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A REMARK ON THE REGULARITY OF THE DISCRETE MAXIMAL OPERATOR

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Abstract

We study the regularity properties of several classes of discrete maximal operators acting on $\mathrm{BV}(\mathbb{Z})$ functions or $\ell^1(\mathbb{Z})$ functions. We establish sharp bounds and continuity for the derivative of these discrete maximal functions, in both the centred and uncentred versions. As an immediate consequence, we obtain sharp bounds and continuity for the discrete fractional maximal operators from $\ell^1(\mathbb{Z})$ to $\mathrm{BV}(\mathbb{Z})$.

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1. Introduction

Considerable attention has been given to the behaviour of differentiability under a maximal operator. Kinnunen [11] first studied the regularity of the usual centred Hardy–Littlewood maximal function $\mathcal{M}$ and showed that $\mathcal{M}$ is bounded on the Sobolev spaces $W^{1,p}(\mathbb{R}^d)$ for all $1 < p \le \infty$. Tanaka [20] (see also [10]) noted that the $W^{1,p}$-bound for the uncentred case of $\mathcal{M}$ also holds by a simple modification of Kinnunen's arguments. This paradigm has been extended to a local version in [12], to a fractional version in [13] and to a multisublinear version in [7, 16]. Due to the lack of reflexivity of L^1, results for $p = 1$ are subtler. A crucial question in this direction was posed by Hajłasz and Onninen in [10].

Question 1.1 [10]. Is the operator $f \mapsto |\nabla \mathcal{M} f|$ bounded from $W^{1,1}(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$?

A standard dilation argument reveals the true nature of this question: whether the variation of the maximal function is controlled by the variation of the original function, that is, whether

$$\|\nabla \mathcal{M} f\|_{L^1(\mathbb{R}^d)} \le C \|\nabla f\|_{L^1(\mathbb{R}^d)}. \tag{1.1}$$

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Progress on this problem has been restricted to dimension $d = 1$. In 2002, Tanaka [20] observed that if $f \in W^{1,1}(\mathbb{R})$, then $\widetilde{\mathcal{M}}f$ is weakly differentiable and (1.1) holds with $d = 1$ and $C = 2$ for the uncentred maximal operator $\widetilde{\mathcal{M}}$. Tanaka's result was later sharpened by Aldaz and Pérez Lázaro [2], who proved that if f is of bounded variation on $\mathbb{R}$, then $\widetilde{\mathcal{M}}f$ is absolutely continuous and

$$\mathrm{Var}(\widetilde{\mathcal{M}}f) \leq \mathrm{Var}(f), \tag{1.2}$$

where $\mathrm{Var}(f)$ denotes the total variation of f. The inequality (1.2) is sharp. A new proof of (1.1) with $d = 1$ and $C = 1$ for $\widetilde{\mathcal{M}}$ was presented by Liu *et al.* in [15]. Very recently, (1.2) was extended to a fractional setting in [6, Theorem 1]. In the remarkable work [14], Kurka showed that if f is of bounded variation on $\mathbb{R}$, then (1.2) holds for $\mathcal{M}$ (with constant $C = 240\,004$). It was also shown in [14] that if $f \in W^{1,1}(\mathbb{R})$, then $\mathcal{M}f$ is weakly differentiable and (1.1) also holds for $\mathcal{M}$ with constant $C = 240\,004$. It is currently unknown whether (1.2) also holds for $\mathcal{M}$. For other interesting related work, we refer the reader to [1, 8, 9, 17, 18].

In this paper, we focus on this topic in the discrete setting. Let us recall some notation and relevant results. For $1 \leq p < \infty$, the ℓ^p-norm and the ℓ^∞-norm of a discrete function $f : \mathbb{Z} \to \mathbb{R}$ are defined by

$$\|f\|_{\ell^p(\mathbb{Z})} = \Big(\sum_{n\in\mathbb{Z}} |f(n)|^p\Big)^{1/p} \quad \text{and} \quad \|f\|_{\ell^\infty(\mathbb{Z})} = \sup_{n\in\mathbb{Z}} |f(n)|.$$

We define the first derivative of f by $f'(n) = f(n+1) - f(n)$ for any $n \in \mathbb{Z}$. The total variation of f recovers the $\ell^1(\mathbb{Z})$-norm of f', that is,

$$\mathrm{Var}(f) = \|f'\|_{\ell^1(\mathbb{Z})} = \sum_{n\in\mathbb{Z}} |f(n+1) - f(n)|.$$

We denote by $\mathrm{BV}(\mathbb{Z})$ the set of functions of bounded variation defined on $\mathbb{Z}$ and write

$$\mathrm{Var}(f; [a,b]) = \|f'\|_{\ell^1([a,b])} = \sum_{n=a}^{b-1} |f(n+1) - f(n)|$$

for the variation of f on the interval $[a, b]$, where a, b are integers (or possibly $\pm\infty$).

