \{a \in G : N_p(a) < \infty\} = \bigcup \{A_{p,k} : k = 1, 2, \cdots\},$$

where

$$A_{p,k} = \{a \in G : N_p(a) \leqq k\}.$$

The lower semicontinuity of $N_p$ shows that $A_{p,k}$ is closed in $G$, and (1.3) that $A_{p,k}$ is symmetric. Therefore $A_p$ is an $F_\sigma$-subgroup of $G$, by (1,3) again.

## 2. The first theorem

THEOREM 1. *Suppose that either*

(i) $A_p$ *is nonnull (relative to Haar measure)*

*or*

(ii) $A_p$ *is nonmeagre.*

*Then*

(iii) $N_p$ *is bounded on* $G$;

(iv) $\hat{\mu} \circ \alpha \in (p, p)$ *for all* $\mu \in M(G)$.

PROOF. If (i) holds, $A_{p,k}$ is nonnull for some $k$. By Steinhaus' theorem, the difference set $A_{p,k} - A_{p,k}$ is a neigbourhood $V$ of 0 in $G$. By (1.3), $N_p(a) \leqq k^2$ for $a \in V$. Since $G$ is compact and connected, a second appeal to (1.3) yields (iii).

If (ii) holds, $A_{p,k}$ has interior points for some $k$, which entails that the difference set $A_{p,k} - A_{p,k}$ is a neighbourhood of 0 in $G$. Whence, as before, $N_p$ is bounded on $G$.

Given (iii), (iv) is derived in the following fashion. Suppose first that $p \neq 1$, so that $1 < p < \infty$. For fixed $f \in L^p(G)$ consider the linear functional

$$g \to \int_G \langle U_{-a} f, g \rangle d\mu(a) \tag{2.1}$$

on $L^{p'}(G)$, where $1/p + 1/p' = 1$ and where $\langle , \rangle$ denotes the usual coupling between $L^p(G)$ and $L^{p'}(G) : \langle u, v \rangle = \int_G u(x) v(-x) dx$. Thanks to (iii), $U_{-a}$ is continuously extendible from $T(G)$ to $L^p(G)$ so as to map the latter continuously and linearly into $L^p(G)$, whence it follows that the linear functional (2.1) is continuous on $L^{p'}(G)$. Since $p' \neq \infty$, there exists a uniquely determined element $Uf$ of $L^p(G)$ such that

$$\langle Uf, g \rangle = \int_G \langle U_{-a} f, g \rangle d\mu(a) \tag{2.2}$$

for all $g \in L^{p'}(G)$. The map $U : f \to Uf$ is linear and continuous from $L^p(G)$ into itself.

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If $p = 1$, so that $p' = \infty$, the last stage of the preceding argument breaks down (since $L^1(G)$ is not the dual of $L^\infty(G)$); however, one may replace $L^\infty(G)$ by its subspace $C(G)$ composed of the continuous functions and so conclude that $U$ maps $L^1(G)$ linearly and continuously into $M(G)$.

In either case, the defining equation (2.2) shows that

$$(Uf)^\wedge = (\hat{\mu} \circ \alpha) \cdot \hat{f} \quad \text{for} \quad f \in L^p(G),$$

so that $U$ is a multiplier operator of type $(p, p)$; in particular, $U$ commutes with translations. In case $p = 1$, it follows that $U$, which is known to map into $M(G)$, actually maps into $L^1(G)$; see [2], § 3. Consequently (2.3) shows that in all cases $\hat{\mu} \circ \alpha \in (p, p)$ and the proof is complete.

*The converse result.* Suppose (iv) to be true. On taking $\mu = \varepsilon_{-a}$ (the Dirac measure at $-a$), it is seen that $\varphi_a = \hat{\varepsilon}_{-a} \circ \alpha$ belongs to $(p, p)$, so that $A_p = G$ and both (i) and (ii) hold. Also, the argument used to show that (i) implies (iii) shows in this case that (iii) holds (whether or not $G$ is connected, since now we know that $N_p(a) < \infty$ for all $a \in G$). Thus (iv) entails (i), (ii), and (iii).

## 3. The case $p = 1$

Since there are no handy criteria for membership of $(p, p)$ unless $p = 1$ or 2, the latter case being trivial, the former is the one in which Theorem 1 is most easily interpreted.

