MULTIPLICITIES AND MINIMAL WIDTHS FOR (0, 1)-MATRICES

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Introduction. In a previous paper **(1)** the notion of the α-width $\epsilon_A(\alpha)$ of a (0, 1)-matrix A was introduced, and a formula for the minimal α-width $\tilde{\epsilon}(\alpha)$ taken over the class $\mathfrak{A}$ of all (0, 1)-matrices having the same row and column sums as A, was obtained. The main tool in arriving at this formula was a block decomposition theorem **(1**, Theorem 2.1; repeated below as Theorem 2.1**)** that established the existence, in the class $\mathfrak{A}$ generated by A, of certain matrices having a simple block structure. The block decomposition theorem does not itself directly involve the notion of minimal α-width, but rather centres around a related class concept, that of multiplicity. We review both of these notions in § 2, together with some other pertinent definitions and results.

The present paper continues the study begun in **(1)**. The principal contribution is a simple construction which produces a single matrix $\tilde{A}$ in the class $\mathfrak{A}$ that has some remarkable properties: the partial row sum vectors of $\tilde{A}$ are as smooth as possible in the sense of majorization (Theorem 3.2); all minimal α-widths and multiplicities for the class $\mathfrak{A}$ can be obtained directly from $\tilde{A}$ (Theorem 3.3 and Corollary 3.4).

In the concluding section we apply the matrix $\tilde{A}$ in the solution of a problem closely related to the minimal width problem. For each A in $\mathfrak{A}$ define $\mu_A(\beta)$ to be the maximal number of columns that can be selected from A in such a way that the resulting submatrix has at most β 1's in each row. It follows readily that the sequences $\epsilon_A(\alpha)$ and $\mu_{A'}(\beta)$, where A' is the complement of A, determine each other, and hence that the class sequence

$$\bar{\mu}(\beta) = \max_{A \text{ in } \mathfrak{A}} \mu_A(\beta)$$

is determined by the minimal width sequence for the complementary class.

1. A basic construction. Let A be a matrix of m rows and n columns whose entries are either 0 or 1. We call A a (0, 1)-matrix of size m by n. Let the sum of row i of A be denoted by r_i and the sum of column j of A by s_j. We call

$$(1.1) \qquad R = (r_1, r_2, \ldots, r_m)$$

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the *row sum vector* of A, and

$$S = (s_1, s_2, \ldots, s_n) \tag{1.2}$$

the *column sum vector* of A. These vectors determine a class

$$\mathfrak{A} = \mathfrak{A}(R, S) \tag{1.3}$$

consisting of all (0, 1)-matrices of size m by n having row sum vector R and column sum vector S. Simple necessary and sufficient conditions on R and S are known in order that the class $\mathfrak{A}(R, S)$ be non-empty **(3; 6)**.

Let A be in $\mathfrak{A}$ and consider the 2 by 2 submatrices of A of the types

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

An *interchange* is a transformation of the elements of A that changes a minor of one of these types into the other, leaving all other elements fixed. The interchange theorem **(6)** asserts that if A and B are in $\mathfrak{A}$, then A is transformable into B by interchanges.

Throughout this paper we suppose and without loss of generality that $\mathfrak{A}$ is non-empty and that

$$r_1 \geqslant r_2 \geqslant \ldots \geqslant r_m > 0, \tag{1.4}$$

$$s_1 \geqslant s_2 \geqslant \ldots \geqslant s_n > 0. \tag{1.5}$$

Such an $\mathfrak{A}$ is termed *normalized.*

Let $A = [a_{ij}]$ be in $\mathfrak{A}$. We call the column vector

$$R_\epsilon = \begin{bmatrix} \sum_{j=1}^{\epsilon} a_{1j} \\ \sum_{j=1}^{\epsilon} a_{2j} \\ \vdots \\ \sum_{j=1}^{\epsilon} a_{mj} \end{bmatrix} \tag{1.6}$$

the ϵth *partial row sum vector* of A. Thus $R_n = R^T$, where R^T denotes the transpose of R.

