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A SPECTRAL THEORY FOR DUALITY SYSTEMS OF OPERATORS ON A BANACH SPACE

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This note is an addendum to my earlier paper [**8**]. The class of adjoint abelian operators discussed there was small because the compatibility relation between the operator and the duality map was too restrictive. (In effect, the relation is appropriate for Hilbert space, but ill-suited for other Banach spaces where the unit ball is not round.) However, the techniques introduced in [**8**] permit us to readily obtain a spectral theory (of the Dunford type) for a wider class of operators on Banach spaces, as we shall show.

A duality system for the operator T is an ordered sextuple

$$\{T, B, \phi, f, g, c(x, y^*)\}, \text{ where}$$

(i) T is a bounded linear operator mapping the Banach space B into B,

(ii) ϕ is a duality map from B to B^*. Thus, for $x \in B$, $\phi(x) = x^* \in B^*$, where $||x|| = ||x^*||$ and $x^*(x) = ||x||^2$. The existence of ϕ follows easily from the Hahn-Banach Theorem. In general, ϕ is not unique, linear or continuous,

(iii) f and g are functions which are defined and analytic in a neighbourhood of $\sigma(T)$, the spectrum of T. Thus, $f(T)$ and $g(T)$ are well-defined.

(iv) $c(x, y^*)$ is a function from $B \times \phi(B)$ into the real numbers. No conditions are placed on the function $c(\cdot\,,\cdot)$ (other than reality).

As a notational convenience we will write $[x, y^*]$ for $y^*(x)$, where $x \in B$ and $y^* \in B^*$. (Since we can replace y^* by y in the semi-inner product, it is possible to delete the *s, and the reader may do so if he wishes. Consistency with [**8**] has dictated the present policy.) We will say that the operator T satisfies a duality system if $[f(T)x, y^*] = c(x, y^*)[x, (g(T)y)^*]$ for all $x \in B$ and $y^* \in \phi(B)$. For example, if B is a Hilbert space and A is a self-adjoint operator, then $\phi(x) = x$ for $x \in B$ (the map is unique), and $[Ax, y^*] = [x, (Ay)^*]$. Thus, every self-adjoint operator satisfies a duality system where $c(x, y^*) \equiv 1$ and $f(z) = g(z) = z$. (I am grateful to D. Koehler for several stimulating conversations and for pointing out the next example.) If we let $B = L^p(d\mu)$, $1 < p < \infty$, then

$$(\phi x)(t) = \operatorname{sgn} x(t)\, \frac{(x(t))^{p-1}}{||x||_p^{p-2}}$$

is the unique duality map. Let M be multiplication by the independent variable

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t, that is $(Mx)(t) = tx(t)$. Then for p an integer, $p > 1$,

$$[M^{p-1}x, y^*] = \frac{1}{||My||_p^{p-2}} [x, (My)^*].$$

Therefore, the operator M satisfies a duality system, with $f(z) = z^{p-1}$ and $g(z) = z$. (Actually, it is clear what M^p should be for p not an integer, and this case can be handled by *ad hoc* methods.)

We are now ready to state our main result on duality systems.

THEOREM. *Let the operator T satisfy a duality system $\{T, B, \phi, f, g, c(x, y^*)\}$ on the reflexive (or weakly complete) Banach space B. Let $h(z) = f(z)g(z)$. Then, $h(T)$ is a scalar operator.*

Proof. Note that for $x \in B$,

$$\begin{aligned}[h(T)x, x^*] &= [f(T)g(T)x, x^*] = c(g(T)x, x^*)[g(T)x, (g(T)x)^*] \\ &= c(g(T)x, x^*)\, ||g(T)x||^2,\end{aligned}$$

which is real. Moreover, when we repeat this argument n times, we find that

$$\begin{aligned}&[(h(T))^n x, x^*] \\ &\quad = c(h(T)^{n-1}g(T)x, x^*) \ldots c((g(T))^n x, (g(T)^{n-1}x)^*) \cdot [g(T)^n x, (g(T)^n x)^*],\end{aligned}$$

which is real. Hence, $[h(T)]^n$ is a Hermitian operator for $n = 1, 2 \ldots$. (An operator A is Hermitian if $[Ax, x^*]$ is real for all $x \in B$.) Since $[h(T)]^n$ is Hermitian, the spectrum of $h(T)$ is a K-spectral set for $h(T)$. (In slightly disguised form, this fact appears in [**7**, Proposition 19; **8**, Lemma 5].) Thus, we may now apply [**8**, Theorem 2] to conclude that $h(T)$ is a scalar operator. The conclusion also follows from [**2**, Theorem 4.1].

COROLLARY 1. *Let T satisfy the above duality system on the reflexive (weakly complete) Banach space B. Let $h(z) = f(z)g(z)$ and assume that $h'(z) \neq 0$ for $z \in \sigma(T)$. Then T is a scalar operator.*

Proof. Simply apply [**1**, Theorem 3] to the scalar operator $h(T)$.

Remark. The corollary can be extended to cover the case when the points $z_i \in \sigma(T)$, where $h'(z_i) = 0$ are isolated points of $\sigma(T)$. Then T is not a scalar but a spectral operator.

Definition. A subspace M of a Banach space B is hyperinvariant for T if $AT = TA$ implies M is invariant for the operator A. (A, T map B into B.)

The next corollary appears to be new, even for B a Hilbert space.

COROLLARY 2. *Let T satisfy the above duality system on the reflexive (weakly complete) Banach space B. If $T \neq \lambda I$, then T has a non-trivial hyperinvariant subspace.*

Proof. Let A commute with T. Then A commutes with the scalar operator $h(T)$. If $\sigma[h(T)]$ consists of more than a single point, then $h(T)$ has a non-trivial spectral projection E, and EB is our hyperinvariant subspace. On the other hand, if $\sigma[h(T)]$ reduces to a singleton, then $h(T) = I$. Thus,

$$g(T) = h(T) - I \equiv 0,$$

where $g(z)$ is analytic in a neighbourhood of $\sigma(T)$. Under these conditions, it is well known that T must satisfy a polynomial, that is, $p(T) \equiv 0$ for a polynomial p. But then T has an eigenvalue α. Set $M = \{x \in B: Tx = \alpha x\}$. Clearly, $M \neq B$ since $T \neq \lambda I$. It is easy to see that M is hyperinvariant; which completes the proof.

Even though Theorem 1 may appear very general, it is still sufficiently strong to recapture the spectral theorem for a positive self-adjoint operator. Indeed, as we have seen, every self-adjoint operator A satisfies a duality system with $f(z) = g(z) = z$. Hence, A^2 is scalar. Since $\sigma(A)$ is positive, and $A^2x = 0$ implies $Ax = 0$, it follows that A is scalar (see [**3**, Theorem 7]). Thus, $A = \int \lambda \, dE(\lambda)$, where $E(\cdot)$ is an idempotent-valued measure supported on $\sigma(A)$. It is not hard to show that $E(\cdot)$ must actually be self-adjoint, that is, projection-valued.

Note that the theorem permits $c(x, y^*)$ to be identically 0. In that case, $h(T) = 0$, and in fact $f(T) = 0$.

Finally, we note that other results from [**8**] can be extended to the present setting, although we shall make no attempt to do so here.

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