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MODIFIED DEFECT RELATIONS FOR THE GAUSS MAP OF MINIMAL SURFACES, III

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Dedicated to Professor N. Tanaka for his 60th birthday

§ 1. Introduction

In [5], the author proved that the Gauss map of a nonflat complete minimal surface immersed in $\mathbf{R}^3$ can omit at most four points of the sphere, and in [7] he revealed some relations between this result and the defect relation in Nevanlinna theory on value distribution of meromorphic functions. Afterwards, Mo and Osserman obtained an improvement of these results in their paper [11], which asserts that if the Gauss map of a nonflat complete minimal surface M immersed in $\mathbf{R}^3$ takes on five distinct values only a finite number of times, then M has finite total curvature. The author also gave modified defect relations for holomorphic maps of a Riemann surface with a complete conformal metric into the n-dimensional complex projective space $\mathbf{P}^n(\mathbf{C})$ and, as its application, he showed that, if the (generalized) Gauss map G of a complete minimal surface M immersed in $\mathbf{R}^m$ is nondegenerate, namely, the image $G(M)$ is not contained in any hyperplane in $\mathrm{P}^{m-1}(C)$, then it can omit at most $m(m+1)/2$ hyperplanes in general position ([8]). Here, the number $m(m+1)/2$ is best-possible for arbitrary odd numbers and some small even numbers m (see [6]). Recently, Ru showed that the "nondegenerate" assumption of the above result can be dropped ([13]). In this paper, we shall introduce a new definition of modified defect and prove a refined Modified defect relation. As its application, we shall give some improvements of the above-mentioned results in [5], [7], [8], [11] and [13].

We roughly explain here the modified defect relations given in this paper. More precise statements are given in § 5. Let M be an open Riemann surface with a conformal metric ds^2 and consider a nondegenerate

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holomorphic map f of M into $\mathbf{P}^n(\mathbf{C})$. We denote by Ω_f the pull-back of Fubini-Study metric on $\mathbf{P}^n(\mathbf{C})$ through f, and by $f^*(H)^{[n]}$ the n-truncated pull-back of a hyperplane H in $\mathbf{P}^n(\mathbf{C})$. In some cases, we regard them as $(1, 1)$-currents on M.

We define the modified H-defect of H for f by

$$D_f(H) := 1 - \inf\{\eta > 0;\ f^*(H)^{[n]} \prec \eta\Omega_f \text{ on } M - K \text{ for some compact set } K\}.$$

Here, by $\Omega_1 \prec \Omega_2$ we mean that there are a divisor ν and a bounded real-valued function k with mild singularities such that $\nu \geq c$ on the support $|\nu|$ of ν for a positive constant c and

$$\Omega_1 + [\nu] = \Omega_2 + dd^c \log k^2$$

holds as currents, where $[\nu]$ denotes the current corresponding to ν.

We also define the H-order of f by

$$\rho_f := \inf\{\rho > 0;\ -\mathrm{Ric}_{ds^2} \prec \rho\Omega_f \text{ on } M - K \text{ for some compact set } K\}.$$

After Chen [2] we say that hyperplanes H_j $(1 \leq j \leq q)$ are located in N-subgeneral position if $H_{j_0} \cap \cdots \cap H_{j_N} = \varnothing$ for all $1 \leq j_0 < \cdots < j_N \leq q$, where $q > N \geq n$. Particularly, we say that H_j $(1 \leq j \leq q)$ are in general position if they are in n-subgeneral position.

The modified defect relation proved in this paper is stated as follows:

THEOREM 1.1. *Let M be an open Riemann surface with a complete conformal metric ds^2, and let f be a nondegenerate holomorphic map of M into $\mathbf{P}^n(\mathbf{C})$ For the particular case where M is biholomorphic with a compact Riemann surface $\overline{M}$ with finitely many points removed, we assume that f cannot be extended to a holomorphic map of $\overline{M}$ into $\mathbf{P}^n(\mathbf{C})$. Then, for arbitrary hyperplanes $H_1, \cdots, H_q$ in $\mathbf{P}^n(\mathbf{C})$ located in N-subgeneral position,*

$$\sum_{j=1}^{q} D_f(H_j) \leq (2N - n + 1)\left(1 + \frac{\rho_f n}{2}\right). \tag{1.2}$$

Let M be a (connected, oriented) minimal surface immersed in $\mathbf{R}^m$. We can regard M as an open Riemann surface with a conformal metric. The set of all oriented 2-planes in $\mathbf{R}^m$ can be identified with the quadric

$$Q_{m-2}(\mathbf{C}) = \{(w_1 : \cdots : w_m);\ w_1^2 + \cdots + w_m^2 = 0\} (\subseteq \mathbf{P}^{m-1}(\mathbf{C})).$$

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By definition, the Gauss map of M is the map $G : M \to Q_{m-2}(\mathbf{C})$ $(\subset \mathbf{P}^{m-1}(\mathbf{C}))$ which maps each $p \in M$ to the point in $Q_{m-2}(\mathbf{C})$ corresponding to the oriented tangent plane of M at p. By the assumption of M, G is a holomorphic map of M into $\mathbf{P}^{m-1}(\mathbf{C})$. We can conclude from Theorem 1.1 the following:

THEOREM 1.3. *Let M be a nonflat complete minimal surface immersed in $\mathbf{R}^m$ with infinite total curvature, and let G be the Gauss map of M. Then, for arbitrary hyperplanes $H_1, \cdots, H_q$ in $\mathbf{P}^{m-1}(\mathbf{C})$ located in general position,*

$$\sum_{j=1}^{q} G_G(H_j) \leq \frac{m(m+1)}{2}.$$

For a holomorphic map f of an open Riemann surface M into $\mathbf{P}^n(\mathbf{C})$ and a hyperplane H in $\mathbf{P}^n(\mathbf{C})$ we can show that $D_f(H) = 1$ if $f^{-1}(H)$ is finite. This yields the following improvement of a result of Ru ([13]):

COROLLARY 1.4. *Let M be a nonflat complete minimal surface immersed in $\mathbf{R}^m$ with infinite total curvature, and let G be the Gauss map of M. If $G^{-1}(H_j)$ are finite for q hyperplanes $H_1, \cdots, H_q$ in $\mathbf{P}^{m-1}(\mathbf{C})$ located in general position, then $q \leq m(m+1)/2$.*

In the case $m = 3$, $Q_1(\mathbf{C})$ can be identified with $\mathbf{P}^1(\mathbf{C})$. The Gauss map G of M is considered as a holomorphic map of M into $\mathbf{P}^1(\mathbf{C})$ with $\rho_G \leq 2$. Theorem 1.1 implies the above-mentioned result given by Mo and Osserman.

In this paper, we also consider minimal surface with branch points. We introduce the new notion of branching H-order for minimal surfaces and give modified defect relations for the Gauss map which relate to this notion.

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§ 2. Nochka weights for hyperplanes in subgeneral position

For later use, we recall an algebraic theorem given by Nochka in [12] (cf. [2]).

We consider q hyperplanes H_j $(1 \leq j \leq q)$ in $\mathbf{P}^n(\mathbf{C})$ which are given by

$$H_j : (A_j, W) = 0 \quad (1 \leq j \leq q)$$

for $A_j \in \mathbf{C}^{n+1} - \{0\}$, where $q > N \geq n$ and (A, W) means $a_0w_0 + \cdots + a_nw_n$ for a vector $A=(a_0, \cdots, a_n)$ and homogeneous coordinates $W=(w_0:\cdots: w_n)$.

After Chen ([2]), we give the following definition.

DEFINITION 2.1. We say that hyperplanes $H_1, \cdots, H_q$ (or vectors $A_1, \cdots, A_q$) are in *N-subgeneral position* if, for every $1 \leq j_0 < \cdots < j_N \leq q$, $A_{j_0}, A_{j_1}, \cdots, A_{j_N}$ generate $\mathbf{C}^{n+1}$, or equivalently,

$$H_{j_0} \cap \cdots \cap H_{j_N} = \varnothing .$$

We assume that $H_1, \cdots, H_q$ are in N-subgeneral position in the following.

Set $Q := \{1, 2, \cdots, q\}$. For each $R \subseteq Q$ we denote by $\sharp R$ the number of elements of R and by $d(R)$ the dimension of the vector subspace of $\mathbf{C}^{n+1}$ generated by $\{A_j; j \in R\}$. For convenience sake, we set $d(\varnothing) = 0$.

In [12], Nochka has given the following theorem to prove Cartan's conjecture.

THEOREM 2.2. *Let $H_1, H_2, \cdots, H_q$ be hyperplanes in $\mathbf{P}^n(\mathbf{C})$ located in N-subgeneral position, where $q > 2N - n + 1$. Then there are some constants $\omega(1), \cdots, \omega(q)$ and θ satisfying the following conditions*;

(i) $0 < \omega(j) \leq \theta \leq 1 \quad (1 \leq j \leq q)$,

(ii) $\sum_{j=1}^{q} \omega(j) = n + 1 + \theta(q - 2N + n - 1)$,

(iii) $\dfrac{n+1}{2N-n+1} \leq \theta \leq \dfrac{n+1}{N+1}$,

(iv) *if $R \subset Q$ and $0 < \sharp R \leq N + 1$, then* $\sum_{j \in R} \omega(j) \leq d(R)$.

For the proof, see [12] or]2].

DEFINITION 2.3. We call constants $\omega(j)$ $(1 \leq j \leq q)$ and θ with the properties (i)~(iv) *Nochka weights* and a *Nochka constant* for $H_1, \cdots, H_q$ respectively.

By definition, H_j $(1 \leq j \leq q)$ are in general position if and only if they are in n-subgeneral position. If $H_1, \cdots, H_q$ are in general position, then we have necessarily $\omega(1) = \cdots = \omega(q) = \theta = 1$.

