A CLASS OF INTEGRAL EQUATIONS INVOLVING LAGUERRE POLYNOMIALS AS KERNEL

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Introduction

Recently Widder (3) has obtained inversion integrals for a convolution transform whose kernel is a Laguerre polynomial. Buschman (1) has considered convolution equations with generalised Laguerre polynomial kernel. His result includes that of Widder. However no attempt has so far been made towards the study of singular integral equations involving Laguerre polynomials as kernel. Here the author obtains an inversion integral for such an equation.

The operator $\mathscr{F}_n$ and its properties

The operator $\mathscr{F}_n$ occurring in this note is defined by the formula

$$\mathscr{F}_n\{f(x)\} = \frac{e^{-x}}{\Gamma(n)} \int_x^1 (y-x)^{n-1} e^y f(y) dy. \tag{1}$$

If n is a positive integer, then by explicit computation we have

$$\frac{d^n}{dx^n}\left[e^x \mathscr{F}_n\{f(v)\} \right] = (-)^n e^x f(x), \tag{2}$$

$$\mathscr{F}_n\{f(x)\} = 0 \text{ for } x = 1. \tag{3}$$

Integral equations and their solutions

Consider the integral equations

$$\int_\sigma^1 \frac{L_n^{-\alpha}(u-\sigma)y(u)}{(u-\sigma)^\alpha} \, du = f(\sigma), \ \sigma \in I, \ n = 1, 2, 3, 4, \ldots \tag{4}$$

where the integral is taken in the Riemann sense, $I = \{\sigma : c \leqq \sigma \leqq 1\}$, $c > 0$ is a constant and $f(\sigma)$ is defined on I. It is assumed that

(a) $0 < \alpha < 1$,

(b) $f^k(1) = 0$ for $0 \leq k \leq 2n$,

(c) $f^{2n+1}(\sigma)$ is piecewise continuous on I. If these conditions are satisfied, then the solution of (4) is given by

$$y(u) = -A(n, \alpha) \int_u^1 \frac{L_n^{\alpha-1}(v-u)}{(v-u)^{1-\alpha}} \mathscr{F}_{2n}\{F(v)\} dv \tag{5}$$

where

$$A(n, \alpha) = \{(n)!\}^2/\Gamma(n-\alpha+1)\Gamma(n+\alpha),$$

$$F(v) = \frac{d^{2n+1}}{dv^{2n+1}}\{f(v)\}$$

and $L_n^{-\alpha}(x)$, $n = 1, 2, 3, \ldots$ are Laguerre polynomials.

As a preparation for the proof of the dual relation (4) and (5) we have to establish the following summation result. It can be easily shown that for $n \geqq 1$

$$\sum_{m=0}^{n} \sum_{r=0}^{n} \frac{z^{m+r}}{(n-m)!(n-r)!(m+r)!(r)!(m)!} = \{(n)!\}^2 L_{2n}(-z). \tag{6}$$

To prove this, we consider the sum

$$S = \sum_{m=0}^{n} \sum_{r=0}^{n} a_{m,r} z^{m+r} = \sum_{p=0}^{2n} c_p z^p$$

where

$$a_{m,r} = [(n-m)!(n-r)!(m+r)!(r)!(m)!]^{-1},$$

and

$$c_p = \sum_{s=0}^{p} a_{s,p-s} \qquad \text{for } p \leqq n;$$

$$= \sum_{s=0}^{2n-p} a_{s+p-n,n-s-1} \quad \text{for } p \geqq n+1.$$

Substituting the value of $a_{m,r}$, we get

$$c_p = \binom{2n}{p} \{(p)!\}^{-1}\{(n)!\}^{-2}.$$

Hence our result follows.

Proof of the dual relation

By virtue of the conditions (a), (b) and (c) it is obvious that the integral (5) exists and that the double integral

$$J = -A(n, \alpha) \int_{\sigma}^{1} \frac{L_n^{-\alpha}(u-\sigma)}{(u-\sigma)^{\alpha}} \left(\int_{u}^{1} \frac{L_n^{\alpha-1}(v-u)}{(v-u)^{1-\alpha}} \mathscr{F}_{2n}\{F(v)\}dv \right) du. \tag{7}$$

obtained by directly substituting (5) in the left-hand side of (4) is convergent. This double integral can be written as

$$J = -A(n, \alpha) \lim_{\varepsilon \to 0} \int_{\sigma+\varepsilon}^{1-\varepsilon} \int_{u+\varepsilon}^{1} \frac{L_n^{-\alpha}(u-\sigma)L_n^{\alpha-1}(v-u)}{(u-\sigma)^{\alpha}(v-u)^{1-\alpha}} \mathscr{F}_{2n}\{F(v)\}dvdu. \tag{8}$$