Given the vectors R and S for a normalized class $\mathfrak{A}$, there is a simple rule for constructing an A in $\mathfrak{A}$. This rule may be stated, somewhat loosely, as follows. *Select any column j and insert its* 1*'s in the positions corresponding to the s_j largest row sums; delete column j, reduce each of these s_j row sums by* 1, *and repeat the entire procedure on another column.*

Example. Let $\mathfrak{A}$ be determined by

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$$R = (7, 6, 3, 2, 2, 2, 2, 2),$$
$$S = (4, 4, 4, 4, 4, 4, 1, 1).$$

Suppose we apply the rule from "right to left," that is, select the last column first, then the next to last, and so on, and give preference to the bottommost positions in a column in case of ties (this keeps the partial row sums monotone). The rule then constructs the matrix $\tilde{A}$ below, having partial row sum vectors given by the matrix $\tilde{M}$.

$$\tilde{A} = \begin{bmatrix} 1&1&1&1&1&1&0&1 \\ 1&0&1&1&1&1&1&0 \\ 1&0&1&0&0&1&0&0 \\ 1&0&0&1&0&0&0&0 \\ 0&1&0&1&0&0&0&0 \\ 0&1&0&0&1&0&0&0 \\ 0&1&0&0&1&0&0&0 \\ 0&0&1&0&0&1&0&0 \end{bmatrix}$$

$$\tilde{M} = \begin{bmatrix} 1&2&3&4&5&6&6&7 \\ 1&1&2&3&4&5&6&6 \\ 1&1&2&2&2&3&3&3 \\ 1&1&1&2&2&2&2&2 \\ 0&1&1&2&2&2&2&2 \\ 0&1&1&1&2&2&2&2 \\ 0&1&1&1&2&2&2&2 \\ 0&0&1&1&1&2&2&2 \end{bmatrix}$$

The validity of the construction can be established by a simple interchange argument, as follows. Let A be in $\mathfrak{A}$ and suppose B has been constructed by the rule. We must show that B is in $\mathfrak{A}$. Assume that in constructing B, the 1's in column j were assigned initially; we may suppose without loss of generality that these 1's occupy the s_j topmost positions in column j. If A has a 0 above a 1 in column j, then by the monotonicity of the row sums of A, there is an interchange that switches this 0 and 1 in column j. Hence we may apply interchanges involving column j of A to obtain a transformed matrix in $\mathfrak{A}$ whose jth column agrees with the jth column of B. We can now suppress column j of the transformed A and B, and repeat the argument. Eventually A has been transformed by interchanges into B, and thus B is in $\mathfrak{A}$.

Of course an analogous procedure in which the roles of rows and columns are reversed also constructs a matrix in the class.

In § 3 we shall apply this construction in the right to left order (as in the example), giving preference to bottommost positions in a column in case of ties (as in the example). The resulting matrix will be denoted by $\tilde{A}$.

2. A review of multiplicity and width. Let $\mathfrak{A} = \mathfrak{A}(R, S)$ be a normalized class and let α and ϵ be integers satisfying

$$0 \leqslant \alpha \leqslant r_m, \tag{2.1}$$

$$1 \leqslant \epsilon \leqslant n. \tag{2.2}$$

We say that a pair α, ϵ in the respective ranges (2.1), (2.2) are *compatible* if there is an A in $\mathfrak{A}$ having an m by ϵ submatrix E^* each of whose row sums is at least α. If α and ϵ are compatible, consider the class of all m by ϵ submatrices E^* of the matrices A in $\mathfrak{A}$ with the row sums of E^* at least α, and let δ^* denote the number of rows of E^* whose sums are precisely α. The non-negative integer

$$\delta = \delta(\alpha, \epsilon) \tag{2.3}$$

equal to the minimum of the integers δ^* is called the *multiplicity* of α with respect to ϵ.

In **(1)** the following theorem was proved.

THEOREM 2.1. *Let α be compatible with ϵ and of multiplicity δ with respect to ϵ. Then there is a matrix A in the normalized $\mathfrak{A}$ of the form*

$$A = \left[\begin{array}{c|c|c} M & J & X \\ \hline F & & \\ \hline E & Y & O \end{array}\right]. \tag{2.4}$$

Here E is of size δ by ϵ with exactly α 1's in each row. M is a matrix of size e by ϵ with $\alpha + 1$ or more 1's in each row. F is a matrix of size $m - (e + \delta)$ by ϵ with exactly $\alpha + 1$ 1's in each row. J is a matrix of 1's of size e by $f - \epsilon$ and O is a zero matrix. The degenerate cases $e = 0$, $e + \delta = m$, $\delta = 0$, $f = \epsilon$, and $f = n$ are not excluded.