We give here a property of Wronskians relating to Nochka weights. Let $f_0, \cdots, f_n$ be holomorphic functions without common zeros on a domain D in $\mathbf{C}$. We denote the Wronskian of $f_0, \cdots, f_n$ by

$$W(f_0, \cdots, f_n) = \det(f_m^{(l)}; 0 \leq l, m \leq n) .$$

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Assume that $f_0, \cdots, f_n$ are linearly independent over $\mathbf{C}$, or $W(f_0, \cdots, f_n) \not\equiv 0$. Consider q vectors

$$A_j = (a_{j0}, a_{j1}, \cdots, a_{jn}) \in \mathbf{C}^{n+1} \quad (1 \leq j \leq q)$$

located in N-subgeneral position and take Nochka weights $\omega(j)$ $(1 \leq j \leq q)$, where $q > 2N - n + 1$. Set

$$F_j = a_{j0}f_0 + \cdots + a_{jn}f_n \quad (1 \leq j \leq q)\,.$$

For a nonzero meromorphic function ψ on D we define the map $\nu_\psi : D \to \mathbf{Z}$ by

$$\nu_\psi(\zeta) := \text{the order of the meromorphic function } \psi \text{ at } \zeta\,.$$

Consider a function

$$\varphi = \frac{|W(f_0, \cdots, f_n)|}{|F_1|^{\omega(1)} \cdots |F_q|^{\omega(q)}}$$

and set

$$\nu_\varphi = \nu_W - \sum_{j=0}^{q} \omega(j)\nu_{F_j}\,,$$

where $W = W(f_0, \cdots, f_n)$.

LEMMA 2.4. *In the above situation, we have*

$$\nu_\varphi + \sum_{j=1}^{q} \omega(j)\min(\nu_{F_j}, n) \geq 0\,.$$

Proof. It suffices to show that

$$\nu_W \geq \sum_{j=1}^{q} \omega(j)(\nu_{F_j} - n)^+\,, \tag{2.5}$$

where x^+ means $\max(x, 0)$ for a real number x. In fact, since

$$\min(\nu_{F_j}, n) + (\nu_{F_j} - n)^+ = \nu_{F_j}\,,$$

we can conclude from (2.5)

$$\begin{aligned}\nu_\varphi + \sum_{j=0}^{q} \omega(j)\min(\nu_{F_j}, n) &= \nu_W - \sum_{j=1}^{q} \omega(j)\nu_{F_j} + \sum_{j=1}^{q} \omega(j)\min(\nu_{F_j}, n) \\ &\geq \sum_{j=1}^{q} \omega(j)((\nu_{F_j} - n)^+ + \min(\nu_{F_j}, n) - \nu_{F_j}) = 0\,.\end{aligned}$$

To prove (2.5), take a point $\zeta \in D$, and set $Q := \{1, 2, \cdots, q\}$ and

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$$S: = \{j \in Q; \nu_{F_j}(\zeta) \geq n + 1\} .$$

We may assume that $S \neq \emptyset$. Then, $\sharp S \leq N$. For, otherwise, by N-subgenerality of A_j's there are $n + 1$ distinct $j_0, \cdots, j_n \in S$ such that $A_{j_0}, \cdots, A_{j_n}$ are linearly independent. So, $f_0, \cdots, f_n$ are represented as linear combinations of $F(H_{j_0}), \cdots, F(H_{j_n})$, whence $f_0, \cdots, f_n$ have a common zero at ζ. We now consider the sets S_τ $(0 \leq \tau \leq t)$ such that

$$S_0: = \emptyset \neq S_1 \subset S_2 \subset \cdots \subset S_t = S$$

and for each $j \in S_\tau - S_{\tau-1}$ $\nu_{F_j}(\zeta)$ equals some constant m_τ, where $m_1 > m_2 > \cdots > m_t \geq n + 1$. Let $V(R)$ be the vector space generated by $\{A_j; j \in R\}$ for each $R \subseteq Q$. Then,

$$V(S_1) \subseteq V(S_2) \subseteq \cdots \subseteq V(S_t) .$$

For each τ take a subset T_τ of S_τ such that $T_{\tau-1} \subset T_\tau$ and $\{A_j; j \in T_\tau\}$ gives a basis of $V(S_\tau)$. We then have $\sharp(T_\tau - T_{\tau-1}) = d(S_\tau) - d(S_{\tau-1})$. For brevity, we set $m_\tau^*: = m_\tau - n$. By the use of Theorem 2.2, (iv) we obtain

$$\begin{aligned}
\sum_{j=1}^{q} \omega(j)(\nu_{F_j} - n)^+ &= \sum_{j \in S} \omega(j)(\nu_{F_j} - n)^+ = \sum_{\tau=1}^{t} \sum_{j \in S_\tau - S_{\tau-1}} \omega(j) m_\tau^* \\
&= \sum_{\tau=1}^{t} \sum_{j \in S_\tau - S_{\tau-1}} \omega(j)\left(\left(\sum_{\sigma=\tau}^{t-1} m_\sigma^* - m_{\sigma+1}^*\right) + m_t^*\right) \\
&= (m_1^* - m_2^*) \sum_{j \in S_1} \omega(j) + (m_2^* - m_3^*) \sum_{j \in S_2} \omega(j) + \cdots + m_t^* \sum_{j \in S_t} \omega(j) \\
&\leq d(S_1)(m_1^* - m_2^*) + d(S_2)(m_2^* - m_3^*) + \cdots + d(S_t) m_t^* \\
&= d(S_1) m_1^* + (d(S_2) - d(S_1)) m_2^* + \cdots + (d(S_t) - d(S_{t-1})) m_t^* \\
&= \sharp T_1 m_1^* + \sharp(T_2 - T_1) m_2^* + \cdots + \sharp(T_t - T_{t-1}) m_t^* .
\end{aligned}$$

Set $T_t = \{j_0, \cdots, j_k\}$, where $k \leq n$. Since $A_{j_0}, \cdots, A_{j_k}$ are linearly independent, after a suitable nonsingular linear transformation of homogeneous coordinates, we may assume that $f_0 = F_{j_0}, \cdots, f_k = F_{j_k}$. Then, by the Laplace' expansion theorem of determinants, W is expanded as the sum of the products of some minors of degree $n - k$ and some minors of degree $k + 1$ whose components consist of the $\leq n$-th derivatives of F_{j_0}, $\cdots, F_{j_k}$. This implies that

$$\nu_W(\zeta) \geq \sum_{j \in T_t} (\nu_{F_j}(\zeta) - n) .$$

Since $\nu_{F_j}(\zeta) = m_\tau$ for every $j \in T_\tau - T_{\tau-1}$ this quantity coincides with the last term of the above inequalities. This completes the proof of Lemma 2.4.

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§ 3. Some properties of the contact functions for derived curves

We recall some known results on the derived curves for a holomorphic map of an open Riemann surface M into $\mathbf{P}^n(\mathbf{C})$.

Let $f: M \to \mathbf{P}^n(\mathbf{C})$ be a nondegenerate holomorphic map. Take a reduced representation $f = (f_0 : \cdots : f_n)$ on M with holomorphic functions f_i on M having no common zeros. Set $\|f\| = (\sum_{0 \le j \le n} |f_i|^2)^{1/2}$ and $F = (f_0, \cdots, f_n)$. For each $p \in M$, taking a holomorphic local coordinate on some neighborhood U of p, we set $F^{(l)} = (f_0^{(l)}, \cdots, f_n^{(l)})$ for each $l = 0, 1, \cdots$, and define the map

$$F_k := F^{(0)} \wedge F^{(1)} \wedge \cdots \wedge F^{(k)} : U \longrightarrow \bigwedge^{k+1} \mathbf{C}^{n+1} \quad (0 \le k \le n-1),$$

where $F_0 = F^{(0)} := F$. Let $\pi : \bigwedge^{k+1} \mathbf{C}^{n+1} - \{0\} \to \mathrm{P}^{N_k}(\mathbf{C})$ denote the canonical projection, where $N_k := \binom{n+1}{k+1} - 1$. By definition, the k-th derived curve of f is the map $f^k = \pi \cdot F_k$, which does not depend on the choice of a holomorphic local coordinate and so is well-defined on the totality of M. We locally define

$$|F_k| = \Big(\sum_{0 \le j_0 < \cdots < j_k \le n} |W(f_{j_0}, \cdots, f_{j_k})|^2\Big)^{1/2} \quad (0 \le k \le n-1).$$

By definition, the pull-back Ω_k of the Fubini-Study metric on $\mathbf{P}^{N_k}(\mathbf{C})$ by f^k is locally given by

$$\Omega_k = dd^c \log |F_k|^2,$$

where $d^c = (\sqrt{-1}/4\pi)(\bar{\partial} - \partial)$. Take a hyperplane

$$H : (A, W) = 0,$$

where $A = (a_0, \cdots, a_n)$ is a unit vector. Set

$$|F_k(H)|^2 = |F_k \vee A|^2 := \sum_{0 \le i_1 < \cdots < i_k \le n} \Big| \sum_{j \ne i_1, \cdots, i_k} a_j W(f_j, f_{i_1}, \cdots, f_{i_k}) \Big|^2,$$

where $\vee$ denotes the interior product as in [14]. We define the contact functions

$$\varphi_k(H)(z) := \frac{|F_k(H)(z)|^2}{|F_k(z)|^2},$$

which are well-defined on the totality of M and do not vanish identically. For $k = 0$, we have $F_0(H) = F(H) := (A, F)$ and $\varphi_0(H) = |F(H)|^2/\|f\|^2$.

We now give the following so-called sums into products estimates.

