J. Austral. Math. Soc. **20** (Series A) (1975), 394–397.

QUASI-FROBENIUS QUOTIENT RINGS OF GROUP RINGS

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(Received 19 March 1974)

1. Introduction

The purpose of this paper is an extension of a theorem of Hughes (1973). He showed:

Let R be a ring which has a right artinian right quotient ring and let G be a group which has a (transfinite) ascending normal series with each factor either finite or cyclic, but only a finite number of finite factors. Then the group ring RG has a right artinian right quotient ring.

In the situation of Hughes' theorem we prove as the main result (Theorem 2.6) of this article that

RG is an order in a quasi-Frobenius ring (QF ring) if R is an order in a QF ring.

This is also a generalization of our result (Horn (1973; 3.9)) that for a polycyclic—by—finite group G the group ring RG is an order in a QF ring if R is an order in a QF ring. The proof of Theorem 2.6 is based on the results and methods developed in Horn (1973). In particular we obtain a different proof of Hughes' theorem.

In the following all considered rings have an identity. $J(R)$ denotes the Jacobson radical of the ring R. Let ρ be an automorphism of R. Then $R[x,\rho]$ is as usual the skew polynomial ring over R.

Let G be a group with normal subgroup N such that G/N is infinite cyclic. If G/N is generated by xN for some $x \in G$ let $\rho_x\colon RN \to RN$ be the automorphism of RN defined by $\rho_x(a) := x^{-1}ax$ for all $a \in RN$. Then the quotient ring of RG is up to isomorphism the quotient ring of $RN[x,\rho_x]$ when it exists (see Horn (1973; page 39)), furthermore it is the quotient ring of $Q[x, \rho_x]$ where Q is the quotient ring of RN.

The direct limit of a directed system $\{R_\alpha \mid \alpha < \gamma\}$ of rings R_α (γ a limit ordinal) is written $R_\gamma = \lim_{\alpha<\gamma}(R_\alpha)$. Tacitly we assume that in a directed system for any $\alpha_1 \leqq \alpha_2 < \gamma$ always R_{α_1} is contained in R_{α_2}, both rings have the same identity, and the ring homomorphism $R_{\alpha_1} \to R_{\alpha_2}$ given by the directed system is the inclusion. Then we identify the direct limit $\lim_{\alpha<\gamma}(R_\alpha)$ with the union $\bigcup_{\alpha<\gamma} R_\alpha$.

For the definition of an ascending normal series of a group we refer to Kurosh (1956). All unexplained notation and terminology is as in Horn (1973).

I would like to thank Professor G. Michler for his suggestions. I am also indebted to the referee for valuable improvements.

2. Proof of the main result

The proof of Theorem 2.6 depends heavily on the following results contained in the paper Horn (1973). Theorem 2.1 generalizes a useful theorem of Shock (1972; Proposition 2.4).

THEOREM 2.1. *Let R be a ring with an automorphism ρ. The right ideal A of R is uniform in R if and only if $AR[x, \rho]$ is a uniform right ideal of $R[x, \rho]$.*

PROOF. See Horn (1973; Satz 1.7).

LEMMA 2.2. *Let R be a ring and let G be a group with a normal subgroup N such that G/N is finite or cyclic. Then*

(a) *If RN is a right order in a right artinian ring then RG is also a right order in a right artinian ring.*

(b) *If RN is an order in a QF ring then RG is also an order in a QF ring.*

PROOF. See Horn (1973; Lemmas 3.2 and 3.8).

Before we now proceed to prove the main result we state two auxiliary lemmas.

LEMMA 2.3. *Let γ be a limit ordinal and let $\{R_\alpha \mid \alpha < \gamma\}$ be a directed system of rings R_α. Then*

(a) *If the regular elements of R_α are regular in $R_{\alpha+1}$ and the right quotient ring Q_α of R_α exists for every $\alpha < \gamma$, then $R_\gamma = \lim_{\alpha<\gamma}(R_\alpha)$ is a right order in $Q_\gamma = \lim_{\alpha<\gamma}(Q_\alpha)$.*

(b) *If U is a uniform right ideal of R_0 such that UR_α is a uniform right ideal of R_α for all $\alpha < \gamma$ then UR_γ is a uniform right ideal of R_γ.*

PROOF. (a) is straightforward.

(b). Suppose UR_γ is not a uniform right ideal of R_γ. Then there are elements $0 \neq a, b \in UR_\gamma$ such that $aR_\gamma \cap bR_\gamma = 0$. Hence for some ordinal $\beta < \gamma$ we have $a, b \in UR_\beta$. Thus $0 \neq aR_\beta, bR_\beta \leqq UR_\beta$ and $aR_\beta \cap bR_\beta = 0$. This contradicts the uniformity of UR_β as a right ideal of R_β.