The *α-width* $\epsilon_A(\alpha)$ of a matrix A in the normalized $\mathfrak{A}$ is the least number of columns that can be selected from A so that the resulting submatrix E^* has row sums at least α. Here $1 \leqslant \alpha \leqslant r_m$. Then

$$\tilde{\epsilon}(\alpha) = \min_{A \text{ in } \mathfrak{A}} \epsilon_A(\alpha) \tag{2.5}$$

is the *minimal α-width* of $\mathfrak{A}$. The integer $\tilde{\epsilon}(\alpha)$, which can also be described as the least ϵ compatible with α, has been explicitly determined in terms of the vectors R and S in **(1)**. This determination used the function

$$N(\epsilon, e, f) = r_{e+1} + \ldots + r_m - (s_{\epsilon+1} + \ldots + s_f) + e(f - \epsilon), \tag{2.6}$$

where ϵ, e, f are integer parameters satisfying

$$0 \leqslant \epsilon \leqslant n, \tag{2.7}$$

$$0 \leqslant e \leqslant m, \tag{2.8}$$

$$\epsilon \leqslant f \leqslant n. \tag{2.9}$$

Precisely, $\tilde{\epsilon}(\alpha)$ is the first ϵ such that

$$N(\epsilon, e, f) \geqslant \alpha(m - e) \tag{2.10}$$

for all e, f satisfying (2.8), (2.9). Note that if A is in $\mathfrak{A}(R, S)$ and if we write

$$A = \begin{bmatrix} * & Y & * \\ X & * & Z \end{bmatrix}, \tag{2.11}$$

with X of size $m - e$ by ϵ and Y of size e by $f - \epsilon$, then

$$N(\epsilon, e, f) = N_1(X) + N_0(Y) + N_1(Z), \tag{2.12}$$

where $N_1(Q)$ $[N_0(Q)]$ denotes the number of 1's (0's) in a (0, 1)-matrix Q.

It was also proved in **(1)** that if α, ϵ are compatible, then

$$\delta(\alpha, \epsilon) = (\alpha + 1)m - s_\epsilon - \min_{\substack{0 \leqslant e \leqslant m \\ \epsilon \leqslant f \leqslant n}} [N(\epsilon - 1, e, f) + \alpha e]. \tag{2.13}$$

(Only the special case of (2.13) in which $\epsilon = \tilde{\epsilon}(\alpha)$ is stated explicitly in **(1)**, but the proof there establishes (2.13) in general.)

3. The matrices $\tilde{A}$ and $\tilde{M}$.

If

$$T = (t_1, t_2, \ldots, t_m)$$
$$T^* = (t_1^*, t_2^*, \ldots, t_m^*)$$

are two vectors of non-negative integers, then T is *majorized* by T^* **(4; 5)**, written

$$T \prec T^*, \tag{3.1}$$

provided that with subscripts renumbered

$$t_1 \geqslant t_2 \geqslant \ldots \geqslant t_m,\ t_1^* \geqslant t_2^* \geqslant \ldots \geqslant t_m^*, \tag{3.2}$$

$$t_1 + t_2 + \ldots + t_e \leqslant t_1^* + t_2^* + \ldots + t_e^*,\ (e = 1, 2, \ldots, m - 1), \tag{3.3}$$

$$t_1 + t_2 + \ldots + t_m = t_1^* + t_2^* + \ldots + t_m^*. \tag{3.4}$$

In connection with this concept, we prove the following lemma, which will be used in the proof of Theorem 3.2.

LEMMA 3.1. *Let $T = (t_1, t_2, \ldots, t_m)$ and $T^* = (t_1^*, t_2^*, \ldots, t_m^*)$ be two vectors of non-negative integers satisfying* (3.2), (3.3), (3.4). *Let U be obtained from T by reducing k of its positive components in positions $i_1, i_2, \ldots, i_k$ by* 1. *Similarly let U^* be obtained from T^* by reducing k of its positive components in positions $j_1, j_2, \ldots, j_k$ by* 1. *If $i_1 \leqslant j_1$, $i_2 \leqslant j_2, \ldots, i_k \leqslant j_k$, then $U \prec U^*$.*

Proof. We proceed by induction on k. Let $k = 1$ and set $i_1 = i$, $j_1 = j$. We may take U and U^* to be monotone non-increasing by assuming that component i' of T has been reduced by 1 to get U, component j' of T^* has been reduced by 1 to get U^*. Here $i' \geqslant i$, $j' \geqslant j$ and