The regularity of discrete maximal operators has attracted the attention of many authors (see [3, 5, 6, 19, 21]). Let M (respectively $\widetilde{M}$) be the discrete centred (respectively uncentred) Hardy–Littlewood maximal operator given by

$$Mf(n) = \sup_{r\in\mathbb{N}} \frac{1}{2r+1} \sum_{k=-r}^{r} |f(n+k)|, \quad \widetilde{M}f(n) = \sup_{r,s\in\mathbb{N}} \frac{1}{r+s+1} \sum_{k=-r}^{s} |f(n+k)|,$$

where $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$. Bober *et al.* [3] proved that

$$\mathrm{Var}(\widetilde{M}f) \leq \mathrm{Var}(f) \tag{1.3}$$

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and

$$\operatorname{Var}(Mf) \le \Big(2 + \frac{146}{315}\Big)\|f\|_{\ell^1(\mathbb{Z})}. \tag{1.4}$$

The inequality (1.3) is sharp. The inequality (1.3) for M was established by Temur in [21] (with constant $C = 294\,912\,004$). Inequality (1.4) is not optimal, and it was asked in [3] whether the sharp constant for (1.4) is $C = 2$. This question was resolved by Madrid in [19]. Recently, Carneiro and Madrid [6] extended (1.3) to the fractional setting. They considered the discrete uncentred fractional maximal operator

$$\widetilde{M}_\alpha f(n) = \sup_{r,s\in\mathbb{N}} \frac{1}{(r+s+1)^{1-\alpha}} \sum_{k=-r}^{s} |f(n+k)|$$

and showed that if $0 \le \alpha < 1$, $q = 1/(1-\alpha)$, $f \in \mathrm{BV}(\mathbb{Z})$ and $\widetilde{M}_\alpha f \not\equiv \infty$, then

$$\|(\widetilde{M}_\alpha f)'\|_{\ell^q(\mathbb{Z})} \le 4^{1/q}\operatorname{Var}(f).$$

To the best of our knowledge, the centred analogue remains an open problem. The motivation for this paper is to investigate the regularity of the discrete centred fractional maximal operator

$$M_\alpha f(n) = \sup_{r\in\mathbb{N}} \frac{1}{(2r+1)^{1-\alpha}} \sum_{k=-r}^{r} |f(n+k)|.$$

More precisely, we shall establish the following theorem.

Theorem 1.2. *Let $0 \le \alpha < 1$. Then M_α is bounded and continuous from $\ell^1(\mathbb{Z})$ to* $\mathrm{BV}(\mathbb{Z})$. *Moreover, if $f \in \ell^1(\mathbb{Z})$, then*

$$\operatorname{Var}(M_\alpha f) \le 2\|f\|_{\ell^1(\mathbb{Z})},$$

and the constant $C = 2$ is the best possible. The same results hold for $\widetilde{M}_\alpha$.

Remark 1.3. Theorem 1.2 extends the result of Madrid in [19, Theorem 1], which corresponds to the case $\alpha = 0$. On the other hand, by the nest property $\ell^{q_1}(\mathbb{Z}) \subsetneq \ell^{q_2}(\mathbb{Z})$ for $0 < q_1 < q_2$, our Theorem 1.2 is new even in the uncentred case. It should be pointed out that $M_\alpha : \mathrm{BV}(\mathbb{Z}) \to \mathrm{BV}(\mathbb{Z})$ is not bounded for $0 < \alpha < 1$. To see this, let $l \in \mathbb{N}\backslash\{0\}$ and $f(n) = \chi_{[-l,l]}(n)$. One can easily check that $\operatorname{Var}(f) = 2$ and $\operatorname{Var}(M_\alpha f) \ge \operatorname{Var}(M_\alpha; [l,\infty)) \ge \frac{1}{2}(4l+1)^\alpha$. This yields our claim by letting $l \to \infty$. The same claim holds for $\widetilde{M}_\alpha$.

We will establish Theorem 1.2 by investigating the end-point regularity of more general discrete maximal operators. Let Φ be a function defined on $(0,\infty)$. For a discrete function $f : \mathbb{Z} \to \mathbb{R}$, we define the discrete centred maximal operator M_Φ with respect to Φ by

$$M_\Phi f(n) = \sup_{r\in\mathbb{N}} \Phi(2r+1) \sum_{k=-r}^{r} |f(n+k)| \tag{1.5}$$

and the uncentred version by

$$\widetilde{M}_\Phi f(n) = \sup_{r,s\in\mathbb{N}} \Phi(r+s+1) \sum_{k=-r}^{s} |f(n+k)|.$$

Clearly, M (respectively $\widetilde{M}$) is the special case of M_Φ (respectively $\widetilde{M}_\Phi$) for $\Phi(t) = t^{-1}$ and, when $\Phi(t) = t^{\alpha-1}$, $0 < \alpha < 1$, M_Φ (respectively $\widetilde{M}_\Phi$) reduces to the operator M_α (respectively $\widetilde{M}_\alpha$).

Our main results can be formulated as follows.

THEOREM 1.4. *Let M_Φ be given as in (1.4). Assume that $\Phi : [1,\infty) \to (0,\infty)$ is convex and decreases to zero. Then M_Φ is bounded and continuous from $\ell^1(\mathbb{Z})$ to* $\mathrm{BV}(\mathbb{Z})$. *Moreover, if $f \in \ell^1(\mathbb{Z})$, then*

$$\mathrm{Var}(M_\Phi f) \le 2\Phi(1)\|f\|_{\ell^1(\mathbb{Z})},$$

and the constant $C = 2\Phi(1)$ is the best possible. The same results hold for $\widetilde{M}_\Phi$.