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THEOREM 3.1. *Let $H_1, \cdots, H_q$ be hyperplanes in $\mathbf{P}^n(\mathbf{C})$ located in N-subgeneral position and $\omega(j)$ $(1 \leq j \leq q)$ be Nochka weights for these hyperplanes. For an arbitrarily given $a > 1$ and $0 \leq k \leq n-1$ set*

$$\Phi_{jk} := \frac{\varphi_{k+1}(H_j)}{\varphi_k(H_j)\log^2(a/\varphi_k(H_j))}.$$

Then, there exists a positive constant C_k depending only on k and H_j such that

$$\sum_{j=1}^{q} \omega(j)\Phi_{jk} \geq C_k\left(\prod_{j=1}^{q} \Phi_{jk}^{\omega(j)}\right)^{1/(n-k)} \tag{3.2}$$

on $M - \bigcup_{1\leq j\leq q}\{z; \varphi_k(H_j)(z) = 0\}$.

For the case where H_j are in general position, this is classical (cf., e.g., [4] or [14]). For the general case, the proof is given by similar arguments by the use of Theorem 2.2 (cf., [2]). Since the description of Chen is lengthy, we give here a shorter proof.

For the proof, we use the following elementary inequality.

(3.3) *For all positive number $x_1, \cdots, x_n$ and $a_1, \cdots, a_n$,*

$$a_1x_1 + \cdots + a_nx_n \geq (a_1 + \cdots + a_n)(x_1^{a_1}\cdots x_n^{a_n})^{1/(a_1+\cdots+a_n)}.$$

Proof of Theorem 3.1. Let

$$H_j : (W, A_j) = 0 \quad (1 \leq j \leq q),$$

where we choose A_j with $|A_j| = 1$ for each j.

We consider the set $\mathscr{R}_k$ of all subsets R of $Q := \{1, 2, \cdots, q\}$ such that $d(R) \leq n-k$. For each $P \in G(n, k)$ we take a nonzero decomposable $(k+1)$-vector E such that $P = \{X \in \mathbf{C}^{n+1}; E \wedge X = 0\}$ and set

$$\psi_k(P) = \max_{R\in\mathscr{R}_k} \min\left\{\frac{|E \vee A_j|^2}{|E|^2}; j \notin R\right\},$$

which depends only on P. We regard ψ_k as a function on the Grassmann manifold $G(n, k)$. For each nonzero vector $E = E_0 \wedge \cdots \wedge E_k$ we set

$$R := \{j \in Q; E \vee A_j = 0\}.$$

Since $E \vee A_j = 0$ means that A_j is contained in the orthogonal complement of the vector space generated by $E_0, \cdots, E_k$, we see $d(R) \leq n-k$, namely, $R \in \mathscr{R}_k$. This yields that ψ_k is positive everywhere on $G(n, k)$.

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Since ψ_k is continuous on the compact space $G(n, k)$, it has a positive lower bound δ on $G(n, k)$.

Take a point z with $F_k(z) \neq 0$. Since $F_k(z)$ determines a point $P \in G(n, k)$, there is a set R in $\mathscr{R}_k$ such that $\varphi_k(H_j) = |E \vee A_j|^2/|E|^2 = \psi_k(P) \geq \delta$ for all $j \notin R$. Therefore, we can choose a finite positive constant K depending only on H_j's such that $\Phi_{jk}(z) \leq K$ for all $j \notin R$. Set $S = \{j; \Phi_{jk} > K\}$. We may assume $S \neq \emptyset$. In fact, otherwise, we have

$$\sum_{j=1}^{q} \omega(j)\Phi_{jk} \geq Kl_0\left(\prod_{j=1}^{q}\left(\frac{\Phi_{jk}}{K}\right)^{\omega(j)/l_0}\right) \geq Kl_0\left(\prod_{j=1}^{q}\left(\frac{\Phi_{jk}}{K}\right)^{\omega(j)}\right)^{1/(n-k)},$$

where $l_0 := \sum_{j=1}^{q} \omega(j)$. We see $S \subseteq R$ and so $d(S) \leq n - k$. Then, $l := \sum_{j \in S} \omega(j) \leq n - k$ by Theorem 2.2, (iv) and $\sharp S \leq N$ by the assumption. By the help of (3.3) we obtain

$$\begin{aligned}\sum_{j=1}^{q} \omega(j)\Phi_{jk} &\geq \sum_{j \in S} \omega(j)\Phi_{jk} \geq Kl \prod_{j \in S}\left(\frac{\Phi_{jk}}{K}\right)^{\omega(j)/l} \\ &\geq Kl \prod_{j \in S}\left(\frac{\Phi_{jk}}{K}\right)^{\omega(j)/(n-k)} \geq C \prod_{j=1}^{q}\left(\frac{\Phi_{jk}}{K}\right)^{\omega(j)/(n-k)}\end{aligned}$$

for some constant C. Eventually, we can find a positive constant C_k satisfying (3.2), which depends only on k and $H_1, \cdots, H_q$. This gives Theorem 3.1.

Now, we give the following improvement of [8, Proposition 3.5].

THEOREM 3.4. *Let $H_1, \cdots, H_q$ be hyperplanes in $\mathbf{P}^n(\mathbf{C})$ located in N-subgeneral position and let $\omega(j)(1 \leq j \leq q)$ and θ be Nochka weights and a Nochka constant for these hyperplanes, where $q > 2N - n + 1$. For every $\varepsilon > 0$ there exist some positive numbers $a(>1)$ and C depending only on ε and H_j $(1 \leq j \leq q)$ such that*

$$\begin{aligned}(3.5)\qquad & dd^c \log \frac{\prod_{k=0}^{n-1} |F_k|^{2\varepsilon}}{\prod_{1 \leq j \leq q, 0 \leq k \leq n-1} \log^{2\omega(j)}(a/\varphi_k(H_j))} \\ & \geq C\left(\frac{|F_0|^{2\theta(q-2N+n-1)}|F_n|^2}{\prod_{j=1}^{q}(|F(H_j)|^2 \prod_{k=0}^{n-1} \log^2(a/\varphi_k(H_j)))^{\omega(j)}}\right)^{2/(n(n+1))} \sqrt{-1}dz \wedge d\bar{z}.\end{aligned}$$

Proof. The proof is similar to that for the case where H_j are in general position. We state here only its outline. As is proved in [4, p. 129] (or [14, p. 122]), for each positive ε there exists a constant a_0 depending only on ε such that for every hyperplane H in $\mathbf{P}^n(\mathbf{C})$ and $a \geq a_0$

$$dd^c \log \frac{1}{\log(a/\varphi_k(H))} \geq \frac{\varphi_{k+1}(H)}{\varphi_k(H)\log^2(a/\varphi_k(H))}\Omega_k - \frac{\varepsilon}{l_0}\Omega_k,$$

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where $l_0 = \sum_{j=1}^{q} \omega(j)$ and $\Omega_k = dd^c \log |F_k|^2$. Choosing some $a \geq a_0$, we denote the left hand side of (3.5) by A. According to Theorem 3.1 and (3.3) we get

$$\begin{aligned} A &\geq \varepsilon \sum_{k=0}^{n-1} \Omega_k + \sum_{j=1}^{q} \omega(j) \sum_{k=0}^{n-1} \left(\frac{2\varphi_{k+1}(H_j)}{\varphi_k(H_j) \log^2(a/\varphi_k(H_j))} - \frac{\varepsilon}{l_0} \right) \Omega_k \\ &= \sum_{k=0}^{n-1} 2 \left(\sum_{j=1}^{q} \omega(j) \Phi_{jk} \right) \Omega_k \geq C_1 \sum_{k=0}^{n-1} \left(\prod_{j=0}^{q} \Phi_{jk}^{\omega(j)} \right)^{1/(n-k)} \Omega_k \\ &\geq C_2 \prod_{k=0}^{n-1} \left(\prod_{j=1}^{q} \Phi_{jk}^{\omega(j)} \lambda_k^{2(n-k)} \right)^{2/(n(n+1))} \sqrt{-1} dz \wedge d\bar{z} \end{aligned}$$

for some positive constants C_1 and C_2, where $\Omega_k = \lambda_k^2 \sqrt{-1} dz \wedge d\bar{z}$. Observe

$$\prod_{k=0}^{n-1} \Phi_{jk} = \frac{|F_0|^2}{|F(H_j)^2} \prod_{k=0}^{n-1} \frac{1}{\log^2(a/\varphi_k(H_j))}$$

and

$$\prod_{k=0}^{n-1} \lambda_k^{2(n-k)} = \frac{|F_n|^2}{|F_0|^{2(n+1)}}$$

as in the previous paper [8, p. 373]. We obtain

$$A \geq C \left(\frac{|F_0|^{2(l_0-n-1)} |F_n|^2}{\prod_{j=1}^{q} (|F(H_j)|^2 \prod_{k=0}^{n-1} \log^2(a/\varphi_k(H_j)))^{\omega(j)}} \right)^{2/(n(n+1))} \sqrt{-1} dz \wedge d\bar{z} .$$

Since $l_0 - n - 1 = \theta(q - 2N + n - 1)$ by Theorem 2.2, (ii), we can conclude Theorem 3.4.

§ 4. Metrics with negative curvature

Let M be an open Riemann surface. We consider a function u on M possibly with singularities in a discrete subset of M.

DEFINITION 4.1. We call u a function with *mild singularities* on an open subset D of M if u is a C^∞ function on D except a discrete subset E of D and for every $a \in D$ we can write

$$|u(z)| = |z|^{\sigma} \prod_i |\log |g_i(z) v_i(z)||^{\tau_i} u^*(z) \tag{4.2}$$

around a with real numbers σ and τ_i, positive C^∞ functions u^*, v_i and some nonzero holomorphic functions $g_i(z)$ satisfying the condition $g_i(0) = 0$, where z is a holomorphic local coordinate with $z(a) = 0$ around a. We mean by a continuous function on D with mild singularities a function with mild singularities which is complex-valued and continuous on the totality of D.

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Let u be a function with mild singularities on D. We define a map ν_u of D into $\mathbf{R}$ by

$\nu_u(a)$: = the number σ with the property (4.2) for some τ and u^*

for each $a \in D$, which does not depend on a choice of a holomorphic local coordinate z. By a divisor on M we mean a map ν of M into $\mathbf{R}$ such that the support $|\nu|$: $= \overline{\{p; \nu(p) \neq 0\}}$ has no accumulation point in M. The above ν_u is a divisor on D, which we call the divisor of u. The divisor of a nonzero meromorphic function ψ is nothing but the divisor ν_ψ stated in § 2.