(1) When $p = 1$, the multiplier functions are simply the Fourier-Stieltjes transforms of elements of $M(G)$. The conclusion of Theorem 1 thus signifies in this case that one has a homomorphism $\Psi$ of $M(G)$ into itself such that $(\Psi\mu)^\wedge = \hat{\mu} \circ \alpha$ for $\mu \in M(G)$. It is known ([1], p. 78, Theorem 4.1.3) that this is the case if and only if $\alpha$ is a piecewise affine map of $X$ into itself. When $G$ is the circle group $R/2\pi Z$, so that $X$ is the additive group $Z$ of integers, this means (loc. cit., p. 95) that there exists a positive integer $q$ and a map $\beta$ of $Z$ into itself having the form

$$\beta(kq+h) = u_h k + v_h$$

for $(k\ h) \in Z \times \{0, 1, \cdots, q-1\}$, where $u_h$, $v_h \in Z$, such that $\beta$ agrees with $\alpha$ except perhaps on a finite subset of $Z$.

It thus appears, for example, that if $\alpha : Z \to Z$ is such that $\alpha(n) \neq O(|n|)$ as $|n| \to \infty$, then the set $S$, composed of all real numbers $s$ such that $\phi'_s : n \to \exp(2\pi i s \alpha(n))$ is the Fourier (-Stieltjes) transform of a measure on the circle, is both null and meagre. Cf. Remark (ii) in § 5.

(2) A simple special case of the preceding remark is that in which $\alpha = P$ is a nonlinear polynomial function mapping $Z$ into $Z$. In this case it is evident that $S$ contains all rational numbers. The referee was kind

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enough to point out and prove that in fact $S$ contains only rational numbers. In fact, if $\phi_s' = \hat{\mu}$, where $\mu$ is a measure on the circle, Wiener's theorems ([3], pp. 107—108) give

$$(3.1) \qquad (2N)^{-1} \sum_{|n| \leqq N} \phi_s'(n) \to \mu(\{x\}) \text{ as } N \to \infty$$

and

$$(3.2) \qquad 1 = \lim_{N \to \infty} (2N)^{-1} \sum_{|n| \leqq N} |\phi_s'(n)|^2 = \sum |\mu(\{x\})|^2.$$

On the other hand, if $s$ is irrational, the numbers $sP(n)$ are equiuniformly distributed modulo 1 ([5], Satz 12), whence it follows that the left hand side of (3.1) converges, as $N \to \infty$, to $\int_0^1 \exp(2\pi i t + inx)dt = 0$. Thus $\mu(\{x\}) = 0$ for all $x$, which contradicts (3.2) and so completes the proof.

## 4. The second theorem

It will now be shown that much stronger variants of Theorem 1 are valid if it be supposed that $\alpha$ has additional properties bearing mainly on its range.

To state this second result, denote by $A$ the set of $a \in G$ such that $\varphi_a \in (p, p)$ for some (possibly $a$-dependent) $p$ satisfying $1 \leqq p \leqq \infty$, $p \neq 2$; in other words, $A = \bigcup \{A_p : 1 \leqq p \leqq \infty, p \neq 2\}$. According to well-known results, $A = \bigcup \{A_p : p \in D\}$, where $D$ is any subset of $[1, 2)$ whose supremum is 2; $D$ may obviously be chosen to be countable. Moreover, since $A_p \subset A_q$ for $1 \leqq p < q \leqq 2$, $A$ is an $F_\sigma$-subgroup of $G$.

THEOREM 2. *Suppose that*

(v) *$\alpha(X)$ is a Sidon subset of $X$*

*and*

(vi) *the set $A$ is either nonnull or nonmeagre.*

*Then*

(vii) *There exists $p \in (1, 2)$ such that $\psi \circ \alpha \in (p, p)$ for all $\psi \in l^\infty(X)$.*

PROOF. Taking $D$ to be a countable subset of (1,2) whose supremum is 2, (vi) ensures that there exists a $p \in (1, 2)$ such that $A_p$ satisfies at least one of (i) and (ii) (see Theorem 1). Hence, by Theorem 1, $N_p$ is bounded on $G$.