THEOREM 1.5. *Let $\widetilde{M}_\Phi$ be given as in (1.5). Assume that $\Phi : [1,\infty) \to (0,\infty)$ satisfies* $\lim_{t\to\infty} \Phi(t) = 0$ *and*

$$\frac{1}{\Phi(s)} + \frac{1}{\Phi(t)} \le \frac{1}{\Phi(t+s)} \tag{1.6}$$

for all $s, t \in [1,\infty)$. Then $\widetilde{M}_\Phi$ is bounded and continuous from $\ell^1(\mathbb{Z})$ to $\mathrm{BV}(\mathbb{Z})$. *Moreover, if $f \in \mathrm{BV}(\mathbb{Z})$, then*

$$\mathrm{Var}(\widetilde{M}_\Phi f) \le \Phi(1)\mathrm{Var}(f).$$

Further, if Φ is also nonincreasing, then the constant $C_\Phi = \Phi(1)$ is the best possible.

REMARK 1.6. Examples of functions satisfying the assumptions on Φ in Theorem 1.4 are $t^{-\alpha}(\alpha > 0)$, $e^{-at}(a > 0)$, $(\ln t)^{-1}$ and so on. An example of a function satisfying the assumptions on Φ in Theorem 1.5 is $(P(t))^{-1}$, where $P(t)$ is a polynomial with positive coefficients. We mention three further examples: (a) $\Phi(t) = t^{-2}$ satisfies the assumptions in Theorems 1.4 and 1.5; (b) $\Phi(t) = t^{-1/2}$ satisfies the assumptions in Theorem 1.4, but Φ does not satisfy the condition (1.6); (c) $\Phi(t) = (1+t^2)^{-1}$ satisfies the assumptions in Theorem 1.5, but Φ is a concave function defined on $[1,\infty)$.

REMARK 1.7. Clearly, Theorem 1.2 follows immediately from Theorem 1.4 when $\Phi(t) = t^{\alpha-1}$ for $0 \le \alpha < 1$. The boundedness part in Theorem 1.4 extends (1.3), which corresponds to the case $\Phi(t) = t^{-1}$. It is not hard to see, by considering the function $f(n) = \chi_{\{1\}}(n)$, that the constant $C_\Phi = 2\Phi(1)$ (respectively $C_\Phi = \Phi(1)$) is best possible in Theorem 1.4 (respectively Theorem 1.5).

The rest of this paper is organised as follows. We prove Theorem 1.4 in Section 2 and Theorem 1.5 in Section 3. The proof of the boundedness part in Theorem 1.4 is

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based on the method of [19]. The main ideas in the proof of the boundedness part of Theorem 1.5 are motivated by [3], but our method is simpler than that of [3]. The proofs of the continuity parts in Theorems 1.4 and 1.5 rely on the boundedness results and an application of the Brezis–Lieb lemma in [4], which was used to prove the continuity of a class of discrete maximal operators by Carneiro and Hughes in [5]. However, our approach is different to and simpler than that of [5].

Throughout this paper, the letter C, sometimes with additional parameters, will stand for positive constants, not necessarily the same at each occurrence but independent of the essential variables.

2. Proof of Theorem 1.4

2.1. Boundedness part. We apply the method in the proof of [19, Theorem 1] to prove the boundedness part of Theorem 1.4. Let $f \in \ell^1(\mathbb{Z})$. Without loss of generality, we may assume that $f \geq 0$.

2.1.1. Centred case. For convenience, let $\Gamma(x) = \Phi(2x+1) - \Phi(2x+3)$ for any $x \geq 0$. One can easily check that $\Gamma(x)$ is decreasing on $[0, \infty)$ and $\sum_{n\in\mathbb{N}} \Gamma(n) = \Phi(1)$. Since $f \in \ell^1(\mathbb{Z})$, then, for any $n \in \mathbb{Z}$, there exists $r_n \in \mathbb{N}$ such that $M_\Phi f(n) = \mathrm{A}_{r_n}(f)(n)$, where

$$\mathrm{A}_r(f)(n) = \Phi(2r+1) \sum_{k=-r}^{r} f(n+k)$$

for any $r \in \mathbb{N}$ and $n \in \mathbb{Z}$. Let

$$X^+ = \{n \in \mathbb{Z} : M_\Phi f(n+1) > M_\Phi f(n)\} \quad \text{and} \quad X^- = \{n \in \mathbb{Z} : M_\Phi f(n) \geq M_\Phi f(n+1)\}.$$

Then

$$\begin{aligned}\mathrm{Var}(M_\Phi f) &= \sum_{n\in X^+} (M_\Phi f(n+1) - M_\Phi f(n)) + \sum_{n\in X^-} (M_\Phi f(n) - M_\Phi f(n+1)) \\ &\leq \sum_{n\in X^+} (\mathrm{A}_{r_{n+1}}(f)(n+1) - \mathrm{A}_{r_{n+1}+1}(f)(n)) + \sum_{n\in X^-} (\mathrm{A}_{r_n}(f)(n) - \mathrm{A}_{r_n+1}(f)(n+1)). \quad (2.1)\end{aligned}$$