DEFINITION 4.3. We call ds^2 a (conformal) *pseudo-metric* on M if for each $a \in M$ we may write $ds^2 = \lambda_z^2 |dz|^2$ around a with a nonnegative function λ_z with mild singularities and a holomorphic local coordinate z. For a pseudo-metric $dz^2 = \lambda_z^2 |dz|^2$ we define the divisor of ds^2 by $\nu_{ds} = \nu_{\lambda_z}$. A continuous pseudo-metric ds^2 means a pseudo-metric such that λ_z is continuous.

DEFINITION 4.4. For a pseudo-metric $ds^2 = \lambda_z^2 |dz|^2$ on M we define the *Ricci form* by

$$\mathrm{Ric}_{ds^2} = -dd^c \log \lambda_z^2$$

as a current, which is well-defined on the totality of M because it does not depend on the choice of holomorphic local coordinate z. We define also the total curvature $C(M)$ of M by $C(M) = -\int_M \mathrm{Ric}_{ds^2}$ for a conformal metric ds^2 on M.

The *Gaussian curvature* $\mathscr{K}_{ds^2}$ of ds^2 is defined by

$$\mathscr{K}_{ds^2} = -\frac{\Delta \log \lambda_z}{\lambda_z^2} = -\frac{4(\partial^2/\partial z \partial \bar{z}) \log \lambda_z}{\lambda_z^2}$$

only on the set where $ds^2 \neq 0$. The curvature $\mathscr{K}_{ds^2}$ is said to be strictly negative if there exists a positive constant C such that $\mathrm{Ric}_{ds^2} \leq -C\Omega_{ds^2}$ on M as currents, where Ω_{ds^2} denotes the area element with respect to the metric ds^2. As is well-known, if the universal covering surface of M is biholomorphic with the unit disc in $\mathbf{C}$, then M has the complete conformal metric with constant curvature -1 which is called the Poincaré metric of M and denoted by $d\sigma_M^2$ in the following.

Let ν be a divisor on M. We denote by $[\nu]$ the current which corresponds to ν, namely the map of $\mathscr{D}$ into $\mathbf{C}$ defined by

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$$[\nu](\varphi) = \int_M \nu\varphi = \sum_{z\in M} \nu(z)\varphi(z)\,,$$

where $\mathscr{D}$ denotes the space of all C^∞ differentiable functions on M with compact supports. In some cases, a (1, 1)-form Ω on M is regarded as a current on M by defining $\Omega(\varphi) := \int_M \varphi\Omega$ for each $\varphi \in \mathscr{D}$.

For two (1, 1)-currents Ω_1, Ω_2 on some open set U and a positive constant c, by the notation $\Omega_1 \prec_c \Omega_2$ we mean that there are divisor ν and a continuous real-valued bounded function k with mild singularities on U such that $\nu(z) \geq c$ for each $z \in |\nu|$ and

$$\Omega_1 + [\nu] = \Omega_2 + dd^c \log k^2$$

on U. For brevity, we mean by $\Omega_1 \prec \Omega_2$ or $\Omega_2 \succ \Omega_1$ that $\Omega_1 \prec_c \Omega_2$ for some positive number c.

Obviously, we have the following:

PROPOSITION 4.5. (i) *If $\Omega_1 \prec_c \Omega_2$ and $\Omega_2 \prec_c \Omega_3$ for a constant $c > 0$, then $\Omega_1 \prec_c \Omega_3$.*

(ii) *If $\Omega_1 \prec \Omega_2$, then $a\Omega_1 + \Omega \prec a\Omega_2 + \Omega$ for every Ω and $a > 0$.*

The purpose of this section is to prove the following:

THEOREM 4.6. *Let M be an open Riemann surface with a complete continuous pseudo-metric ds^2 and let $d\tau^2$ be a continuous pseudo-metric on M whose curvature is strictly negative outside a compact set K. Assume that there exists a constant p with $0 < p < 1$ such that*

$$-\operatorname{Ric}_{ds^2} \prec_{1-p} p(-\operatorname{Ric}_{d\tau^2})$$

on $M - K$. Then M is of finite type, namely M is biholomorphic with a compact Riemann surface from which finitely many points have been removed.

For the proof of Theorem 4.6, we recall the following generalization of Schwarz lemma.

LEMMA 4.7. *Let $d\tau^2$ be a continuous pseudo-metric on $\Delta_R := \{w; |w| < R\}$ whose curvature is strictly negative. Then, for some positive constant C_0,*

$$d\tau^2 \leq C_0 d\sigma^2_{\Delta_R}\,.$$

For the proof, see [1, pp. 12$\sim$14].

We need also the following result given by A. Huber.

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THEOREM 4.8. *For an open Riemann surface M, if there is a complete conformal metric ds^2 on M such that*

$$\int_M \max(-\mathscr{K}_{ds^2}, 0)\Omega_{ds^2} < +\infty ,$$

then M is of finite type, where Ω_{ds^2} is the area element with respect to ds^2.

The proof is omitted. For the details, refer to [10, Theorem 13, p. 61, Theorem 15, p. 71], [16, Theorem 1, p. 316] and [3].

Proof of Theorem 4.6. By assumption, there are a divisor ν and a continuous real-valued bounded function k on $M-K$ with mild singularities such that $\nu(z) \geq 1 - p$ for every $z \in |\nu|$ and

$$-\operatorname{Ric}_{ds^2} + [\nu] = p(-\operatorname{Ric}_{d\tau^2}) + dd^c \log k^2 .$$

Here, we may assume that the divisor ν and the continuous function k with mild singularities are defined on the totality of M and $0 \leq k \leq 1$. Take a nowhere zero holomorphic 1-form ω and write

$$ds^2 = \lambda^2|\omega|^2 , \qquad d\tau^2 = \eta^2|\omega|^2 ,$$

where λ, η are continuous functions with mild singularities on M. Set

$$u := \frac{k\eta^p}{\lambda} .$$

Then, $v := \log u$ is harmonic on $M - (K \cup |\nu|)$, $\nu_u \geq 1 - p$ on $|\nu| - K$ and

$$\lambda = e^{-v}k\eta^p \leq e^{-v}\eta^p . \tag{4.9}$$

Define a new pseudo-metric

$$d\rho^2 := e^{-2v/(1-p)}|\omega|^2$$

on M and set $M_1 := \{a \in M; \nu_{d\rho}(a) \geq 0\}$. Then, $d\rho^2$ is a metric on M_1 which is flat on $M_1 - K$ and $\nu_{d\rho} \leq -1$ on $M-(K \cup M_1)$ because $\nu_u \geq 1-p$ on $|\nu_u| - K$. For the proof of Theorem 4.6, it suffices to prove that

(4.10) *The surface M_1 is complete with respect to the metric $d\rho^2$.*

In fact, if (4.10) is true, then M_1 is of finite type by Theorem 4.8, because

$$\int_{M_1} \max(-\mathscr{K}_{d\rho^2}, 0)\Omega_{d\rho^2} = \int_K \max(-\mathscr{K}_{d\rho^2}, 0)\Omega_{d\rho^2} < \infty .$$

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This implies that M is also of finite type.

Assume that (4.10) is not true. Then

$$d_0 := \mathrm{dist}_{d\rho}(K, \partial M_1) < +\infty ,$$

where $\mathrm{dist}_{d\rho}(K, \partial M_1)$ denotes the distance between K and the boundary ∂M_1 of M_1. So, for a sufficiently small positive number δ there is a rectifiable curve $\gamma_0(t)(0 \leq t < 1)$ such that $\gamma_0(0) \in K$, $\gamma_0(t)$ tends to ∂M_1 as $t \to 1$ and $L_{d\rho}(\gamma_0) < d_0 + \delta$, where $L_{d\rho}(\gamma_0)$ denotes the length of the curve γ_0 with respect to the metric $d\rho^2$. If we take some t_0 sufficiently near to 1, the point $p_0 := \gamma_0(t_0)$ satisfies the inequalities

$$\mathrm{dist}_{d\rho}(K, p_0) > \frac{d_0}{2} , \qquad L_{d\rho}(\gamma_0 | [t_0, 1)) < \frac{d_0}{2} ,$$

where $\gamma | [\alpha, \beta)$ denotes the part of γ from $t = \alpha$ to $t = \beta$.

Since $d\rho^2$ is flat on $M_1 - K$, there is an isometry Φ of a disc $\Delta_R := \{w \in \mathbf{C}; |w| < R\}$ with the standard metric onto an open neighborhood of p_0 in $M_1 - K$ with the metric $d\rho^2$ such that $\Phi(0) = p_0$. We take the largest R $(\leq +\infty)$ such that there is a local isometry Φ of Δ_R onto some open set in $M_1 - K$ with $\Phi(0) = p_0$. Then, we see easily $R \leq L_{d\rho}(\gamma_0 | [t_0, 1)) < d_0/2$. By the definition of R, we can find a point a_0 in $\partial \Delta_R$ such that, for the line segment

$$\Gamma : w = ta_0 \quad (0 \leq t < 1) ,$$

the image $\gamma := \Phi(\Gamma)$ tends to the boundary of $M_1 - K$ as t tends to 1. Then, γ cannot tend to the set K nor $M - (K \cup M_1)$. In fact, if γ tends to K, then we have an absurd conclusion

$$R \geq \mathrm{dist}_{d\rho}(K, p_0) > d_0/2$$

and, if γ tends to the set $M - (K \cup M_1)$, then $R = L_{d\rho}(\gamma) = +\infty$ because $\nu_{d\rho} \leq -1$ on $M - (K \cup M_1)$. Therefore, γ tends to the boundary of M.

