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Part 1. Risk theory

RUIN PROBABILITIES IN A DIFFUSION ENVIRONMENT

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RUIN PROBABILITIES IN A DIFFUSION ENVIRONMENT

BY JAN GRANDELL AND HANSPETER SCHMIDLI

Abstract

We consider an insurance model, where the underlying point process is a Cox process. Using a martingale approach applied to diffusion processes, finite-time Lundberg inequalities are obtained. By change-of-measure techniques, Cramér–Lundberg approximations are derived.

Keywords: Ruin probability; Lundberg inequality; Cox process; diffusion process; martingale method; change-of-measure technique

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1. Introduction

Björk and Grandell [2] derived by a 'martingale approach' a general (infinite-time) Lundberg inequality when the occurrence of the claims is described by a Cox process. They applied this general result to the 'independent jump intensity' and 'Markov renewal intensity' cases and got rather explicit results. Both of these classes contain Markovian intensities and in those (Markovian) cases they considered a 'modified' martingale approach in order to obtain improved Lundberg inequalities. Grandell [5, appendix] derived finite-time Lundberg inequalities in the above cases of Markovian intensities. All results referred to in [2] can also be found in [5]. Embrechts *et al.* [3] extended the finite-time results for Markovian intensities to non-Markovian intensities within the classes mentioned. Their method is to 'Markovize' by introducing auxiliary processes and to apply the theory of piecewise-deterministic Markov processes. Similar results were proved by Schmidli [14] for the more general case where the intensity levels build a Markov renewal process. Grigelionis [7], [8] extended that approach, so that, for example, diffusion intensities were also included. A series of conditions similar to the conditions used in the present paper have to be fulfilled in [7] and [8].

It is well known that any stochastic process can be approximated by Markov renewal processes. We could therefore argue that considering Cox models with a piecewise constant intensity is sufficient for applications. However, Cox models are often motivated by an auxiliary process driving the intensity. This auxiliary process typically describes some state of nature or of the economy. We would therefore expect that changes of these states should occur continuously, maybe accompanied by shocks. We therefore consider in this paper intensities of diffusion type. For simplicity, we consider only one-dimensional diffusion Markov processes. The approach used, however, could easily be generalised to multidimensional diffusions.

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a complete probability space on which all stochastic quantities are defined. It is assumed to carry the following independent objects: (i) a point process $N = \{N_t; t \geq 0\}$ with $N_0 = 0$; (ii) a sequence $\{U_k\}_1^\infty$ of independent and identically distributed random variables, having common distribution function F, with $F(0) = 0$ and mean value μ. The *risk process*, X, is defined by

$$X_t = ct - \sum_{k=1}^{N_t} U_k \qquad \left(\sum_{k=1}^{0} U_k := 0\right), \tag{1.1}$$

where c is a positive real constant. If N is a stationary point process with intensity α, i.e. $\mathrm{E}[N_t] = \alpha t$, then $\mathrm{E}[X_t] = ct - \mathrm{E}[N_t]\,\mathrm{E}[U_k] = (c - \alpha\mu)t$. The relative *safety loading* ρ is defined by $\rho = (\alpha\mu)^{-1}(c - \alpha\mu)$. Let $h(r) := \int_0^\infty \mathrm{e}^{rz}\,\mathrm{d}F(z) - 1$. Assume that the distribution $F(x)$ is *light tailed*, i.e. that there exists $r_\infty > 0$ such that $h(r) \uparrow +\infty$ as $r \uparrow r_\infty$ (we allow for the possibility that $r_\infty = +\infty$). This means that the tail of F decreases at least exponentially fast, and, thus, heavy-tailed distributions like the lognormal and the Pareto distributions are excluded. It is easily seen that $h(0) = 0$, and that h is increasing, convex, and continuous on $(-\infty, r_\infty)$.