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$$t_i = t_{i+1} = \ldots = t_{i'} > t_{i'+1}, \tag{3.5}$$

$$t_j^* = t_{j+1}^* = \ldots = t_{j'}^* > t_{j'+1}^*, \tag{3.6}$$

where $t_{i'+1} = 0$ if $i' = m$ and $t_{j'+1}^* = 0$ if $j' = m$. If now $i' \leqslant j'$, then clearly $U \prec U^*$. Suppose that $i' > j'$. Thus

$$1 \leqslant i \leqslant j \leqslant j' < i' \leqslant m. \tag{3.7}$$

If U is not majorized by U^*, there is an integer e satisfying

$$j' \leqslant e < i' \tag{3.8}$$

for which

$$t_1 + t_2 + \ldots + t_e = t_1^* + t_2^* + \ldots + t_e^*. \tag{3.9}$$

By assumption,

$$t_1 + t_2 + \ldots + t_{e-1} \leqslant t_1^* + t_2^* + \ldots + t_{e-1}^*, \tag{3.10}$$

$$t_1 + t_2 + \ldots + t_{e+1} \leqslant t_1^* + t_2^* + \ldots + t_{e+1}^*. \tag{3.11}$$

Subtracting (3.10) from (3.9), and (3.9) from (3.11) yields

$$t_e \geqslant t_e^*, \tag{3.12}$$

$$t_{e+1} \leqslant t_{e+1}^*. \tag{3.13}$$

By (3.5), (3.7), (3.8), we have

$$t_e = t_{e+1}. \tag{3.14}$$

Thus (3.12), (3.13), (3.14) and $t_e^* \geqslant t_{e+1}^*$ imply

$$t_e^* = t_e = t_{e+1}^*. \tag{3.15}$$

If $e = j'$, this contradicts (3.6). If, on the other hand, $j' < e$, we have from (3.15) and (3.9),

$$t_1 + t_2 + \ldots + t_{e-1} = t_1^* + t_2^* + \ldots + t_{e-1}^*. \tag{3.16}$$

We may now repeat the argument with $e - 1$ in place of e. Eventually (3.6) is contradicted. This verifies Lemma 3.1 for $k = 1$.

Assume the validity of the lemma for $k - 1$. Let P and P^* be obtained from T and T^* by reducing components $i_2, i_3, \ldots, i_k$ of T and components $j_2, j_3, \ldots, j_k$ of T^*. By the induction assumption, we have $P \prec P^*$. Of course P and P^* may not be in monotone non-increasing order, but such rearrangements of them can be secured without disturbing the i_1 position of P or the j_1 position of P^*. Applying the argument used for $k = 1$ to these rearrangements shows that $U \prec U^*$, thus proving Lemma 3.1.

Let the vectors R and S be given for a normalized class $\mathfrak{A}(R, S)$ and let $\tilde{A}$ denote the matrix in $\mathfrak{A}(R, S)$ constructed by the rule of § 1, proceeding column-wise from right to left and giving preference within a column to bottommost positions in case of ties. We now prove

THEOREM 3.2. *Let A be arbitrary in the normalized class $\mathfrak{A}$ and let A have partial row sum vectors $R_1, R_2, \ldots, R_n$. Let the matrix $\tilde{A}$ in $\mathfrak{A}$ have partial row sum vectors $\tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_n$. Then $\tilde{R}_\epsilon \prec R_\epsilon$, $\epsilon = 1, 2, \ldots, n$.*

Proof. We prove Theorem 3.2 by induction. Note that

$$\tilde{R}_n = R_n = R^T, \tag{3.17}$$

and hence the theorem is valid with $\epsilon = n$. Assume that

$$\tilde{R}_{\epsilon+1} \prec R_{\epsilon+1}, \tag{3.18}$$

and consider the vectors $\tilde{R}_\epsilon$, R_ϵ. The vector R_ϵ is obtained from a non-increasing rearrangement $R^*_{\epsilon+1}$ of $R_{\epsilon+1}$ by reducing $s_{\epsilon+1}$ distinct components of $R^*_{\epsilon+1}$ by 1. A rearrangement of $\tilde{R}_\epsilon$ is obtained from the monotone $\tilde{R}_{\epsilon+1}$ by reducing the first $s_{\epsilon+1}$ components of $\tilde{R}_{\epsilon+1}$ by 1. By Lemma 3.1, we have

$$\tilde{R}_\epsilon \prec R_\epsilon. \tag{3.19}$$

This proves Theorem 3.2.

