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Laws in finite strictly simple loops

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It is shown that a finite loop with no proper nontrivial subloops has a finite basis for its laws.

1. Introduction

As was mentioned in the survey paper [5] the question of whether a finite loop has a finite basis for its laws appears to be a test case for the conjecture that a finite algebra belonging to a variety all of whose algebras have modular congruence lattices has a finite basis for its laws. So far the only result known is that of Evans [4] which shows that a finite commutative Moufang loop has a finite basis for its laws. The main result of this paper is:

THEOREM. *A finite loop which has no proper nontrivial subloops has a finite basis for its laws.*

(Such a loop will be called a strictly simple loop.)

2. Definitions and preliminary results

Critical algebra and Cross variety of algebras are defined as in [5]. If $\underline{V}$ is a variety, then $\underline{V}^{(n)}$ denotes the variety defined by the laws of $\underline{V}$ involving at most n variables.

If $\underline{V} = \text{var}(A)$ where A is a finite algebra then a result of Birkhoff [1] shows that $\underline{V}^{(n)}$ is finitely based. Thus if we can find an n such that $\underline{V}^{(n)}$ has the other two attributes of a Cross variety, namely

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locally-finiteness and only finitely many (non-isomorphic) critical algebras we will have that $\underline{\underline{V}}$, as a subvariety of a Cross variety, is itself Cross.

We now consider the special case in which A is a finite strictly simple loop. Such a loop is necessarily monogenic. Since the result is well known for cyclic groups of prime order we can assume A has a trivial centre. Definitions and properties of loops used here may be found in Bruck [2].

3. The variety $\underline{\underline{V}}$

LEMMA 3.1. *A finitely generated loop in* $\underline{\underline{V}}$ *is isomorphic to a direct product of a finite number of copies of* A .

Proof. Let B be such a loop; then B is a homomorphic image of a subloop of a direct product of a finite number of copies of A (Birkhoff [1]). Thus it is sufficient to prove that subloops and homomorphic images of finite direct products of copies of A again have the same form.

First we consider subloops. Let

$$B \leq A_1 \times \ldots \times A_r ,$$

where $A_i \simeq A$, and proceed by induction on r ; the result is clearly true if $r = 1$ since A has no proper nontrivial subloops. The projection of B on each factor is either A or 1 and the intersection of B with each factor is either A or 1 . If the intersection with any factor is 1 then B is isomorphic to its projection on the remaining factors and so has the stated form and if the intersection of B with all factors is A then $B = A_1 \times \ldots \times A_r$.

Now suppose $B \simeq G/N$ where $G = A_1 \times \ldots \times A_r$ and $N \trianglelefteq G$. To show B has the required form it is sufficient to show that N is the direct product of some of the A_i , since B will then be isomorphic to the direct product of the remaining factors. This will follow if we can show that N has nontrivial intersection with any factor on which it has nontrivial projection. So suppose $N \trianglelefteq A \times D$ and that N contains a pair (a, d) with $a \neq 1$. Since every inner mapping of A yields an inner mapping of $A \times D$ we have that $(a\theta, d)$ is in N for all θ in $I(A)$. Since A has no centre and $a \neq 1$ there exists a θ such that $a\theta = a' \neq a$ and then $(a, d)\backslash(a', d) = (a\backslash a', 1)$ is in N, so that $N \cap A$ contains the nontrivial element $a\backslash a'$.

DEFINITION 3.2. Let H be a subloop of a loop G, then the centraliser of H in G, $C_G(H)$ is defined by

$$C_G(H) = \{x \mid xh = hx,\ x(h_1h_2) = (xh_1)hx_2,\ (h_1x)h_2 = h_1(xh_2),\ (h_1h_2)x = h_1(h_2x),\ \forall h, h_1, h_2 \in H\} .$$

Note that in general $C_G(H)$ is not a subloop. However, if $H = G$ then it reduces to the centre of G.

LEMMA 3.3. *The centraliser of any subloop of a finite loop* G *in* $\underline{\underline{V}}$ *is a normal subloop of* G.

Proof. Let $H \leq G = A_1 \times \ldots \times A_r$, $A_i \simeq A$. Then H has projection 1 or A on every A_i. Clearly $C_G(H)$ will contain the direct product of those factors on which H has projection 1. On the other hand the projection of any element of $C_G(H)$ on a factor on which H has projection A must lie in the centre of A, and so must be 1. It follows that $C_G(H)$ is the direct product of those factors on which H has projectic 1 1, and so is certainly a normal subloop of G.

LEMMA 3.4. $\underline{\underline{V}}$ *satisfies an antiassociative law* $x \cdot p(x) = 1$ *where* $p(x)$ *is a commutator-associator word.*

Proof. This follows immediately from Theorem 4.1 in Evans [3] since $\underline{\underline{V}}$ contains no nontrivial groups.

4. The variety $\underline{\underline{V}}^{(n)}$

Let $n \geq 6$ and consider $\underline{\underline{V}}^{(n)}$.

LEMMA 4.1. $\underline{\underline{V}}^{(n)}$ *contains no nontrivial groups.*

Proof. Since $n \geq 1$, $\underline{\underline{V}}^{(n)}$ satisfies the antiassociative law of Lemma 3.4 and so contains only the trivial group.