On the other hand, for a fixed $n \in \mathbb{Z}$,

$$\begin{aligned}&\mathrm{A}_{r_{n+1}}(f)(n+1) - \mathrm{A}_{r_{n+1}+1}(f)(n) \\ &\quad = \Phi(2r_{n+1}+1) \sum_{k\in\mathbb{Z}} f(k)\chi_{[n+1-r_{n+1}, n+r_{n+1}+1]}(k) \\ &\qquad - \Phi(2r_{n+1}+3) \sum_{k\in\mathbb{Z}} f(k)\chi_{[n-r_{n+1}-1, n+r_{n+1}+1]}(k) \\ &\quad \leq \sum_{k\in\mathbb{Z}} f(k)\Gamma(r_{n+1})(\chi_{\{n<k\leq n+r_{n+1}+1\}}(n) + \chi_{\{n+1-r_{n+1}\leq k\leq n\}}(n)) \\ &\quad \leq \sum_{k\in\mathbb{Z}} f(k)(\Gamma(k-n-1)\chi_{\{n<k\}}(k) + \Gamma(n-k)\chi_{\{n\geq k\}}(k)), \quad (2.2)\end{aligned}$$

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$$
\begin{aligned}
&\mathrm{A}_{r_n}(f)(n) - \mathrm{A}_{r_n+1}(f)(n+1) \\
&\quad = \Phi(2r_n+1)\sum_{k\in\mathbb{Z}} f(k)\chi_{[n-r_n,n+r_n]}(k) \\
&\qquad - \Phi(2r_n+3)\sum_{k\in\mathbb{Z}} f(k)\chi_{[n-r_n,n+r_n+2]}(k) \\
&\quad \le \sum_{k\in\mathbb{Z}} f(k)\Gamma(r_n)(\chi_{\{n<k\le n+r_n\}}(n) + \chi_{\{n-r_n\le k\le n\}}(n)) \\
&\quad \le \sum_{k\in\mathbb{Z}} f(k)(\Gamma(k-n-1)\chi_{\{n<k\}}(k) + \Gamma(n-k)\chi_{\{n\ge k\}}(k)). \qquad (2.3)
\end{aligned}
$$

It follows from (2.1)–(2.3) that

$$
\begin{aligned}
\mathrm{Var}(M_\Phi f) &\le \sum_{k\in\mathbb{Z}} f(k)\bigg(\sum_{n\in X^+,\,n<k}\Gamma(k-n-1) + \sum_{n\in X^+,\,n\ge k}\Gamma(n-k) \\
&\quad + \sum_{n\in X^-,\,n<k}\Gamma(k-n-1) + \sum_{n\in X^-,\,n\ge k}\Gamma(n-k)\bigg) \\
&= \sum_{k\in\mathbb{Z}} f(k)\bigg(\sum_{n<k}\Gamma(k-n-1) + \sum_{n\ge k}\Gamma(n-k)\bigg) \le 2\Phi(1)\|f\|_{\ell^1(\mathbb{Z})}.
\end{aligned}
$$

2.1.2. Uncentred case. The proof follows similar lines to Section 2.1.1 and we only need to make some modifications. Let $\Upsilon(x) = \Phi(x+1) - \Phi(x+2)$ for any $x \ge 0$. Observe that $\Upsilon(x)$ is decreasing on $[0,\infty)$ and $\sum_{n\in\mathbb{N}}\Upsilon(n) = \Phi(1)$. Since $f \in \ell^1(\mathbb{Z})$, for all $n \in \mathbb{Z}$ there exist $r_n,\ s_n \in \mathbb{N}$ such that

$$
\widetilde{M}_\Phi f(n) = \mathrm{B}_{r_n,s_n}(f)(n) := \Phi(r_n+s_n+1)\sum_{k=-r_n}^{s_n} f(n+k).
$$

Fix $n \in \mathbb{Z}$. By arguments similar to those used to derive (2.2)–(2.3),

$$
\begin{aligned}
&\mathrm{B}_{r_{n+1},s_{n+1}}(f)(n+1) - \mathrm{B}_{r_{n+1},s_{n+1}+1}(f)(n) \\
&\quad \le \sum_{k\in\mathbb{Z}} f(k)(\Upsilon(k-n-1)\chi_{\{n<k\}}(k) + \Upsilon(n-k)\chi_{\{n\ge k\}}(k)), \qquad (2.4) \\
&\mathrm{B}_{r_n,s_n}(f)(n) - \mathrm{B}_{r_n+1,s_n}(f)(n+1) \\
&\quad \le \sum_{k\in\mathbb{Z}} f(k)(\Upsilon(k-n-1)\chi_{\{n<k\}}(k) + \Upsilon(n-k)\chi_{\{n\ge k\}}(k)). \qquad (2.5)
\end{aligned}
$$