THEOREM 3.3. *The matrix $\tilde{A}$ is of form* (2.4) *for all compatible pairs α, ϵ.*

Proof. Let α and ϵ be compatible and let

$$A = \left[\begin{array}{c|c|c} M & J & X \\ \hline F & & \\ \cline{1-1} E & Y & O \end{array}\right] \tag{3.20}$$

be the matrix whose existence is given by Theorem 2.1. Thus E is of size $\delta = \delta(\alpha, \epsilon)$ by ϵ with exactly α 1's in each row; M is of size e by ϵ with at least $\alpha + 1$ 1's in each row; F is of size $m - (\delta + e)$ by ϵ with exactly $\alpha + 1$ 1's in each row; J is a matrix of size e by $f - \epsilon$ consisting entirely of 1's and O is a zero matrix.

Consider the first ϵ columns of $\tilde{A}$. Each of the row sums of these ϵ columns must be at least α, for otherwise we may use the matrix A to contradict $\tilde{R}_\epsilon \prec R_\epsilon$. By the definition of multiplicity, the first ϵ columns of $\tilde{A}$ cannot have fewer than δ rows with exactly α 1's in each row. Nor can these ϵ columns have more than δ rows with exactly α 1's in each row. For if this were the case, again $\tilde{R}_\epsilon \prec R_\epsilon$ would be contradicted. Hence, since $\tilde{R}_\epsilon$ is monotone, $\tilde{A}$ has a δ by ϵ matrix of form E in the lower left corner, and the portion of $\tilde{A}$ corresponding to M and F of A must contain at least $\alpha + 1$ 1's in each row. But

$$N(\epsilon, e, f) = N_1(F) + N_1(E) + N_0(J) + N_1(O)$$

is a class invariant. Hence the portions of $\tilde{A}$ corresponding to F, J, and O of A are of the desired form. This completes the proof.

Define $\tilde{M}$ to be the m by n matrix of non-negative integers whose column vectors are the partial row sum vectors of $\tilde{A}$,

$$\tilde{M} = [\tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_n]. \tag{3.21}$$

We call $\tilde{M}$ the *multiplicity matrix* of the normalized class $\mathfrak{A}$. Corollary 3.4 collects some immediate consequences of Theorem 3.3 that justify this nomenclature.

COROLLARY 3.4. *Let $\tilde{M} = [\tilde{R}_1, \tilde{R}_2, \ldots, \tilde{R}_n]$ be the multiplicity matrix of the normalized class $\mathfrak{A}$. Then α and ϵ are compatible if and only if the last component of $\tilde{R}_\epsilon$ is at least α. If α and ϵ are compatible, the multiplicity $\delta(\alpha, \epsilon)$ of α with respect to ϵ is equal to the number of components of $\tilde{R}_\epsilon$ that are equal to α.*

Example. To illustrate Theorem 3.3, consider the example of § 1 corresponding to the compatible pair $\alpha = 1$, $\epsilon = 3$:

$$\tilde{A} = \left[\begin{array}{ccc|ccc|cc}
1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\
\hline
1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
\cline{1-3}
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0
\end{array}\right],$$

$$\tilde{M} = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 6 & 7 \\
1 & 1 & 2 & 3 & 4 & 5 & 6 & 6 \\
1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 \\
1 & 1 & 1 & 2 & 2 & 2 & 2 & 2 \\
0 & 1 & 1 & 2 & 2 & 2 & 2 & 2 \\
0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\
0 & 1 & 1 & 1 & 2 & 2 & 2 & 2 \\
0 & 0 & 1 & 1 & 1 & 2 & 2 & 2
\end{bmatrix}.$$

From $\tilde{M}$ the multiplicity $\delta(\alpha, \epsilon)$ for each compatible α, ϵ may be determined as in Corollary 3.4,

	$\alpha \backslash \epsilon$	1	2	3	4	5	6	7	8
	0	4	1	0	0	0	0	0	0
$\delta(\alpha, \epsilon)$:	1	×	×	(5)	3	1	0	0	0 .
	2	×	×	×	×	×	(5)	5	5

Here a crossed-out cell in the array means that α and ϵ are incompatible. Since the minimal α-width $\tilde{\epsilon}(\alpha)$ is the first ϵ compatible with α, bracketed entries in the array pick out $\tilde{\epsilon}(1) = 3$, $\tilde{\epsilon}(2) = 6$. In terms of the matrix $\tilde{A}$, $\tilde{\epsilon}(\alpha)$ can be read off by looking at its last row: the αth 1 of this row occurs in column $\tilde{\epsilon}(\alpha)$.