LEMMA 2.4. *Let R be a ring and let G be a group with a (transfinite) ascending normal series $\{G_\alpha \mid \alpha \leqq \gamma\}$ from the subgroup G_0 to $G = G_\gamma$ (γ an ordinal) such that for every $\alpha < \gamma$ the factor group $G_{\alpha+1}/G_\alpha$ is infinite cyclic and RG_α is a right order in a right artinian ring Q_α. Then*

(a) *RG_γ has a right quotient ring Q_γ and $Q_\alpha = \lim_{\beta<\alpha}(Q_\beta)$ for every limit ordinal $\alpha \leqq \gamma$.*

(b) *If U is a uniform right ideal of Q_0 then UQ_α is a uniform right ideal of Q_α for all $\alpha \leqq \gamma$.*

(c) *$J_\alpha = J(Q_\alpha) = J(Q_0)Q_\alpha$ for all $\alpha \leqq \gamma$ and $J_\alpha = \lim_{\beta<\alpha}(J_\beta)$ for every limit ordinal $\alpha \leqq \gamma$.*

(d) *$J(Q_0)^k = J_0^k = 0$ implies $J_\alpha^k = 0$ for all $\alpha \leqq \gamma$.*

(e) *$\bar{Q}_\alpha = Q_\alpha / J_\alpha$ is a semisimple artinian ring for every $\alpha \leqq \gamma$.*

PROOF. We prove the statements by transfinite induction on α. (a) follows immediately from the preceding Lemma 2.3 (a) and Lemma 2.2, (b) is a direct consequence of Theorem 2.1 and Lemma 2.3 (b), and (d) follows from (c).

For the proof of (c) and (e) first suppose that λ is not a limit ordinal and that for $\alpha < \lambda$ (c) and (e) are valid. Then the assertions hold also for λ: (c) follows from Korollar 2.10 of Horn (1973) and Theorem 1.9 of Small (1966) while Lemma 2.2 (a) yields (e).

Now we assume that λ is a limit ordinal. By induction hypothesis we have $J_{\alpha+1} = J_\alpha Q_{\alpha+1}$ for all $\alpha < \lambda$. Since $J_{\alpha+1}$ is nilpotent it follows that $J_{\alpha+1} \cap Q_\alpha = J_\alpha$ and $K \cap Q_\alpha = J_\alpha$ with $K = \bigcup_{\alpha<\lambda} J_\alpha \subseteq J_\lambda$. Therefore Q_λ / K is a direct limit of the von Neumann regular rings $K + Q_\alpha / K \cong Q_\alpha / J_\alpha$ for $\alpha < \lambda$, whence Q_λ / K has the same property and therefore has zero Jacobson radical. Hence $K = J_\lambda$, which implies (c). Moreover from Lemma 2.3 we obtain that for a decomposition $1 = e_1 + \cdots + e_n$ with primitive orthogonal idempotents $e_1, \cdots, e_n$ in Q_0 the right ideal $e_i\bar{Q}_\lambda$ of $\bar{Q}_\lambda = Q_\lambda / J_\lambda$ is uniform in $\bar{Q}_\lambda$ for every $i = 1, \cdots, n$. Therefore the ring $\bar{Q}_\lambda$ has finite right Goldie dimension. Thus we conclude that $\bar{Q}_\lambda$ is a semisimple artinian ring.

THEOREM 2.5 (Hughes (1973)). *Let R be a ring which is a right order in a right artinian ring and let G be a group which has a (transfinite) ascending normal series with each factor either finite or cyclic, but only a finite number of finite factors. Then RG is a right order in a right artinian ring.*

PROOF. Let γ be an ordinal and let $\{G_\alpha \mid \alpha < \gamma\}$ be the ascending normal series of G with $G_0 = (1)$ and $G = G_\gamma$ as assumed. By transfinite induction we show that RG_α is a right order in a right artinian ring for every $\alpha \leqq \gamma$.

Let $\lambda \leqq \gamma$ be an ordinal such that RG_α is a right order in a right artinian ring for all $\alpha < \lambda$. From Lemma 2.2 we obtain the result in case λ is not a limit ordinal. Thus we may assume that λ is a limit ordinal. Since the ascending normal series in G has only a finite number of finite factors, there is an ordinal $\beta < \lambda$ such that all factors between G_β and G_λ are infinite cyclic. Therefore, applying Lemma 2.4, it follows that RG_λ is a right order in $Q_\lambda = \lim_{\alpha<\lambda}(Q_\alpha)$. In particular $J(Q_\lambda) = J(Q_\beta)Q_\lambda$ is a finitely generated nilpotent right ideal of Q_λ and Q_λ/J_λ is a semisimple artinian ring. Hence Q_λ is right artinian.

THEOREM 2.6. *Let R be a ring which is an order in a QF ring and let G be a group which has a (transfinite) ascending normal series with each factor either finite or cyclic, but only a finite number of finite factors. Then RG is an order in a QF ring.*

PROOF. As in the proof of Theorem 2.5 we proceed by transfinite induction and use the same meanings of α, β, γ, λ. By Theorem 2.5 RG_α is an order in an artinian ring Q_α for each $\alpha \leqq \gamma$. The case that λ is not a limit ordinal results from Lemma 2.2. Now let λ be a limit ordinal. Using the characterization of QF rings given by Hajarnavis (1971; page 336) it is sufficient to show that

(i) the left annihilator and the right annihilator of $J(Q_\lambda)$ in Q_λ coincide.

(ii) Q_λ is a direct sum of uniform right (and left) ideals.

The proof of (i) is routine applying Lemma 2.4 (c). Now we consider the decomposition $1 = e_1 + \cdots + e_n$ with primitive orthogonal idempotents $e_1, \cdots, e_n$ of Q_β. By the induction hypothesis Q_β is a QF ring, whence e_iQ_β is a uniform right ideal of Q_β for $i = 1, \cdots, n$. Therefore from Lemma 2.4 (b) it follows that Q_λ is the direct sum of the uniform right ideals e_iQ_λ of Q_λ. This finishes the proof of the theorem.

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