Let T_u be the time of ruin with initial capital u, i.e. $T_u = \inf\{t \geq 0 \mid u + X_t < 0\}$. We have

$$\Psi(u, t) := \mathrm{P}\{u + X(s) < 0 \text{ for some } s \in (0, t]\} = \mathrm{P}\{T_u \leq t\}$$

and

$$\Psi(u) := \mathrm{P}\{u + X(s) < 0 \text{ for some } s > 0\} = \mathrm{P}\{T_u < \infty\}.$$

In order to simplify the presentation, from now on we assume that all processes are càdlàg, i.e. right continuous and the limits from the left exist. For any process $Y = \{Y_t; t \geq 0\}$, the natural filtration $\mathbb{F}^Y = (\mathcal{F}_t^Y; t \geq 0)$ is the smallest right-continuous filtration $\mathbb{F}$ such that Y_t is $\mathcal{F}_t$-measurable for all $t \geq 0$.

The rest of this paper is organised as follows. In Section 2 Cox models are introduced and the martingale approach is discussed. In Section 3 we review change-of-measure techniques that are useful in order to prove Cramér–Lundberg approximations. In Section 4 that approach is, by way of example, applied to Ornstein–Uhlenbeck intensities. We show that it is possible to verify the conditions we need in our general results. The example of Section 4.1 has also been treated in [11], where the Lundberg inequalities were also obtained.

2. Cox models

Let $\lambda = \{\lambda_t; t \geq 0\}$ be a nonnegative stochastic process. We call it the *intensity process*. Define $\Lambda_t := \int_0^t \lambda(s)\,\mathrm{d}s$. Let $\tilde{N}$ be a homogeneous Poisson process with rate 1 independent of Λ. The claim number process N is given by $N_t = \tilde{N}_{\Lambda_t}$. Then N is a Cox process. A *Cox model* is a risk process X, cf. (1.1), where N is a Cox process.

For stationary intensity processes, we let $\alpha = \mathrm{E}[\lambda_1]$ and $r_\lambda(t) = \mathrm{cov}[\lambda_s, \lambda_{s+t}]$. It readily follows that $\mathrm{E}[N_t] = \alpha t$ and $\mathrm{var}[N_t] = \alpha t + \mathrm{var}[\Lambda_t]$. Hence, a natural measure of the variability of Λ is σ_Λ^2, defined by $\sigma_\Lambda^2 := \lim_{t\to\infty} t^{-1}\,\mathrm{var}[\Lambda_t] = \int_{-\infty}^{\infty} r_\lambda(t)\,\mathrm{d}t$, provided that the integral is well defined. A detailed discussion of Cox processes and their impact on risk theory can be found in [5, Chapter 4].

We will now consider Cox models where the intensity process is generated by a Markov diffusion process Z,

$$\mathrm{d}Z_t = a(Z_t)\,\mathrm{d}t + b(Z_t)\,\mathrm{d}W_t, \qquad Z_0 = z,$$

where $a(z)$ and $b(z)$ are Lipschitz continuous functions. Let $\ell(\cdot)$ be a nonnegative monotone function on the state space. Then the intensity process is defined by $\lambda_t = \ell(Z_t)$.

Now we consider the vector-valued process Y given by $Y_t = (X_t, Z_t, t)$, which is a Markov process. The generator, $\mathcal{A}$, of Y is given by

$$\mathcal{A} f(x, z, t) = c\frac{\partial f(x, z, t)}{\partial x} + \ell(z)\int_0^\infty (f(x-y, z, t) - f(x, z, t))\,\mathrm{d}F(y) + a(z)\frac{\partial f(x, z, t)}{\partial z} + \frac{1}{2}b^2(z)\frac{\partial^2 f(x, z, t)}{\partial z^2} + \frac{\partial f(x, z, t)}{\partial t}, \tag{2.1}$$

provided that f is twice continuously differentiable with respect to z, and continuously differentiable with respect to x and t.