We conclude this section by listing some properties of $\delta(\alpha, \epsilon)$:

$$\delta(\alpha, \epsilon) \geqslant \delta(\alpha, \epsilon + 1), \tag{3.22}$$

$$\delta(\alpha - 1, \tilde{\epsilon}(\alpha)) = 0, \tag{3.23}$$

$$\delta(\alpha, \tilde{\epsilon}(\alpha)) > 0, \tag{3.24}$$

$$\delta(\alpha, \tilde{\epsilon}(\alpha)) = \delta(\alpha - 1, \tilde{\epsilon}(\alpha) - 1) + m - s_{\tilde{\epsilon}(\alpha)}. \tag{3.25}$$

The first three of these are evident, either from the definition of multiplicity or from the multiplicity matrix $\tilde{M}$. The last is easily proved using $\tilde{M}$; it can also be established from the formula (2.13) for $\delta(\alpha, \epsilon)$, but this approach is more complicated.

4. The sequences $\mu_A(\beta)$ and $\bar{\mu}(\beta)$. Let $\mathfrak{A} = \mathfrak{A}(R, S)$ be a normalized class and suppose that β is an integer parameter in the range

$$1 \leqslant \beta \leqslant r_1. \tag{4.1}$$

For each A in $\mathfrak{A}$ let $\mu_A(\beta)$ denote the maximal number of columns of A all of whose row sums are at most β. (For example, if A is the line-point incidence matrix of a projective plane, then $\mu_A(2)$ is the maximal number of points no three of which are collinear, that is, the size of a maximal oval in the plane.) In this section we point out the close connection between this concept and that of width. In particular, we show that the preceding discussion on multiplicity and minimal width solves the problem of determining the class sequence

$$\bar{\mu}(\beta) = \max_{A \text{ in } \mathfrak{A}} \mu_A(\beta). \tag{4.2}$$

It will simplify matters in this section if we extend the range of β in (4.1) to include $\beta = 0$ by defining $\mu_A(0) = 0$. We also take $\epsilon_A(0) = 0$.

By the *complementary class* $\mathfrak{A}' = \mathfrak{A}(R', S')$ of $\mathfrak{A} = \mathfrak{A}(R, S)$ we mean the class of all (0, 1)-matrices of size m by n with row sum vector

$$R' = (n - r_m, n - r_{m-1}, \ldots, n - r_1), \tag{4.3}$$

and column sum vector

$$S' = (m - s_n, m - s_{n-1}, \ldots, m - s_1). \tag{4.4}$$

For the purposes of this discussion we take $\mathfrak{A}$ so that $r_1 < n$, $s_1 < m$. This is no real restriction and makes the complementary class normalized. There is, of course, a natural correspondence between the matrices of $\mathfrak{A}$ and those of $\mathfrak{A}'$, given by taking the complement of A and reversing the order of its rows and columns. We denote the resulting matrix by A′ and call it the *class complement* of A.

LEMMA 4.1. *Let α and β be integers in the respective intervals*

$$0 \leqslant \alpha \leqslant n - r_1, \tag{4.5}$$

$$0 \leqslant \beta \leqslant r_1, \tag{4.6}$$

and let A' be the class complement of A. Then

$$\epsilon_{A'}(\alpha) \leqslant \alpha + \beta \tag{4.7}$$

if and only if

$$\mu_A(\beta) \geqslant \alpha + \beta. \tag{4.8}$$

Proof. Note that $\alpha + \beta$ ranges over the interval $0 \leqslant \alpha + \beta \leqslant n$.

Assume (4.7). Then there are $\alpha + \beta$ columns of A' having at least α 1's in each row. Hence there are $\alpha + \beta$ columns of A having at most $\beta = (\alpha+\beta)-\alpha$ 1's in each row. Thus (4.8) holds. Conversely, if (4.8) holds, so that A has $\alpha + \beta$ columns with at most β 1's in each row, then A' has $\alpha + \beta$ columns with at least α 1's in each row. Hence (4.7).

We use this lemma a number of times in the proof of Theorem 4.2, which shows that the sequences $\epsilon_{A'}(\alpha)$ and $\mu_A(\beta)$ determine each other.