Take $f \in L^p(G)$ and $g \in L^{p'}(G)$, and write $h$ for the function $a \to \langle U_{-a} f, g\rangle$ ($U_{-a}$ having been continuously extended into a linear map of $L^p(G)$ into itself). If $(f_i)$ is a net of trigonometric polynomials converging in $L^p(G)$ to $f$ one has, since the $U_{-a}$ are equicontinuous as a consequence of the boundedness of $N_p$ on $G$,

$$\begin{aligned} h(a) &= \lim_\iota \langle U_{-a} f_\iota, g\rangle \\ &= \lim_\iota \sum_{\chi\in X} \varphi_{-a}(\chi)\hat{f}_\iota(\chi)\hat{g}(\chi) \\ &= \lim_\iota \sum_{\chi\in X} \hat{f}_\iota(\chi)\hat{g}(\chi)\cdot(-a, \alpha(\chi)) \end{aligned}$$

uniformly for $a \in G$. This shows that $h$ is continuous and that $\hat{h}$ vanishes off $-\alpha(X)$. In view of (v), $h \in A(G)$ ([1], Theorem 5.7.3). Furthermore, two applications of the closed graph theorem show that

$$\|h\|_{A(G)} \leqq \text{const.}\ \|f\|_p \|g\|'_p,$$

where

$$\|h\|_{A(G)} = \sum_{\chi\in X} |\hat{h}(\chi)|$$

denotes the usual norm on $A(G)$. As a result, if $\sigma$ is any pseudomeasure on $G$, there is a continuous linear map $U$ of $L^p(G)$ into itself such that

$$\langle Uf, g\rangle = \int_G h\,d\sigma = \int_G \langle U_{-a} f, g\rangle\, d\sigma(a) \tag{4.1}$$

for $f \in L^p(G)$ and $g \in L^{p'}(G)$; cf. the corresponding portion of the proof of Theorem 1. From (4.1) it appears that $(Uf)^\wedge = (\hat{\sigma}\circ\alpha)\cdot\hat{f}$, so that $\hat{\sigma}\circ\alpha \in (p, p)$. As $\sigma$ ranges over $P(G)$, $\hat{\sigma}$ ranges over $l^\infty(X)$, so that (vii) is established.

## 5. Discussion of Theorem 2

The set $F_\alpha$ of functions of the form $\psi\circ\alpha$, where $\psi$ ranges over $l^\infty(X)$, comprises exactly those bounded, complex-valued functions on $X$ which are constant on each of the sets $\alpha^{-1}(\{\chi\})$; in particular, $F_\alpha = l^\infty(X)$ if $\alpha$ is one-to-one.

Theorem 2 asserts that $F_\alpha \subset (p, p)$ for some $p \in (1, 2)$ whenever conditions (v) and (vi) hold.

Now the standard theory of random Fourier series (see, for example, [3] pp. 212—222, and [4]) is easily developed to show that the relation $F_\alpha \subset (p, p)$ is false for $p \neq 2$, at least if $\alpha$ satisfies the following condition:

$$\text{(v}'\text{)} \qquad \sup_{\chi\in X} \#\alpha^{-1}(\{\chi\}) < \infty,$$

where $\#S$ (a nonnegative integer or $\infty$) denotes the number of elements of the set $S$. One thus obtains the following corollary.

COROLLARY. *Suppose that $\alpha$ satisfies* (v) *and* (v′). *Then there exists a null and meagre $F_\sigma$-subgroup $A$ of $G$ such that, if $a \in G\backslash A$, $\varphi_a$ belongs to $(p, p)$ for no $p \neq 2$.*

A simple special case is that in which $G = R/2\pi Z$, $X = Z$, and $\alpha(n) = 2^{|n|}$ for $n \in Z$.

More generally, suppose that $G$ is first countable. Enumerate $X$ as $\{\chi_k : k = 1, 2, \cdots\}$. Then ([1], p. 126) one can define many sequences $n_1 < n_2 < \cdots$ of positive integers such that $\{\chi_{n_k} : k = 1, 2, \cdots\}$ is a Sidon subset of $X$. Defining $\alpha$ by $\alpha(\chi_k) = \chi_{n_k}$, the corollary shows in particular that for almost all $a \in G$ the function $\varphi_a : \chi_k \to (a, \chi_{n_k})$ belongs to $(p, p)$ for no $p \neq 2$.