For fixed r, we look for a positive $\mathbb{F}^Y$-martingale M of the form

$$M_t = \mathrm{e}^{-\theta t} g(Z_t)\mathrm{e}^{-rX_t}, \tag{2.2}$$

with a twice continuously differentiable function g. Without loss of generality, we assume that $\mathrm{E}[g(Z_0)] = 1$. For $f(x, z, t) = \mathrm{e}^{-rx} g(z)\mathrm{e}^{-\theta t}$, we obtain (see (2.1))

$$\mathcal{A} f(x, z, t) = \big[-crg(z) + \ell(z)h(r)g(z) + a(z)g'(z) + \tfrac{1}{2}b^2(z)g''(z) - \theta g(z)\big]\mathrm{e}^{-rx-t\theta},$$

and, thus, in order that $\mathcal{A} f(x, z, t) = 0$, g is a solution to

$$\tfrac{1}{2}b^2(z)g''(z) + a(z)g'(z) - [\theta + cr - \ell(z)h(r)]g(z) = 0. \tag{2.3}$$

In general, (2.3) is hard to solve. Suppose, however, that we found a positive solution g and a θ-value. Obviously, θ will depend on r, and since r will be the 'important' parameter, we write $\theta(r)$ in order to make this dependence explicit. Suppose further that $\theta(0) = 0$ (which always holds if Z is stationary), and that $\theta(r)$ is well defined, differentiable, and convex. We can now obtain estimates of the ruin probability by a martingale technique, first introduced in [4], and described in detail in [3] and [6]; therefore, we will only give the main steps.

Choose $\underline{y}$ and $\overline{y}$ such that $0 \le \underline{y} \le \overline{y} < \infty$, and consider $\overline{y}u \wedge T_u$, which is a bounded $\mathbb{F}^Y$-stopping time. Since M is positive, it follows by optional stopping that

$$1 = \mathrm{E}^{\mathcal{F}_0^Y}[M(\overline{y}u \wedge T_u)] \ge \mathrm{E}^{\mathcal{F}_0^Y}[M(T_u) \mid \underline{y}u < T_u \le \overline{y}u]\mathrm{P}^{\mathcal{F}_0^Y}\{\underline{y}u < T_u \le \overline{y}u\}.$$

Since $u + X(T_u) \le 0$ on $\{T_u < \infty\}$, we obtain

$$\mathrm{P}^{\mathcal{F}_0^Y}\{\underline{y}u < T_u \le \overline{y}u\} \le \frac{\exp\{-u\min(r - \underline{y}\theta(r), r - \overline{y}\theta(r))\}}{\mathrm{E}^{\mathcal{F}_0^Y}[g(Z(T_u)) \mid \underline{y}u < T_u \le \overline{y}u]}.$$

Provided that $\mathrm{E}^{\mathcal{F}_0^Y}[g(Z(T_u)) \mid \underline{y}u < T_u \le \overline{y}u]$ is bounded away from 0, we obtain

$$\Psi(u, \overline{y}u) - \Psi(u, \underline{y}u) \le C_1\exp\{-u\min(r - \underline{y}\theta(r), r - \overline{y}\theta(r))\}$$

for some constant $C_1 < \infty$. Define the *finite-time Lundberg exponent* $R(\underline{y}, \overline{y})$ by

$$R(\underline{y}, \overline{y}) = \sup_{r\ge 0}\min(r - \underline{y}\theta(r), r - \overline{y}\theta(r)),$$

yielding the *finite-time Lundberg inequality*

$$\Psi(u, \overline{y}u) - \Psi(u, \underline{y}u) \le C_1\mathrm{e}^{-R(\underline{y},\overline{y})u}. \tag{2.4}$$

Assume that $\theta(r) = 0$ has a positive solution R and that $\theta(r) < \infty$ for some $r > R$. Set $f_y(r) = r - y\theta(r)$. Let r_y denote the solution to $f_y'(r_y) = 0$. We call $y_0 := 1/\theta'(R)$ the

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critical value. Then (see [3])

$$R(\underline{y}, \overline{y}) = \begin{cases} R & \text{if } \underline{y} \le y_0 \le \overline{y}, \\ f_{\overline{y}}(r_{\overline{y}}) & \text{if } \overline{y} \le y_0, \\ f_{\underline{y}}(r_{\underline{y}}) & \text{if } y_0 \le \underline{y}. \end{cases}$$