Let

$$
\tilde{X}^+ = \{n\in\mathbb{Z} : \widetilde{M}_\Phi f(n+1) > \widetilde{M}_\Phi f(n)\} \quad \text{and} \quad \tilde{X}^- = \{n\in\mathbb{Z} : \widetilde{M}_\Phi f(n) \ge \widetilde{M}_\Phi f(n+1)\}.
$$

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Then, from (2.4)–(2.5),

$$\begin{aligned}\operatorname{Var}(\widetilde{M}_\Phi f) &= \sum_{n\in\tilde{X}^+}(\widetilde{M}_\Phi f(n+1)-\widetilde{M}_\Phi f(n)) + \sum_{n\in\tilde{X}^-}(\widetilde{M}_\Phi f(n)-\widetilde{M}_\Phi f(n+1))\\ &\le \sum_{n\in\tilde{X}^+}(\mathrm{B}_{r_{n+1},s_{n+1}}(f)(n+1)-\mathrm{B}_{r_{n+1},s_{n+1}+1}(f)(n))\\ &\quad+\sum_{n\in\tilde{X}^-}(\mathrm{B}_{r_n,s_n}(f)(n)-\mathrm{B}_{r_n+1,s_n}(f)(n+1))\\ &\le \sum_{k\in\mathbb{Z}} f(k)\Big(\sum_{n<k}\Upsilon(k-n-1)+\sum_{n\ge k}\Upsilon(n-k)\Big)\le 2\Phi(1)\|f\|_{\ell^1(\mathbb{Z})}.\end{aligned}$$

2.2. Continuity part. Let $f_j \to f$ in $\ell^1(\mathbb{Z})$ as $j\to\infty$. Without loss of generality, we may assume that $f_j \ge 0$ for all j and that $f\ge 0$, since $\||f_j|-|f|\| \le |f_j - f|$.

2.2.1. Centred case. Let $\Gamma(x)$ and A_r be as in Section 2.1.1. We want to show that

$$\lim_{j\to\infty}\|(M_\Phi f_j)'-(M_\Phi f)'\|_{\ell^1(\mathbb{Z})}=0. \tag{2.6}$$

For any $\epsilon\in(0,1)$, there exists $N_1 = N_1(\epsilon, f)>0$ such that

$$\|f_j - f\|_{\ell^\infty(\mathbb{Z})}\le \|f_j-f\|_{\ell^1(\mathbb{Z})}<\epsilon \tag{2.7}$$

and

$$\|f_j\|_{\ell^\infty(\mathbb{Z})}\le\|f_j\|_{\ell^1(\mathbb{Z})}\le\|f_j-f\|_{\ell^1(\mathbb{Z})}+\|f\|_{\ell^1(\mathbb{Z})}<\|f\|_{\ell^1(\mathbb{Z})}+1 \tag{2.8}$$

for any $j\ge N_1$. Fix $n\in\mathbb{Z}$ and $j\ge N_1$. It follows from (2.7) that

$$|M_\Phi f_j(n)-M_\Phi f(n)|\le\sup_{r\in\mathbb{N}}\Phi(2r+1)\sum_{k=n-r}^{n+r}|f_j(k)-f(k)|\le\Phi(1)\|f_j-f\|_{\ell^1(\mathbb{Z})}<\Phi(1)\epsilon$$

for any $j\ge N_1$, which implies that $M_\Phi f_j\to M_\Phi f$ pointwise as $j\to\infty$ and

$$\lim_{j\to\infty}(M_\Phi f_j)'(n)=(M_\Phi f)'(n) \tag{2.9}$$

for all $n\in\mathbb{Z}$. By the boundedness part of Theorem 1.4, $(M_\Phi f)'\in\ell^1(\mathbb{Z})$. By the classical Brezis–Lieb lemma in [4], to derive (2.6) it suffices to show that

$$\lim_{j\to\infty}\|(M_\Phi f_j)'\|_{\ell^1(\mathbb{Z})}=\|(M_\Phi f)'\|_{\ell^1(\mathbb{Z})}. \tag{2.10}$$

By (2.9) and Fatou's lemma,

$$\|(M_\Phi f)'\|_{\ell^1(\mathbb{Z})}\le\liminf_{j\to\infty}\|(M_\Phi f_j)'\|_{\ell^1(\mathbb{Z})}.$$

Thus, (2.10) reduces to showing that

$$\limsup_{j\to\infty}\|(M_\Phi f_j)'\|_{\ell^1(\mathbb{Z})}\le\|(M_\Phi f)'\|_{\ell^1(\mathbb{Z})}. \tag{2.11}$$