Now, we shall estimate the length of γ with respect to the metric ds^2. To this end, we define a function $\tilde{\eta}$ with mild singularities by $d\tau = \tilde{\eta} d\rho$. Then, since

$$d\tau = \eta |\omega| = \tilde{\eta} \mathrm{e}^{-v/(1-p)} |\omega| ,$$

we have

$$\eta = \tilde{\eta} \mathrm{e}^{-v/(1-p)} .$$

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By the use of (4.9), the length of γ with respect to ds^2 is estimated by

$$L_{ds}(\gamma) \leq \int_\gamma e^{-v}\eta^p|\omega| \leq \int_\gamma e^{-v}\tilde{\eta}^p e^{-pv/(1-p)}|\omega| = \int_\gamma \tilde{\eta}^p d\rho$$
$$= \int_\Gamma (\tilde{\eta}\cdot\Phi)^p \Phi^*(d\rho) = \int_\Gamma (\tilde{\eta}\cdot\Phi)^p|dw|\,.$$

On the other hand, by the assumption of $d\tau$ the curvature of $\Phi^*(d\tau)$ is strictly negative on Δ_R. It follows from Lemma 4.7 that

$$\Phi^*(d\tau) = (\tilde{\eta}\cdot\Phi)|dw| \leqq C_0\, d\sigma_{\Delta R}$$

for a positive constant C_0. Since the Poincaré metric $d\sigma^2_{\Delta R}$ is given by

$$d\sigma^2_{\Delta R} = C_1\left(\frac{R}{R^2-|w|^2}\right)^2|dw|^2$$

for some positive constant C_1, we see

$$L_{ds}(\gamma) \leq C_2\int_\Gamma \left(\frac{R}{R^2-|w|^2}\right)^p|dw| \leq \frac{C_3}{1-p}R^{1-p} < \infty\,,$$

where each C_i denotes some constant. This contradicts the completeness of M with respect to ds^2. Therefore, we have (4.10) and so the proof of Theorem 4.6 is completed.

§ 5. Modified defect relations

Let f be a nondegenerate holomorphic map of M into $\mathbf{P}^n(\mathbf{C})$ and let

$$H : (A, W) = 0$$

be a hyperplane in $\mathbf{P}^n(\mathbf{C})$, where A is a unit vector. Take a reduced representation $f = (f_0 : \cdots : f_n)$ on M and set $F = (f_0, \cdots, f_n)$, where $f_0, \cdots, f_n$ are holomorphic functions without common zeros. We consider the divisor $\nu(f, H) = \nu_{F(H)}$ for the function $F(H) := (A, F)$. The n-truncated pull-back $f^*(H)^{[n]}$ of the divisor corresponding to H stated in § 1 is defined by

$$f^*(H)^{[n]} = [\min(\nu(f, H), n)]\,.$$

We have always

$$f^*(H)^{[n]} \prec \Omega_f\,. \tag{5.1}$$

In fact, if we take a nonzero holomorphic function g with $\nu_g = \min(\nu(f, H), n)$, for $k := |F(H)|/\|f\|$ and $u := |F(H)/g|$ we see $0 \leq k \leq 1$ and

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$$f^*(H)^{[n]} + dd^c \log u^2 = dd^c \log \|f\|^2 + dd^c \log k^2 .$$

As $dd^c \log u^2 = [\nu_u]$ and $\nu_u \geq 1$ on $|\nu_u|$ by Poincaré-Lelong formula, we get (5.1).

DEFINITION 5.2. For an arbitrary compact subset K of M we set

$$\eta_f(H)^K := \inf\{\eta > 0; f^*(H)^{[n]} \prec \eta\Omega_f \text{ on } M - K\} .$$

We define the *modified H-defect* of H for f by

$$D_f(H) := 1 - \inf\{\eta_f(H)^K; K \text{ is an arbitrary compact subset of } M\} .$$

Remark 5.3. The notion of H-defect defined in the previous paper [8] is essentially equal to the quantity $\eta_f(H)^K$ for the case where $K = \emptyset$.

The modified H-defect has the following properties.

PROPOSITION 5.4. (i) $0 \leq D_f(H) \leq 1$.

(ii) *If there exists a bounded nonzero holomorphic function g on $M - K$ for a compact set K such that $\nu_g \geq \min(\nu(f, H), n)$, or particularly, if $\sharp f^{-1}(H)$ is finite, then $D_f(H) = 1$.*

(iii) *If $\nu(f, H)(a) \geq m$ at every $a \in f^{-1}(H) - K$ for some compact set K, then $D_f(H) \geq 1 - n/m$.*

Proof. The assertion (i) is obvious. The function $k := |g|$ is a bounded continuous function with mild singularities. Since

$$f^*(H)^{[n]} + [\nu_g - \min(\nu(f, H), n)] = \eta\Omega_f + dd^c \log k^2$$

on $M - K$ for $\eta = 0$, we have (ii). To see (iii), consider the bounded function

$$k := \left(\frac{|F(H)|}{\|f\|}\right)^{n/m} .$$

By the assumption, k is a continuous function with mild singularities and satisfies

$$f^*(H)^{[n]} + \left[\frac{n}{m}\nu(f, H) - \min(\nu(f, H), n)\right] = \frac{n}{m} dd^c \log \|f\|^2 + dd^c \log k^2$$

on $M - K$. Since $(n/m)\nu(f, H) \geq n \geq \min(\nu(f, H), n)$ on $f^{-1}(H) - K$ by the assumption, we have $D_f(H) \geq 1 - n/m$.

Now, we recall the classical defect for a nondegenerate holomorphic map of an open neighborhood of $\Delta_{R,\infty} := \{z; R \leq |z| < +\infty\}$ into $\mathbf{P}^n(\mathbf{C})$.

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The order function of f and the counting function (truncated by n) of a hyperplane H for f are defined by

$$T_f(r) = \int_R^r \frac{dt}{t} \int_{R \leq |z| < t} \Omega_f \quad (R < r < +\infty),$$

$$N_f(r)^{[n]} = \int_R^r \frac{dt}{t} \int_{R \leq |z| < t} f^*(H)^{[n]} \quad (R < r < +\infty)$$

respectively. The classical defect (truncated by n) is defined by

$$\delta_f(H)^{[n]} = 1 - \limsup_{r \to \infty} \frac{N_f(r)^{[n]}}{T_f(r)}.$$

We can easily prove the following relation by the same arguments as in the previous paper [7, § 1].

PROPOSITION 5.5. *Let f be a nondegenerate holomorphic map of an open Riemann surface M into $\mathbf{P}^n(\mathbf{C})$. Assume that, for some open subset G of M with compact boundary ∂G, there is a biholomorphic map Φ of a neighborhood of $M - G$ onto a neighborhood of $\Delta_{R,\infty}$ such that $\Phi(M - G) = \Delta_{R,\infty}$ and $\Phi(\partial G) = \{z; |z| = R\}$, and assume that the restriction $\tilde{f} := f \cdot \Phi^{-1} | \Delta_{R,\infty}$ has an essential singularity at ∞, namely, the map $\tilde{f}(1/z) : \{z; 0 < |z| < 1/R\} \to \mathbf{P}^n(\mathbf{C})$ has no holomorphic extension to $\{z; |z| < 1/R\}$. Then, for every hyperplane H*

$$0 \leq D_f(H) \leq \delta_{\tilde{f}}(H)^{[n]} \leq 1.$$

The classical defect relation improved by Nochka is given as follows:

THEOREM 5.6. *Let f be a nondegenerate holomorphic map of an open neighborhood of $\Delta_{R,\infty}$ into $\mathbf{P}^n(\mathbf{C})$ such that it has an essential singularity at ∞. Take q $(> 2N - n + 1)$ hyperplanes H_j $(1 \leq j \leq q)$ in $\mathbf{P}^n(\mathbf{C})$ located in N-subgeneral position and consider Nochka constants $\omega(j)$. Then,*

$$\sum_{j=1}^{q} \omega(j) \delta_f(H_j)^{[n]} \leq n + 1.$$

For the proof, see [12] or [2].

Now, we consider a nonconstant holomorphic map f of an open Riemann surface M with a pseudo-metric ds^2 into $\mathbf{P}^n(\mathbf{C})$.

DEFINITION 5.7. For an arbitrary compact set K we set

$$\rho_f^K := \inf\{\rho > 0; -\mathrm{Ric}_{ds^2} \prec \rho \Omega_f \text{ on } M - K\},$$

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where we set $\rho_f^K = +\infty$ if there is no ρ with the above property. We define the *H-order* of f by

$$\rho_f = \inf\{\rho_f^K;\ K \text{ is an arbitrary compact subset of } M\}\,.$$

To state the modified defect relation, we give another definition.

DEFINITION 5.8. Let M be an open Riemann surface of finite type, namely M is biholomorphic with a compact Riemann surface $\overline{M}$ with finitely many points removed. A holomorphic map of M into $\mathbf{P}^n(\mathbf{C})$ is said to be *transcendental* if f has no holomorphic extension to $\overline{M}$.

THEOREM 5.9. *Let M be an open Riemann surface with a complete continuous pseudo-metric ds^2 and let f be a nondegenerate holomorphic map of M into $\mathbf{P}^n(\mathbf{C})$. Take q $(> 2N - n + 1)$ hyperplanes in N-subgeneral position and Nochka constants $\omega(j)$ $(1 \leq j \leq q)$. If M is not of finite type or else f is transcendental, then*

$$\sum_{j=1}^{q} \omega(j) D_f(H_j) \leq n + 1 + \frac{\rho_f n(n+1)}{2}\,. \tag{5.10}$$

Before proving Theorem 5.9, we notice that Theorem 1.1 stated in § 1 is an easy consequence of Theorem 5.9. Let A denote the right hand side of (1.2). Then we have

$$A \geq 2N - n + 1 + \frac{\rho_f \sigma_n}{\theta} = q + \frac{1}{\theta}\Big(n + 1 + \rho_f \sigma_n - \sum_{j=1}^{q} \omega(j)\Big)$$

$$\geq q + \sum_{j=1}^{q} \frac{\omega(j)}{\theta}(D_f(H_j) - 1) \geq q + \sum_{j=1}^{q} (D_f(H_j) - 1) = \sum_{j=1}^{q} D_f(H_j)$$

by the help of Theorem 2.2, where $\sigma_n := n(n+1)/2$. This gives Theorem 1.1.