REMARKS. (i) The corollary can be stated in an apparently different way. To this end we write, for $h \in L^1(G)$,

$$\|h\|_{p,p} = \text{Sup}\,\{\|h * f\|_p : f \in T(G),\ \|f\|_p \leqq 1\}. \tag{5.1}$$

Suppose chosen any sequence or net $(k_\iota)$ of functions in $L^2(G)$ such that $\lim_\iota \hat{k}_\iota = 1$ pointwise on $X$. With these conventions, the corollary asserts that

$$\lim_\iota \|\sum_{\chi \in X} (a, \alpha(\chi))\hat{k}_\iota(\chi)\chi\|_{p,p} = \infty \quad (a \in G\backslash A,\ p \neq 2). \tag{5.2}$$

In case $G = R/2\pi Z$, this conclusion (5.2) may be alternatively expressed in the form

$$\lim_{N\to\infty} \|\sum_{|n|\leqq N} (1-|n|/N)e^{2\pi i s\alpha(n)+inx}\|_{p,p} = \infty$$

for $s \in R\backslash B$ and $p \neq 2$, where $B$ is a certain null and meagre $F_\sigma$-subgroup of $R$.

(ii) Notwithstanding the foregoing corollary and § 3,(2), if $G$ is the circle group one can easily construct one-to-one maps $\alpha$ of $Z$ into itself such that $\alpha(Z)$ is a Sidon set and such that the additive group $S$ of numbers $s \in R$, such that $n \to \varphi'_s(n) = \exp\,(2\pi i s\alpha(n)) \in (p, p)$ for all $p \in [1, \infty]$, contains uncountably many transcendental numbers.

Thus, suppose that $s \in R$ admits rational approximations $p_r/q_r$:

$$s = p_r/q_r + \varepsilon_r, \tag{5.3}$$

where $r, p_r, q_r \in Z$, $r > 0$, $q_r > 0$, and $|p_r|$ and $q_r$ are coprime; and that $\alpha$ is a map of $Z$ into itself such that to each $n \in Z$ corresponds $r(n) \in Z$, $r(n) > 0$, for which

$$\alpha(n)/q_{r(n)} \in Z,\ \sum_{n\in Z} (\alpha(n)\varepsilon_{r(n)})^2 < \infty. \tag{5.4}$$

Then

$$\varphi'_s(n) = \exp\,(2\pi i\alpha(n)\varepsilon_{r(n)}) = 1 + O(\alpha(n)\varepsilon_{r(n)}),$$

which makes it apparent that $\varphi'_s \in (p, p)$ for all $p \in [1, \infty]$.

On the hand other, conditions (5.3) and (5.4) can be satisfied for suitable one-to-one maps $\alpha$ of $Z$ into itselt for which $\alpha(Z)$ is a Sidon set, and for corresponding uncountable sets of Liouville numbers $s$. To do this it suffices, for example, to begin with an ultimately strictly increasing sequence $(\nu_k)$ of positive integers such that $\nu_k$ divides $\nu_{k+1}$ and to define $\alpha$ by $\alpha(0) = 0$ and $\alpha(n) = \operatorname{sgn} n \cdot \nu_{|n|}$ for $n \in Z$, $n \neq 0$. Having thus fixed $\alpha$, (5.3) and (5.4) will be satisfied by each number $s$ of the form

$$s = \sum_{k=1}^{\infty} b_k/\nu_k \tag{5.5}$$

for which $b_k \in Z$ and

$$\sum_{n=1}^{\infty} (\nu_n \sum_{k>n}^{\infty} b_k/\nu_k)^2 < \infty.$$

If the $\nu_k$ are suitably chosen (for example, $\nu_k = 2^{k^2}$, or $\nu_k = 2^{[k^\gamma]}$ for some fixed $\gamma > 1$, or $\nu_k = 2^{[k \cdot \log^\gamma k]}$ for some fixed $\gamma > 0$), all will be well for any bounded sequence $(b_k)$ of integers. Such sequences $(b_k)$ will generate, via (5.5), uncountably many transcendental Liouville numbers $s$ for each of which $\varphi'_s \in (p, p)$ for all $p \in [1, \infty]$.

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