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We now prove (2.11). Note that there exists a sufficiently large positive integer $R_1 = R_1(\epsilon, f)$ such that

$$\sum_{|n|\geq R_1} f(n) < \epsilon. \tag{2.12}$$

One can easily check that

$$\lim_{|n|\to\infty} M_\Phi f(n) = 0.$$

It follows that there exists an integer $R_2 = R_2(\epsilon) > 0$ such that $M_\Phi f(n) < \epsilon$ for all $|n| \geq R_2$. There also exists an integer $R_3 > 0$ such that $\Phi(2r+1) < \epsilon$ if $r \geq R_3$. Let $R = \max\{R_1, R_2, R_3, 1\}$. By (2.9), there exists $N_2 = N(\epsilon, R) > 0$ such that

$$|(M_\Phi f_j)'(n) - (M_\Phi f)'(n)| \leq \frac{\epsilon}{4R+2} \tag{2.13}$$

for any $j \geq N_2$ and $|n| \leq 2R$. From (2.13),

$$\begin{aligned} \|(M_\Phi f_j)'\|_{\ell^1(\mathbb{Z})} &\leq \sum_{|n|\leq 2R} |(M_\Phi f_j)'(n) - (M_\Phi f)'(n)| + \|(M_\Phi f)'\|_{\ell^1(\mathbb{Z})} + \sum_{|n|\geq 2R} |(M_\Phi f_j)'(n)| \\ &\leq \|(M_\Phi f)'\|_{\ell^1(\mathbb{Z})} + \epsilon + \sum_{|n|\geq 2R} |(M_\Phi f_j)'(n)| \end{aligned} \tag{2.14}$$

for any $j \geq N_2$. Fix $j \geq N_2$ and set

$$X_j^+ = \{|n| \geq 2R : M_\Phi f_j(n+1) > M_\Phi f_j(n)\}, \quad X_j^- = \{|n| \geq 2R : M_\Phi f_j(n) \geq M_\Phi f_j(n+1)\}.$$

Since $f_j \in \ell^1(\mathbb{Z})$, for $n \in \mathbb{Z}$ there exists $r_n \in \mathbb{N}$ such that $M_\Phi f_j(n) = \mathrm{A}_{r_n}(f_j)(n)$. So,

$$\begin{aligned} \sum_{|n|\geq 2R} |(M_\Phi f_j)'(n)| &= \sum_{n\in X_j^+} (M_\Phi f_j(n+1) - M_\Phi f_j(n)) + \sum_{n\in X_j^-} (M_\Phi f_j(n) - M_\Phi f_j(n+1)) \\ &\leq \sum_{n\in X_j^+} (\mathrm{A}_{r_{n+1}}(f_j)(n+1) - \mathrm{A}_{r_{n+1}+1}(f_j)(n)) \\ &\quad + \sum_{n\in X_j^-} (\mathrm{A}_{r_n}(f_j)(n) - \mathrm{A}_{r_n+1}(f_j)(n+1)). \end{aligned} \tag{2.15}$$

By arguments similar to those used in deriving (2.2) and (2.3),

$$\mathrm{A}_{r_{n+1}}(f_j)(n+1) - \mathrm{A}_{r_{n+1}+1}(f_j)(n) \leq \sum_{k\in\mathbb{Z}} f_j(k)(\Gamma(k-n-1)\chi_{\{n<k\}}(k) + \Gamma(n-k)\chi_{\{n\geq k\}}(k)), \tag{2.16}$$

$$\mathrm{A}_{r_n}(f_j)(n) - \mathrm{A}_{r_n+1}(f_j)(n+1) \leq \sum_{k\in\mathbb{Z}} f(k)(\Gamma(k-n-1)\chi_{\{n<k\}}(k) + \Gamma(n-k)\chi_{\{n\geq k\}}(k)). \tag{2.17}$$

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It follows from (2.7), (2.8), (2.12) and (2.15)–(2.17) that

$$
\begin{aligned}
\sum_{|n|\ge 2R} |(M_\Phi f_j)'(n)| &\le \sum_{k\in\mathbb{Z}} f_j(k)\Big(\sum_{n<k,\,|n|\ge 2R} \Gamma(k-n-1) + \sum_{n\ge k,\,|n|\ge 2R} \Gamma(n-k)\Big)\\
&\le \sum_{|k|\ge R} f_j(k)\Big(\sum_{n<k,\,|n|\ge 2R} \Gamma(k-n-1) + \sum_{n\ge k,\,|n|\ge 2R} \Gamma(n-k)\Big)\\
&\quad + \sum_{|k|<R} f_j(k)\Big(\sum_{n<k,\,|n|\ge 2R} \Gamma(k-n-1) + \sum_{n\ge k,\,|n|\ge 2R} \Gamma(n-k)\Big)\\
&\le 2\Phi(1)\|f_j\chi_{\{|k|\ge R\}}\|_{\ell^1(\mathbb{Z})} + \sum_{|k|<R} f_j(k)\Big(\sum_{n\le -2R} \Gamma(k-n-1) + \sum_{n\ge 2R} \Gamma(n-k)\Big)\\
&\le C\epsilon + C\Phi(2R+1) \le C\epsilon
\end{aligned}
$$

for any $j \ge N_1$. Combining this inequality with (2.14) yields

$$\|(M_\Phi f_j)'\|_{\ell^1(\mathbb{Z})} \le \|(M_\Phi f)'\|_{\ell^1(\mathbb{Z})} + C\epsilon$$

for any $j \ge \max\{N_1, N_2\}$, which gives (2.11).

2.2.2. *Uncentred case.* Similar arguments to those in Section 2.2.1 yield the continuity of $\widetilde{M}_\Phi$.