Proof of Theorem 5.9. It suffices to prove (5.10) for the case $\rho_f < +\infty$. Moreover, we may assume that M is not of finite type. In fact, if M is of finite type and f is transcendental, Theorem 5.9 is obvious from Theorem 5.6.

Take arbitrary constants $\rho > 0$ and η_j $(1 \leq j \leq q)$ satisfying the condition

$$-\operatorname{Ric}_{ds^2} \prec \rho \Omega_f\,, \qquad f^*(H_j)^{[n]} \prec \eta_j \Omega_f \tag{5.11}$$

on $M - K$ for a compact set K. By definition, there are divisors $[\nu_j]$ and bounded continuous real-valued functions k_j with mild singularities such that $\nu_j \geq c_j$ on $|\nu_j|$ for some positive constant c_j and

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$$f^*(H_j)^{[n]} + [\nu_j] = \eta_j \Omega_f + dd^c \log k_j^2$$

on $M - K$. Here, we may assume that $0 \leq k_j \leq 1$. Set $h_j := k_j \|f\|^{\eta_j}$. Then, each $\log h_j$ is harmonic on $M-(K \cup |\nu_{h_j}| \cup f^{-1}(H_j))$ and $\nu_{h_j} - \min(\nu(f, H_j), n) = \nu_j$.

Set

$$\gamma := \theta(q - 2N + n - 1) - \sum_{j=1}^{q} \omega(j)\eta_j .$$

We shall prove

$$\gamma \leq \rho\sigma_n . \tag{5.12}$$

Assume that $\gamma > \rho\sigma_n$. Take a positive number ε with $\gamma > \varepsilon\sigma_{n+1}$ and set

$$p := \frac{\rho(\sigma_n + \tau_n\varepsilon)}{\gamma - \varepsilon\sigma_{n+1}} . \tag{5.13}$$

For an arbitrary holomorphic local coordinate z, using the same notation as in § 3, we set

$$\eta_z := \left(\frac{\|f\|^{\gamma - \varepsilon\sigma_{n+1}} |F_n| \prod_{j=1}^{q} |h_j|^{\omega(j)} \prod_{k=0}^{n} |F_k|^{\varepsilon}}{\prod_{j=1}^{q} (|F(H_j)| \prod_{k=0}^{n-1} \log(a/\varphi_k(H_j)))^{\omega(j)}} \right)^{1/(\sigma_n + \varepsilon\tau_n)} .$$

Then, the pseudo-metric $d\tau^2 := \eta_z^2 |dz|^2$ is defined on $M - K$ independently of the choice of holomorphic local coordinate z. Here, $\eta_z = 0$ if $\varphi_k(H_j) = 0$. On the other hand, if we set

$$\varphi := \frac{|F_n|}{\prod_{j=1}^{q} |F(H_j)|^{\omega(j)}}$$

and $\nu_0 := \nu_\varphi + \sum_{j=1}^{q} \omega(j)\nu_{h_j}$, then $\nu_0 \geq c'$ on $|\nu_0|$ for some $c' > 0$, because

$$\begin{aligned} \nu_0 &= \sum_{j=1}^{q} \omega(j)(\nu_{h_j} - \min(\nu(f, H_j), n)) + \left(\nu_\varphi + \sum_{j=1}^{q} \omega(j)\min(\nu(f, H_j), n)\right) \\ &\geq \sum_{j=1}^{q} \omega(j)(\nu_{h_j} - \min(\nu(f, H_j), n)) \end{aligned}$$

by Lemma 2.4. This implies that $d\tau^2$ is a continuous pseudo-metric on $M - K$. We suitably extend $d\tau^2$ to a continuous pseudo-metric on the totality of M.

We shall next prove that $d\tau^2$ has strictly negative curvature on $M - K$. To this end, we recall the following inequality given in [8, Proposition 3.8]:

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$$dd^c \log |F_0|^2 |F_1|^2 \cdots |F_{n-1}|^2 \geq \frac{\tau_n}{\sigma_n} \left(\frac{|F_0|^2 \cdots |F_n|^2}{|F_0|^{2\sigma_{n+1}}} \right)^{1/\tau_n} \sqrt{-1} dz \wedge d\bar{z} .$$

This and Theorem 3.4 yield that

$$\begin{aligned} dd^c \log \eta_z^2 &\geq \frac{\gamma - \varepsilon\sigma_{n+1}}{\sigma_n + \varepsilon\tau_n} \Omega_f + \frac{\varepsilon}{2(\sigma_n + \varepsilon\tau_n)} dd^c \log (|F_0|^2 \cdots |F_{n-1}|^2) \\ &\quad + \frac{1}{2(\sigma_n + \varepsilon\tau_n)} dd^c \log \frac{\prod_{k=0}^{n-1} |F_k|^{2\varepsilon}}{\prod_{1 \leq j \leq q, 0 \leq k \leq n-1} \log^{2\omega(j)} (a/\varphi_k(H_j))} \\ &\geq \frac{\varepsilon}{2} \frac{\tau_n}{\sigma_n(\sigma_n + \varepsilon\tau_n)} \left(\frac{|F_0|^2 \cdots |F_n|^2}{|F_0|^{2\sigma_{n+1}}} \right)^{1/\tau_n} dz \wedge d^c z \\ &\quad + C_0 \left(\frac{|F_0|^{2\theta(q-2N+n-1)} |F_n|^2}{\prod_{j=1}^{q} (|F(H_j)|^2 \prod_{k=0}^{n-1} \log^2 (a/\varphi_k(H_j)))^{\omega(j)}} \right)^{1/\sigma_n} \sqrt{-1} dz \wedge d\bar{z} . \\ &\geq C_1 \left(\frac{\|f\|^{\theta(q-2N+n-1) - \varepsilon\sigma_{n+1}} |F_n| \prod_{k=0}^{n} |F_k|^{\varepsilon}}{\prod_{j=1}^{q} (|F(H_j)| \prod_{k=0}^{n-1} \log (a/\varphi_k H_j)))^{\omega(i)}} \right)^{2/(\sigma_n + \varepsilon\tau_u)} \sqrt{-1} dz \wedge d\bar{z} , \end{aligned}$$

where C_0, C_1 are some positive constants. On the other hand, since $|h_j| \leq \|f\|^{\eta_j}$,

$$\|f\|^{\theta(q-2N+n-1) - \varepsilon\sigma_{n+1}} \geq \|f\|^{\gamma - \varepsilon\sigma_{n+1}} h_1^{\omega(1)} \cdots h_q^{\omega(q)}$$

on $M - K$. This concludes that $dd^c \log \eta^2 \geq C_2 \eta^2$ for some positive constant C_2. Therefore, $d\tau^2$ has strictly negative curvature on $M - K$.

Now, we represent each hyperplane H_j $(1 \leq j \leq q)$ as

$$H_j : a_{j0} w_0 + \cdots + a_{jn} w_n = 0 .$$

For some holomorphic local coordinate z and each pair of indices j, k, we choose indices $i_1, \cdots, i_k$ with $1 \leq i_1 < \cdots < i_k \leq q$ such that

$$\psi_{kj}^z := \sum_{l \neq i_1, \cdots, i_k} a_{jl} W(f_l, f_{i_1}, \cdots, f_{i_k}) \not\equiv 0 .$$

For convenience' sake, we set $\psi_{j0}^z = F(H_j)$. By the theorem of identity, $\psi_{jk}^z \not\equiv 0$ for every holomorphic local coordinate z. We now define

$$k := \left(\frac{\prod_{1 \leq j \leq q, 0 \leq k \leq n-1} |\psi_{jk}^z|^{\varepsilon/q} \log^{\omega(j)} (a/\varphi_k(H_j))}{\prod_{0 \leq k \leq n-1} |F_k|^{\varepsilon}} \right)^{1/(\sigma_n + \varepsilon\tau_n)}$$

Then, k is bounded because

$$\begin{aligned} \frac{|\psi_{jk}^z|^{\varepsilon/q} \log^{\omega(j)} (a/\varphi_k(H_j))}{|F_k|^{\varepsilon/q}} &\leq \left(\frac{|F_k(H_j)|}{|F_k|} \right)^{\varepsilon/q} \log^{\omega(j)} (a/\varphi_k(H_j)) \\ &\leq \sup_{0 < x \leq 1} x^{\varepsilon/q} \log^{\omega(j)} \left(\frac{a}{x} \right) < + \infty . \end{aligned}$$

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Set

$$v_1 := |\varphi| \prod_{j=1}^{q} |h_j|^{\omega(j)}, \quad v_2 := \prod_{1\le j\le q, 0\le k\le n-1} |\psi_{jk}^z|^{\varepsilon/q}, \quad v := v_1 v_2 .$$

Then $\log v$ is harmonic on $M - (K \cup |\nu_v|)$, $\nu_{v_1} \geq c''$ on $|\nu_{v_1}|$ for some c'', not depending on each ε, and $\nu_{v_2} \geq \varepsilon/q$ on $|\nu_{v_2}|$. We consider the function η_z for some ε such that

$$\frac{\gamma - \rho\sigma_n}{\sigma_{n+1} + \rho\tau_n} > \varepsilon > \max\left(\frac{\gamma - \rho\sigma_n}{\rho/q + \sigma_{n+1} + \rho\tau_n}, \frac{\gamma - \rho\sigma_n - c''\rho}{\sigma_{n+1} + \rho\tau_n}\right).$$

and, moreover, $-\operatorname{Ric}_{ds^2} \prec_{1-p} \rho\Omega_f$. Then, we have $0 < p < 1$ and

$$\nu_v \geq \min\left(\frac{\varepsilon}{q}, c''\right) \geq \frac{(\gamma - \varepsilon\sigma_{n-1})(1-p)}{\rho}$$

on $|\nu_v| - K$. Since

$$k\eta_z = \|f\|^{(\gamma - \varepsilon\sigma_{n+1})/(\sigma_n + \varepsilon\tau_n)} v^{1/(\sigma_n + \varepsilon\tau_n)},$$

we have

$$\rho\Omega_f + \left[\frac{\rho}{\gamma - \varepsilon\sigma_{n+1}}\nu_v\right] = p\, dd^c \log \eta_z^2 + dd^c \log k^{2p} .$$

This gives that

$$-\operatorname{Ric}_{ds^2} \prec_{1-p} p(-\operatorname{Ric}_{dr^2})$$

on $M - K$. This contradicts the assumption by Theorem 4.6. Thus, we conclude (5.12).