THEOREM 4.2. (i) *Let α be fixed in the interval* (4.5) *and let β be the least integer in the interval* (4.6) *for which $\mu_A(\beta) - \beta \geqslant \alpha$. Then $\epsilon_{A'}(\alpha) = \alpha + \beta$. Conversely, if α is fixed in the interval* (4.5) *and if $\epsilon_{A'}(\alpha) = \alpha + \beta$, then β is the least integer in* (4.6) *for which $\mu_A(\beta) - \beta \geqslant \alpha$.*

(ii) *Let β be fixed in the interval* (4.6) *and let α be the largest integer in the interval* (4.5) *for which $\epsilon_{A'}(\alpha) - \alpha \leqslant \beta$. Then $\mu_A(\beta) = \alpha + \beta$. Conversely, if β is fixed in the interval* (4.6) *and if $\mu_A(\beta) = \alpha + \beta$, then α is the largest integer in* (4.5) *for which $\epsilon_{A'}(\alpha) - \alpha \leqslant \beta$.*

Proof. Observe that both sequences $\epsilon_{A'}(\alpha) - \alpha$ and $\mu_A(\beta) - \beta$ are monotone non-decreasing. We now prove (i). Let α be fixed in (4.5) and let β be the least integer in (4.6) for which $\mu_A(\beta) - \beta \geqslant \alpha$. Such a β exists since $\mu_A(r_1) - r_1 = n - r_1 \geqslant \alpha$. By Lemma 4.1, we have $\epsilon_{A'}(\alpha) \leqslant \alpha + \beta$. If $\beta = 0$, then $\alpha = 0$, and the conclusion follows. If $\beta > 0$, then $\mu_A(\beta - 1) - (\beta - 1) < \alpha$. Hence by Lemma 4.1, $\epsilon_{A'}(\alpha) > \alpha + \beta - 1$. Thus $\alpha + \beta - 1 < \epsilon_{A'}(\alpha) \leqslant \alpha + \beta$, and hence $\epsilon_{A'}(\alpha) = \alpha + \beta$.

Conversely, suppose $\epsilon_{A'}(\alpha) = \alpha + \beta$. We show first that $0 \leqslant \beta \leqslant r_1$. Clearly $0 \leqslant \beta$. Since also $\epsilon_{A'}(n - r_1) \leqslant n$, then $\beta = \epsilon_{A'}(\alpha) - \alpha \leqslant \epsilon_{A'}(n - r_1) - (n - r_1) \leqslant r_1$. Now by hypothesis and Lemma 4.1, we have $\mu_A(\beta) - \beta \geqslant \alpha$. If also $\mu_A(\beta - 1) - (\beta - 1) \geqslant \alpha$, Lemma 4.1 implies $\epsilon_{A'}(\alpha) \leqslant \alpha + \beta - 1$, a contradiction. Hence β is the least integer in (4.6) for which $\mu_A(\beta) - \beta \geqslant \alpha$.

The proof of (ii) is similar.

Let $\mathfrak{A}$ be a normalized class and let the complementary normalized class $\mathfrak{A}'$ have minimal width sequence $\tilde{\epsilon}(\alpha)$. The discussion of the preceding section shows that the matrix $\tilde{A}$ in $\mathfrak{A}'$ has width $\tilde{\epsilon}(\alpha)$ for each $\alpha = 0, 1, \ldots, n - r_1$. It follows that the matrix $\tilde{A}'$ in $\mathfrak{A}$ yields the sequence $\bar{\mu}(\beta)$:

$$\bar{\mu}(\beta) = \max_{A \text{ in } \mathfrak{A}} \mu_A(\beta) = \mu_{\tilde{A}'}(\beta), \quad \beta = 0, 1, \ldots, r_1. \tag{4.9}$$

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The sequences $\bar{\epsilon}(\alpha)$ for $\mathfrak{A}'$ and $\bar{\mu}(\beta)$ for $\mathfrak{A}$ determine each other in the manner outlined in Theorem 4.2. In terms of the matrix $\tilde{A}$ in $\mathfrak{A}'$, the integer $\bar{\mu}(\beta)$ for $\mathfrak{A}$ can be singled out as follows: if the $(\beta + 1)$st zero of the last row of $\tilde{A}$ occurs in column j, then $\bar{\mu}(\beta) = j - 1$.

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