3. Proof of Theorem 1.5

Before giving the proof of Theorem 1.5, we recall an important definition and present two lemmas, which will play a key role in our proof. For a discrete function $f : \mathbb{Z} \to \mathbb{R}$, we say that a point n is a local maximum of f if

$$f(n-1) \le f(n) \quad \text{and} \quad f(n) > f(n+1).$$

LEMMA 3.1. *Let $f : \mathbb{Z} \to \mathbb{R}$ be a bounded function and Φ be as in Theorem 1.5. If n is a local maximum of $\widetilde{M}_\Phi f$, then $\widetilde{M}_\Phi f(n) = \Phi(1)|f(n)|$.*

PROOF. First we claim that there exist $s_0, r_0 \in \mathbb{N}$ such that $s_0 + r_0 \ne 0$ and

$$\widetilde{M}_\Phi f(n) = \Phi(r_0 + s_0 + 1) \sum_{k=-r_0}^{s_0} |f(n+k)|. \tag{3.1}$$

Suppose no such $s_0, r_0 \in \mathbb{N}$ exist with $s_0 + r_0 \ne 0$ and (3.1) holds. We may assume without loss of generality that $\widetilde{M}_\Phi f(n)$ is not attained for any $r \in \mathbb{N}$. Let $\{N_k\}_{k=1}^\infty$ be an increasing sequence of positive integers with $\lim_{k\to\infty} N_k = \infty$. By our assumption,

$$\widetilde{M}_\Phi f(n) = \sup_{r\ge N_k,\, s\in\mathbb{N}} \Phi(r+s+1) \sum_{m=-r}^{s} |f(n+m)| \quad \forall k \ge 1.$$

It follows that for any $\epsilon > 0$ and $k \ge 1$, there exist $r_k \ge N_k$ and $s_k \in \mathbb{N}$ such that

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$$\begin{aligned}\widetilde{M}_\Phi f(n) &\le \Phi(r_k+s_k+1)\sum_{m=-r_k}^{s_k}|f(n+m)|+\epsilon\\ &= \Phi(r_k+s_k+1)\sum_{m=-r_k}^{s_k}|f(n+1+m)|\\ &\quad+\Phi(r_k+s_k+1)(|f(n-r_k)|-|f(n+s_k+1)|)+\epsilon\\ &\le \widetilde{M}_\Phi f(n+1)+\Phi(r_k+s_k+1)\|f\|_{\ell^\infty(\mathbb{Z})}+\epsilon. \end{aligned}\tag{3.2}$$

If $\epsilon \to 0$ and $k \to \infty$, (3.2) leads to $\widetilde{M}_\Phi f(n) \le \widetilde{M}_\Phi f(n+1)$, which is a contradiction.

We may assume without loss of generality that $s_0 \ge 1, r_0 \ge 1$ (since the other cases can be obtained by a simple modification of our arguments). By (3.1), our assumption and the properties of Φ,

$$\begin{aligned}\widetilde{M}_\Phi f(n) &= \Phi(r_0+s_0+1)\Big(|f(n)|+\sum_{k=-r_0+1}^{0}|f(n-1+k)|+\sum_{k=0}^{s_0-1}|f(n+1+k)|\Big)\\ &\le \Phi(r_0+s_0+1)\Big(\frac{1}{\Phi(1)}+\frac{1}{\Phi(s_0)}+\frac{1}{\Phi(r_0)}\Big)\widetilde{M}_\Phi f(n)\\ &\quad+\ \Phi(r_0+s_0+1)\Big(|f(n)|-\frac{1}{\Phi(1)}\widetilde{M}_\Phi f(n)\Big)\\ &\le \widetilde{M}_\Phi(f)(n)+\Phi(r_0+s_0+1)\Big(|f(n)|-\frac{1}{\Phi(1)}\widetilde{M}_\Phi f(n)\Big),\end{aligned}$$

which yields $\widetilde{M}_\Phi f(n) \le \Phi(1)|f(n)|$. Thus, $\widetilde{M}_\Phi f(n) = \Phi(1)|f(n)|$. □

Remark 3.2. Lemma 3.1 implies [3, Lemma 3] when $\Phi(t) = t^{-1}$.

Lemma 3.3. *Let $[a, b]$ be an interval with a, b being integers (or possibly $\pm\infty$) and $f \in \mathrm{BV}(\mathbb{Z})$. Assume that Φ satisfies the conditions in Theorem 1.5. Then*

$$\mathrm{Var}(\widetilde{M}_\Phi f;[a,b]) \le \Phi(1)\mathrm{Var}(f;[a,b]).$$

Proof. We only consider the bounded interval $[a, b]$, since the assertion of Lemma 3.3 for unbounded intervals $[a, b]$ follows easily from this and the fact that $\mathrm{Var}(\widetilde{M}_\Phi f;[a, b])$ is the supremum of $\mathrm{Var}(\widetilde{M}_\Phi f;[a', b'])$ over bounded subintervals $[a', b'] \subset [a, b]$. Suppose that $-\infty < a < b < \infty$. We may assume without loss of generality that a_1 (respectively a_ℓ $(\ell \ge 1)$) is the first (respectively last) local maximum of $\widetilde{M}_\Phi f$. Invoking Lemma 3.1, we have $\widetilde{M}_\Phi f(a_k) = \Phi(1)|f(a_k)|$. Then