By Theorem 2.2, we can rewrite (5.12) as

$$\sum_{j=1}^{q} \omega(j)(1 - \eta_j) \leq n + 1 + \rho\sigma_n .$$

Taking the supremum of the left hand side and the infimum of the right hand side for all admissible constants η_j, ρ and compact set K, we can conclude

$$\sum_{j=1}^{q} \omega(j) D_f(H_j) \leq n + 1 + \rho_f \sigma_n .$$

The proof of Theorem 5.9 is completed.

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§ 6. Modified defect relations for Gauss maps

Let $x = (x_1, \cdots, x_m) : M \to \mathbf{R}^m$ be a (possibly branched) minimal surface. By S we denote the set of all branch points, namely points where the Jacobi matrix of x are not of maximal rank. The set of all oriented 2-planes in $\mathbf{R}^m$ may be identified with the quadric $Q_{m-2}(\mathbf{C})$ in $\mathbf{P}^{m-1}(\mathbf{C})$. By definition, the Gauss map G of M is the map of $M - S$ into $Q_{m-2}(\mathbf{C})$ which maps each $p \in M - S$ to the point in $Q_{m-2}(\mathbf{C})$ corresponding to the oriented tangent plane of M at p. The surface M is considered an open Riemann surface with the conformal pseudo-metric ds^2 induced from the standard metric on $\mathbf{R}^m$. Taking a holomorphic local coordinate z, we set $f_i := \partial x_i/\partial z$, which are holomorphic by the assumption of M. The set S coincides with the set of common zeros of the functions f_i $(1 \leq i \leq m)$.

DEFINITION 6.1. We now define the *branching divisor* of M by

$$\nu_{ds} = \min\{\nu_{f_i}; 1 \leq i \leq m\},$$

which is well-defined on the totality of M independently of holomorphic local coordinate z.

The Gauss map G is locally given by

$$G = (f_1 : \cdots : f_m)$$

on $M - S$. Take a nonzero holomorphic function h on M with $\nu_h = \nu_{ds}$. If we set $g_i := f_i/h$ $(1 \leq i \leq m)$, we have

$$G = (g_1 : g_2 : \cdots : g_m)$$

on the common domain of g_i's except S. The right hand side of this has the holomorphic extension across S. We define the Gauss map G of M to be the holomorphic map of M into $\mathbf{P}^{m-1}(\mathbf{C})$ locally given by the right hand side of the above identity.

DEFINITION 6.2. For an arbitrary compact subset K of M we set

$$\rho_{ds}^K := \inf\{\rho; [\nu_{ds}] \prec \rho\Omega_G \text{ on } M - K\}.$$

We define the *branching H-order* of M by

$$\rho_{ds} := \inf\{\rho_{ds}^K; K \text{ is an arbitrary compact subset of } M\}.$$

Obviously, if $x : M \to \mathbf{R}^m$ is an immersion, then $\rho_{ds} = 0$.

THEOREM 6.3. *Let $x : M \to \mathbf{R}^m$ be a complete minimal surface and*

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$G : M \to \mathbf{P}^N(\mathbf{C})$ the Gauss map of M, where $N = m - 1$. Consider the smallest projective linear subspace $\mathbf{P}^n(\mathbf{C})$ of $\mathbf{P}^N(\mathbf{C})$ which includes $G(M)$. Take q hyperplanes $H_1, \cdots, H_q$ in $\mathbf{P}^N(\mathbf{C})$ such that $\tilde{H}_j := H_j \cap \mathbf{P}^n(\mathbf{C})\,(1 \leq j \leq q)$ are hyperplanes in $\mathbf{P}^n(\mathbf{C})$ located in N-subgeneral position, where $q > 2N - n + 1$. If M is not of finite type, then for Nochka weight $\omega(j)$ for $\tilde{H}_j$

$$\sum_{j=1}^{q} \omega(j) D_G(\tilde{H}_j) \leq n + 1 + \frac{(1 + \rho_{ds})n(n+1)}{2}.$$

Proof. By assumption, the Gauss map G is nondegenerate as the map into $\mathbf{P}^n(\mathbf{C})$. Take a system of homogeneous coordinate $(w_0 : \cdots : w_N)$ on $\mathbf{P}^N(\mathbf{C})$ such that $\mathbf{P}^n(\mathbf{C}) := \{(w_0 : \cdots : w_N);\ w_{n+1} = \cdots = w_N = 0\}$ and a reduced representation $G = (g_0 : \cdots : g_n)$ of G as a map into $\mathbf{P}^n(\mathbf{C})$. So, we have

$$ds^2 = 2\|f\|^2 |dz|^2 \leq C|h|^2 \|g\|^2 |dz|^2$$

for a constant $C > 0$ and $\|g\| := (|g_0|^2 + \cdots + |g_n|^2)^{1/2}$. Then, f_i are linear combinations of $hg_0, \cdots, hg_n$ for a holomorphic function h with $\nu_h = \nu_{ds}$. So, for each $\rho \geq 0$ such that $[\nu_{ds}] \prec \rho\Omega_G$ on M outside a compact set K, we have

$$-\mathrm{Ric}_{ds^2} \prec (\rho + 1)\Omega_G .$$

Taking the infimum of the right hand side for various ρ, we obtain

$$\rho_G \leq \rho_{ds} + 1 . \tag{6.4}$$

Theorem 6.3 is now immediate consequence of Theorem 5.9.

COROLLARY 6.5. *Let $x : M \to \mathbf{R}^m$ be a complete nonflat minimal surface with infinite total curvature, let $G : M \to \mathbf{P}^N(\mathbf{C})$ be the Gauss map of M and let n be the least dimension of projective linear subspaces of $\mathbf{P}^N(\mathbf{C})$ which include $G(M)$, where $N = m - 1$. Then, for arbitrary hyperplanes $H_1, \cdots, H_q$ in $\mathbf{P}^N(\mathbf{C})$ located in general position and not including $G(M)$,*

$$\sum_{j=1}^{q} D_G(H_j) \leq 2N - n + 1 + \frac{(1 + \rho_{ds})n(2N - n + 1)}{2} . \tag{6.6}$$

Proof. Let us denote by $\mathbf{P}^n(\mathbf{C})$ the smallest projective linear subspace which includes $G(M)$. Then, as is easily seen, hyperplanes $H_1 \cap \mathbf{P}^n(\mathbf{C})$, $\cdots, H_q \cap \mathbf{P}^n(\mathbf{C})$ of $\mathbf{P}^n(\mathbf{C})$ are located in N-subgeneral position. If M is biholomorphic with a compact Riemann surface $\overline{M}$ with finitely many points removed and G is not transcendental, then Ric_{ds^2} is continuously

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extended to $\overline{M}$ and so $C(M) = C(\overline{M}) < +\infty$, which contradicts the assumption. So, we have the situation in Theorem 1.1 for the map G considered as a map into $\mathbf{P}^n(\mathbf{C})$. Since $\rho_G \leq \rho_{ds} + 1$ by (6.4), Corollary 6.5 is a direct result of Theorem 1.1.

We shall give here the proof of Theorem 1.3. By assumption the Gauss map G is not a constant. We have (6.6) for some n with $1 \leq n \leq N$. Here, $\rho_{ds} = 0$ because M has no branch points. These imply that

$$\begin{aligned}\sum_{j=1}^{q} D_G(H_j) &\leq 2N - n + 1 + \frac{n(2N - n + 1)}{2} \\ &= \frac{(N+1)(N+2) - (N-n)(N-n-1)}{2} \\ &\leq \frac{(N+1)(N+2)}{2} = \frac{m(m+1)}{2}.\end{aligned}$$

This gives Theorem 1.3.

Now, we consider a holomorphic curve in $\mathbf{C}^m$ given by a nonconstant holomorphic map $w = (w_1, w_2, \cdots, w_m) : M \to \mathbf{C}^m$. The space $\mathbf{C}^m$ is identified with $\mathbf{R}^{2m}$ by associating a point $(x_1 + \sqrt{-1}y_1, \cdots, x_m + \sqrt{-1}y_m) \in \mathbf{C}^m$ with $(x_1, y_1, \cdots, x_m, y_m)$. The curve $w : M \to \mathbf{C}^m$ is considered as a minimal surface $w = (x_1, y_1, \cdots, x_m, y_m) : M \to \mathbf{R}^{2m}$. By Cauchy-Riemann's equations, we know

$$f_i := \frac{\partial x_i}{\partial z} = \sqrt{-1}\frac{\partial y_i}{\partial z} \quad (1 \leq i \leq m).$$

Therefore, the Gauss map of M is given by

$$G = (f_1 : -\sqrt{-1}f_1 : \cdots : f_m : -\sqrt{-1}f_m)$$

and so $G(M)$ is included in the projective linear subspace

$$\mathbf{P}^{m-1}(\mathbf{C}) := \{(u_1 : v_1 : \cdots : u_m : v_m) \in \mathbf{P}^{2m-1}(\mathbf{C});\ u_i = \sqrt{-1}v_i\ (1 \leq i \leq m)\}.$$

Particularly, if M is not included in any proper affine subspace of $\mathbf{C}^m$, G is nondegenerate as a map into $\mathbf{P}^{m-1}(\mathbf{C})$. The Gauss map G considered as a map of M into $\mathbf{P}^{m-1}(\mathbf{C})$ is the same as the complex Gauss map of M defined in the previous paper [8, p. 369].