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LEMMA 4.2. *A finitely-generated loop in* $\underline{\underline{V}}^{(n)}$ *is generated by a finite number of loops isomorphic to* A .

Proof. Let $G = \langle x_1, \ldots, x_r \rangle$; then $\langle x_i \rangle \in \underline{\underline{V}}$ (since $n \geq 1$) and so $\langle x_i \rangle$ is a direct product of a finite number of loops isomorphic to A . The totality of all such loops is clearly finite and generates G .

LEMMA 4.3. *Let* $G \in \underline{\underline{V}}^{(n)}$ *and* $H \leq G$ *, then* $C_G(H) \trianglelefteq G$.

Proof. Let $h_1, h_2 \in H$, $c_1, c_2 \in C_G(H)$ and $x, y \in G$. Then $L = \langle h_1, h_2, c_1, c_2, x, y \rangle \in \underline{\underline{V}}$, (since $n \geq 6$). Now $c_1, c_2 \in C_L(\langle h_1, h_2 \rangle)$ and, by Lemma 3.3, this is a normal subloop of L , so that $\langle c_1, c_2 \rangle$, $c_1\theta$, $c_2\theta$ are in $C_L(\langle h_1, h_2 \rangle)$ for all inner mappings θ of L , in particular, for $\theta = R(x)R(y)R(xy)^{-1}$, $L(x)L(y)L(yx)^{-1}$ and $R(x)L(x)^{-1}$. Since h_1, h_2; c_1, c_2; x, y are arbitrary elements of H , $C_G(H)$ and G respectively, it follows that $C_G(H)$ is a normal subloop of G .

LEMMA 4.4. *If* $G = \langle I, J \rangle$ *where* $I \simeq A$ *and* $J \simeq A_1 \times \ldots \times A_r$ *with* $A_i \simeq A$ *, then* G *is a direct product of a finite number of loops isomorphic to* A .

Proof. We proceed by induction on r . The result is certainly true for $r = 1$ since then $\langle I, J \rangle$ belongs to $\underline{\underline{V}}$, so assume $r > 1$, and the result is true for $r - 1$. Then $J = A_1 \times J_2$ and $\langle I, A_1 \rangle = K_1$, $\langle I, J_2 \rangle = K_2$ where K_1 and K_2 are finite direct products of loops isomorphic to A . Let $X = C_G(A_1)$, $Y = C_G(X)$; then both X and Y are normal subloops of G . We now show that $G = XY$. Since $A_2 \times \ldots \times A_r \leq X$ it is sufficient to prove that $K_1 = \langle I, A_1 \rangle \leq XY$. Since $K_1 \in \underline{\underline{V}}$ we have, as in Lemma 3.3, that $C_{K_1}(A_1)$ is the direct product of those factors of K_1 on which A_1 has projection 1 , and so these factors belong to X . It remains to prove that the factors on which

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A_1 has projection A belong to Y. Let $x_1, x_2 \in X$ and consider $H = \langle K_1, x_1, x_2 \rangle$. Since H has four generators it is in $\underline{\underline{V}}$ and so again is a direct product of loops isomorphic to A. If $D_1, \ldots, D_s$ are the factors of K_1 on which A_1 has projection A, then A_1 will have projection A on precisely those factors of H on which some D_i has projection A. Thus x_1, x_2 as elements of $C_H(A_1)$ must belong to those factors of H on which every D_i has projection 1. It follows that if $c \in D_i$, then $cx_1 = x_1c$, $c(xy) = (cx_1)x_2$, $(x_1c)x_2 = x_1(cx_2)$, $(x_1x_2)c = x_1(x_2c)$ so that, since x_1, x_2, were arbitrary elements of X, $D_i \le Y$ as required. Note that this also implies that $A_1 \le Y$.

Now $X \cap Y$ is an abelian group and so is trivial. Thus $G = X \times Y = K_1X = K_2Y$ (since $A_2 \times \ldots \times A_r \le X$, $A_1 \le Y$). Thus $X \simeq K_2Y/Y \simeq K_2/Y \cap K_2$ is a finite direct product of loops isomorphic to A, and so also is Y. It follows that G has the required form.

COROLLARY 4.5. *A finitely generated loop in* $\underline{\underline{V}}^{(n)}$ *is a direct product of a finite number of loops isomorphic to* A.

Proof. By Lemma 4.2, $G = \langle A_1, \ldots, A_s \rangle$ for some (finite) s. Induction on s using Lemma 4.4, now gives the required result.

THEOREM 4.6. $\underline{\underline{V}}^{(n)}$ *is a Cross variety.*

Proof. By Birkhoff's result, [1] Theorem 11, $\underline{\underline{V}}^{(n)}$ has a finite basis for its laws, and Corollary 4.5 shows that finitely generated loops in $\underline{\underline{V}}^{(n)}$ are finite and that A is the only critical loop in $\underline{\underline{V}}^{(n)}$.

Since a subvariety of a Cross variety is Cross the theorem stated in the introduction follows immediately.

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