$$\begin{aligned}\mathrm{Var}(\widetilde{M}_\Phi f;[a,b]) &= \mathrm{Var}(\widetilde{M}_\Phi f;[a,a_1])+\mathrm{Var}(\widetilde{M}_\Phi f;[a_\ell,b])+\sum_{k=0}^{\ell}\mathrm{Var}(\widetilde{M}_\Phi f;[a_k,a_{k+1}])\\ &\le \widetilde{M}_\Phi f(a_1)-\widetilde{M}_\Phi f(a)+\widetilde{M}_\Phi f(a_\ell)-\widetilde{M}_\Phi f(b)\\ &\quad+\sum_{k=1}^{\ell-1}(\widetilde{M}_\Phi f(a_k)-\widetilde{M}_\Phi f(b_{k+1})+\widetilde{M}_\Phi f(a_{k+1})-\widetilde{M}_\Phi f(b_{k+1}))\end{aligned}$$

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$$\begin{aligned}
&\le \Phi(1)\Big(|f(a_1)| - |f(a)| + |f(a_\ell)| - |f(b)| \\
&\qquad + \sum_{k=1}^{\ell-1}(|f(a_k)| - |f(b_{k+1})| + |f(a_{k+1})| - |f(b_{k+1})|)\Big) \\
&\le \Phi(1)\Big(\mathrm{Var}(f;[a,a_1]) + \mathrm{Var}(f;[a_\ell,b]) \\
&\qquad + \sum_{k=1}^{\ell-1}(\mathrm{Var}(f;[a_k,b_{k+1}]) + \mathrm{Var}(f;[b_{k+1},a_{k+1}]))\Big) \\
&\le \Phi(1)\mathrm{Var}(f;[a,b]).
\end{aligned}$$

This completes the proof of Lemma 3.3. □

Proof of Theorem 1.5. The boundedness part of Theorem 1.5 can be seen as a special case of Lemma 3.3. It remains to show the continuity part. We will indicate here the modifications that have to be made to the proof for the continuity part in Theorem 1.4. Let $f_j \to f$ in $\ell^1(\mathbb{Z})$ as $j \to \infty$. We want to show that

$$\lim_{j\to\infty} \|(\widetilde{M}_\Phi f_j)' - (\widetilde{M}_\Phi f)'\|_{\ell^1(\mathbb{Z})} = 0. \tag{3.3}$$

By the same arguments as in Section 2.2.1, for any $\epsilon \in (0,1)$, there exists $N_1 = N_1(\epsilon, f) > 0$ such that

$$\|f_j - f\|_{\ell^\infty(\mathbb{Z})} \le \|f_j - f\|_{\ell^1(\mathbb{Z})} < \epsilon \tag{3.4}$$

for any $j \ge N_1$. It follows easily from the same arguments as in Section 2.2.1 that in order to establish (3.3), it suffices to show that

$$\limsup_{j\to\infty} \|(\widetilde{M}_\Phi f_j)'\|_{\ell^1(\mathbb{Z})} \le \|(\widetilde{M}_\Phi f)'\|_{\ell^1(\mathbb{Z})}. \tag{3.5}$$

By the same arguments used to derive (2.14), there exist $R > 0$ and $N_2 > 0$ such that

$$\sum_{|n|\ge 2R} f(n) < \epsilon \tag{3.6}$$

and

$$\|(\widetilde{M}_\Phi f_j)'\|_{\ell^1(\mathbb{Z})} \le \|(\widetilde{M}_\Phi f)'\|_{\ell^1(\mathbb{Z})} + \epsilon + \sum_{|n|\ge 2R} |(\widetilde{M}_\Phi f_j)'(n)| \tag{3.7}$$

for any $j \ge N_2$. On the other hand, by Lemma 3.3, (3.4) and (3.6),

$$\begin{aligned}
\sum_{|n|\ge 2R} |(\widetilde{M}_\Phi f_j)'(n)| &\le \mathrm{Var}(\widetilde{M}_\Phi f_j;[2R,\infty)) + \mathrm{Var}(\widetilde{M}_\Phi f_j;(-\infty,-2R]) \\
&\le \Phi(1)(\mathrm{Var}(f_j;[2R,\infty)) + \mathrm{Var}(f_j;(-\infty,-2R])) \\
&\le \Phi(1)(\mathrm{Var}(f_j - f;(-\infty,-2R] \cup [2R,\infty)) + \Phi(1)\mathrm{Var}(f;(-\infty,-2R] \cup [2R,\infty))) \\
&\le 2\Phi(1)\|f_j - f\|_{\ell^1} + 2\Phi(1)\sum_{|n|\ge 2R} f(n) \le 4\Phi(1)\epsilon
\end{aligned} \tag{3.8}$$

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for any $j \geq N_1$. Combining (3.8) with (3.7) yields

$$\|(\widetilde{M}_\Phi f_j)'\|_{\ell^1(\mathbb{Z})} \leq \|(\widetilde{M}_\Phi f)'\|_{\ell^1(\mathbb{Z})} + (4\Phi(1) + 1)\epsilon$$

for all $j \geq \max\{N_1, N_2\}$. This gives (3.5) and completes the proof of Theorem 1.5. □

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