As an easy consequence of Theorem 6.3, we have the following improvement of [8, Theorem 2.10].

COROLLARY 6.7. *Let $w : M \to \mathbf{C}^m$ be a holomorphic curve in $\mathbf{C}^m$ which*

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is complete and not included in any affine hyperplane, and let G be the Gauss map of M considered as a map of M into the above-mentioned space $\mathbf{P}^{m-1}(\mathbf{C})$. If M is not of finite type, then

$$\sum_{j=1}^{q} D_G(H_j) \leq m + \frac{(\rho_{ds} + 1)m(m-1)}{2}$$

for arbitrary hyperplanes $H_1, \cdots, H_q$ in $\mathbf{P}^{m-1}(\mathbf{C})$ in general position.

We next consider a minimal surface $x = (x_1, x_2, x_3) : M \to \mathbf{R}^3$. In this case, the quadric $Q_1(\mathbf{C})$ is canonically biholomorphic with $\mathbf{P}^1(\mathbf{C})$. Instead of the Gauss map $G : M \to Q_1(\mathbf{C})$ we may study the classical Gauss map $g : M \to \mathbf{P}^1(\mathbf{C})$ defined by

$$g = (f_3 : f_1 - \sqrt{-1} f_2),$$

where $f_i = \partial x_i / \partial z$ $(i = 1, 2, 3)$. Then, the metric of M is given by

$$ds^2 = |h|^2(|g_0|^2 + |g_1|^2)^2 |dz|^2$$

for a reduced representation $g = (g_0 : g_1)$ and a nonzero holomorphic function h with $\nu_h = \min(\nu_1, \nu_2, \nu_3)$ (cf., [7, p. 254]). Since $\nu_{ds} = \nu_h$, we have $-\mathrm{Ric}_{ds} \prec (\rho + 2)\Omega_g$ whenever $[\nu_{ds}] \prec \rho \Omega_g$. This yields

$$\rho_g \leq \rho_{ds} + 2 .$$

According to Theorem 6.9, we can easily prove the following:

THEOREM 6.8. *Let $x : M \to \mathbf{R}^3$ be a nonflat complete minimal surface which has infinite total curvature, and let $g : M \to \mathbf{P}^1(\mathbf{C})$ be the classical Gauss map. Then, for arbitrary distinct points $\alpha_1, \cdots, \alpha_q$ in $\mathbf{P}^1(\mathbf{C})$,*

$$\sum_{j=1}^{q} D_g(\alpha_j) \leq 4 + \rho_{ds} . \tag{6.9}$$

EXAMPLE 6.10. We give here an example of a minimal surface with infinite total curvature, for which the branching H-order is equal to two and the equality holds for the above modified defect relation (6.9).

Our construction is due to the Weierstrass-Enneper representation theorem for minimal surfaces in $\mathbf{R}^3$. We shall suitably choose meromorphic functions f and g on a simply connected open Riemann surface $\tilde{M}$ and show that, for the meromorphic 1-forms

$$\varphi_1 := \frac{(1 - g^2) f dz}{2}, \quad \varphi_2 := \sqrt{-1}\frac{(1 + g^2) f dz}{2}, \quad \varphi_3 := g f dz$$

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and a point $z_0 \in \tilde{M}$, the map

$$(x_1, x_2, x_3) := \left(\mathrm{Re}\int_{z_0}^{\zeta} \varphi_1(z), \mathrm{Re}\int_{z_0}^{\zeta} \varphi_2(z), \mathrm{Re}\int_{z_0}^{\zeta} \varphi_3(z)\right) \tag{6.11}$$

of $\tilde{M}$ into $\mathbf{R}^3$ gives a minimal surface with the desired properties. To this end, we first take a meromorphic function ϕ on $\mathbf{C}$ such that ϕ takes the values a_i $(1 \leq i \leq 3)$ with multiplicity three and all zeros of $\phi(z) - w$ are simple for every $w \in \mathbf{C} - \{a_1, a_2\}$, where $a_3 = \{\infty\}$. For the existence of such a function, see [9, p. 45]. Consider the domain $D := \mathbf{C} - (\phi^{-1}(a_1) \cup \phi^{-1}(a_2))$ and the analytic subset $V^* := \{(z, w); \phi(z) = w^3\}$ of $D \times \mathbf{C}$. Let $\hat{M}$ be the normalization of the closure of V^* in $D \times \mathbf{P}^1(\mathbf{C})$ and $\tilde{M}$ the universal covering of $\hat{M}$. Then $\tilde{M}$ is canonically regarded as a covering surface over D with the projection $F : \tilde{M} \to D$.

Now, consider the many-valued meromorphic function

$$\tilde{g}(\zeta) := (\phi \cdot F)^{1/3}(\zeta)$$

on $\tilde{M}$. By the definition of $\tilde{M}$, $\tilde{g}$ has single-valued branch on $\tilde{M}$, by which we define the function g. We next consider the many-valued meromorphic function

$$\tilde{f}(\zeta) := \frac{F'(\zeta)}{((\phi \cdot F)(\zeta) - a_1)^{1/3}((\phi \cdot F)(\zeta) - a_2)^{1/3}}\,. \tag{6.12}$$

Since $(\phi \cdot F) - a_1$ and $(\phi \cdot F) - a_2$ have poles of order three at every point of $(\phi \cdot F)^{-1}(\infty)$, $\tilde{f}$ has a single-valued branch on $\tilde{M}$, by which we define the function f. For these functions f and g, we shall prove that the minimal surface $x = (x_1, x_2, x_3) : \tilde{M} \to \mathbf{R}^3$ defined by (6.11) has all desired properties.

Take a reduced representation $g = g_1/g_0$, or $g = (g_0 : g_1)$ as a map into $\mathbf{P}^1(\mathbf{C})$. The induced metric is given by

$$ds^2 = \frac{1}{4}|f|^2(1 + |g|^2)^2|d\zeta|^2 = \frac{|f|^2}{4|g_0|^4}(|g_0|^2 + |g_1|^2)^2|d\zeta|^2\,.$$

Obviously, $\nu_{ds} = 0$ at each point where g has no zero or no pole. The same is true at each point where g has a pole, because g_0 has a zero of order one and the denominator of the right hand side of (6.12) has a pole of order two at such a point. Moreover, $\nu_{ds}(\zeta) = \nu_{F'}(\zeta) = 2$ whenever $g(\zeta) = 0$. These show that $\nu_{ds} = 2\nu_{g_1}$. If we consider a bounded function $k := |g_1|^2/(|g_0|^2 + |g_1|^2)(\leq 1)$, we obtain

$$[\nu_{ds}] = dd^c \log |g_1|^4 = 2dd^c \log \|g\|^2 + dd^c \log k^2\,.$$

This gives $\rho_{ds} \leq 2$.

We next show that $\tilde{M}$ is complete. Assume that there is a curve Γ in $\tilde{M}$ which is of finite length and tends to the boundary. For $\Gamma' := F(\Gamma)$ in D we have

$$L := L_{ds}(\Gamma) = \int_{\Gamma} \frac{|F'(\zeta)|(1 + |(\phi \cdot F)(\zeta)|^{2/3})}{|(\phi \cdot F)(\zeta) - a_1|^{1/3}|(\phi \cdot F)(\zeta) - a_2|^{1/3}} |d\zeta|$$
$$= \int_{\Gamma'} \frac{1 + |\phi(z)|^{2/3}}{|\phi(z) - a_1|^{1/3}|\phi(z) - a_2|^{1/3}} |dz| ,$$

and Γ' tends to the set $\partial D = \{\infty\} \cup \phi^{-1}(a_1) \cup \phi^{-1}(a_2)$. On the other hand, we can easily find a positive constant C_0 such that

$$\frac{1 + |\phi(z)|^{2/3}}{|\phi(z) - a_1|^{1/3}|\phi(z) - a_2|^{1/3}} \geq C_0$$

for every $z \in D$. If Γ' tends to ∞, then we get an absurd conclusion

$$L \geq C_0 \int_{\Gamma'} |dz| = + \infty .$$

Otherwise, there exists some point $z_0 \in \phi^{-1}(a_1) \cup \phi^{-1}(a_2)$ such that Γ' tends to z_0. Changing indices if necessary, we may assume that $\phi(z_0) = a_1$. Then, we find a neighborhood U of z_0 and a positive constant C_1 such that the following inequality holds on U

$$\frac{1 + |\phi(z)|^{2/3}}{|\phi(z) - a_1|^{1/3}|\phi(z) - a_2|^{1/3}} \geq C_1 \frac{1}{|z - z_0|} ,$$

because $\nu_{\phi - a_1}(z_0) = 3$. For the portion $\tilde{\Gamma}$ of Γ' in U, we obtain

$$L \geq \int_{\tilde{\Gamma}} \frac{1 + |\phi(z)|^{2/3}}{|\phi(z) - a_1|^{1/3}|\phi(z) - a_2|^{1/3}} |dz| \geq C_1 \int_{\tilde{\Gamma}} \frac{|dz|}{|z - z_0|} = + \infty .$$

This concludes the completeness of $\tilde{M}$.

On the other hand, since $\phi(z) \neq a_1, a_2$ on D, we see

$$g(\zeta) \notin \{a_1^{1/3}, a_1^{1/3}\omega, a_1^{1/3}\omega^2, a_2^{1/3}, a_2^{1/3}\omega, a_2^{1/3}\omega^2\}$$

for every $\zeta \in \tilde{M}$, where ω denotes one of the primitive third roots of unity. So, the modified defects of these values are all one. This shows that the inequality (6.9) is best-possible in this case.

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