Since the integrals are uniformly bounded and there are at most a finite number of discontinuities of $F(v)$ in the region

$$R: \begin{cases} u+\varepsilon \leqq v \leqq 1, \\ \sigma+\varepsilon \leqq u \leqq 1, \end{cases}$$

it is justifiable to interchange the order of integration in R. Thus we obtain

$$J = - A(n, \alpha) \lim_{\varepsilon \to 0} \int_{\sigma+2\varepsilon}^{1} \mathscr{F}_{2n}\{F(v)\}dv \int_{\sigma+\varepsilon}^{\sigma-\varepsilon} \frac{L_n^{-\alpha}(u-\sigma)L_n^{\alpha-1}(v-u)}{(u-\sigma)^{\alpha}(v-u)^{1-\alpha}} du. \qquad (9)$$

Because $L_n^{-\alpha}(u-\sigma)L_n^{\alpha-1}(v-u)$ is continuous and finite in the interval $v \leqq u \leqq \sigma$ and

$$\int_{\sigma}^{v} (u-\sigma)^{\alpha}(v-u)^{1-\alpha}du = \pi/\sin \pi\alpha,$$

the integral

$$\int_{\sigma}^{v} \frac{L_n^{-\alpha}(u-\sigma)L_n^{\alpha-1}(v-u)}{(u-\sigma)^{\alpha}(v-u)^{1-\alpha}} du$$

exists, so that we can write

$$J = - A(n, \alpha) \int_{\sigma}^{1} \mathscr{F}_{2n}\{F(v)\}dv \int_{\sigma}^{v} \frac{L_n^{-\alpha}(u-\sigma)L_n^{\alpha-1}(v-u)}{(u-\sigma)^{\alpha}(v-u)^{1-\alpha}} du. \qquad (10)$$

From ((2), p. 188 (7)) we have

$$L_n^{\alpha}(x) = \sum_{m=0}^{n} \binom{n+\alpha}{n-m} \frac{(-x)^m}{(m)!}.$$

Hence

$$L_n^{-\alpha}(u-\sigma)L_n^{\alpha-1}(v-u) = \sum_{m=0}^{n} \sum_{r=0}^{n} \frac{\Gamma(n-\alpha+1)\Gamma(n+\alpha)(-)^{m+r}(u-\sigma)^m(v-u)^r}{(n-m)!(n-r)!\Gamma(m-\alpha+1)\Gamma(\alpha+r)(r)!(m)!}.$$

$$(11)$$

Substituting (11) in (10), writing $u-\sigma = x(v-\sigma)$ and making use of the following Euler's integral of the first kind

$$\int_{\sigma}^{v} (u-\sigma)^{m-\alpha}(v-u)^{\alpha+r-1}du = (v-\sigma)^{m+r} \frac{\Gamma(m-\alpha+1)\Gamma(\alpha+r)}{(m+r)!}, \qquad (12)$$

(10) reduces to

$$J = - \{(n)!\}^2 \int_{\sigma}^{1} \mathscr{F}_{2n}\{F(v)\}B(v-\sigma)dv \qquad (13)$$

where

$$B(v-\sigma) = \sum_{m=0}^{n} \sum_{r=0}^{n} \frac{(\sigma-v)^{m+r}}{(n-m)!(n-r)!(m+r)!(r)!(n)!}.$$

By (6), we have after using Rodrigues' formula for the Laguerre polynomial

$$B(v-\sigma) = L_{2n}(v-\sigma)/\{(n)!\}^2$$

$$= \frac{e^v}{(2n)!\{(n)!\}^2} \frac{d^{2n}}{dv^{2n}} \{e^{-v}(v-\sigma)^{2n}\}. \qquad (14)$$

Introducing (14) in (13), we get

$$J = - \frac{1}{(2n)!} \int_{\sigma}^{1} \frac{d^{2n}}{dv^{2n}} \{e^{-v}(v-\sigma)^{2n}\}[e^v\mathscr{F}_{2n}\{F(v)\}]dv.$$

Successive integrations by parts and application of the properties (2) and (3) give

$$J = - \frac{1}{(2n)!} \int_{\sigma}^{1} (v-\sigma)^{2n} \frac{d^{2n+1}}{dv^{2n+1}} \{f(v)\} dv.$$

Further successive integrations by parts and application of the conditions $f^{k}(1) = 0, 0 \leq k \leq 2n$, finally yield

$$J = f(\sigma).$$

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