We have $R(\underline{y}, \overline{y}) > 0$ if $\overline{y} \le y_0$ or $y_0 \le \underline{y}$. For $\underline{y} = 0$ and $\overline{y} \to \infty$, we obtain

$$\Psi(u) \le C_1 e^{-Ru}, \tag{2.5}$$

which is the 'ordinary' *Lundberg inequality* with R the *Lundberg exponent*. A complication that may occur is if $\theta(r)$ is defined for $r \in [0, \tilde{r}]$ only, and $\theta(\tilde{r}) < 0$. Since in that case y_0 is not defined, we consider only the infinite-time case. Provided that $C_1 < \infty$, (2.5) holds with $R = \tilde{r}$.

3. The change-of-measure technique

Suppose that the martingale M given by (2.2) is well defined. Then $E[M_t] = 1$ and $M_t > 0$. It is therefore possible to define the equivalent measure Q on $\mathcal{F}_t^Y$ by $Q\{A\} = E_P[M_t; A]$. It is possible to extend the measure Q to $\mathcal{F}$. It follows that, for any stopping time T and any set $A \in \mathcal{F}_T$ such that $A \subset \{T < \infty\}$, the formula $Q\{A\} = E_P[M_t; A]$ holds. For an introduction to change-of-measure techniques, see [1, pp. 160–168], [9], or [13, Chapter 10].

Let $\tilde{h}(r) = h(r) + 1$. The next result gives us the law of the risk process under the measure Q. A proof can be found in [12].

Lemma 3.1. *Suppose that the martingale M defined by (2.2) is well defined and that g is twice continuously differentiable. Then the process $Y = (X, Z, t)$ is a Cox model under the measure Q with intensity process $\tilde{\lambda}_t = \ell(Z_t)\tilde{h}(r)$ and claim size distribution $d\tilde{F}(x) = e^{rx}\, dF(x)/\tilde{h}(r)$. The process Z is a Markov diffusion process with generator*

$$\tilde{\mathcal{A}} f = \frac{ga + b^2 g'}{g} f' + \frac{1}{2} b^2 f'' \tag{3.1}$$

on the set of twice continuously differentiable functions f.

Typically, the function $\theta(r)$ will be convex. Since $E_Q[X_t] = E_P[X_t g(Z_t) e^{-rX_t} e^{-\theta(r)t}]$, we have, provided that the derivative and expectation can be interchanged,

$$0 = \frac{d}{dr} E_P[g(Z_t) e^{-rX_t} e^{-\theta(r)t}] = E_P\Big[\Big(\frac{d}{dr} g(Z_t)\Big) e^{-rX_t} e^{-\theta(r)t}\Big] - E_Q[X_t] - t\theta'(r).$$

Typically, dividing by t and letting $t \to \infty$, the first term on the right-hand side will vanish. Thus, $t^{-1} E_Q[X_t]$ will converge to $-\theta'(r)$. Hence, the safety loading condition will not be fulfilled for $r \ge r_0$, where r_0 is the solution to $\theta'(r) = 0$. This means that $Q\{T_u < \infty\} = 1$ if and only if $r \ge r_0$.

Expressing the ruin probability under the measure Q yields

$$\Psi(u) = E_Q\Big[\frac{1}{g(Z_{T_u})} e^{r(X_{T_u}+u)} e^{\theta(r)T_u}; T_u < \infty\Big] e^{-ru}.$$

Choosing $r = R$, the ruin probability simplifies to

$$\Psi(u) = E_Q\Big[\frac{1}{g(Z_{T_u})} e^{R(X_{T_u}+u)}\Big] e^{-Ru}.$$

Note that $\theta'(R) > 0$ by convexity and, thus, $R > r_0$. By the definition of T_u, $X_{T_u} + u < 0$. If $\mathrm{E_Q}[1/g(Z_{T_u})]$ can be bounded from above, a Lundberg inequality is found.

Let $f(u) = \mathrm{E_Q}[\mathrm{e}^{R(X_{T_u}+u)}g(Z_{T_u})]$. Suppose first that $Z_0 = z$ for some fixed z. Choose $\varepsilon > 0$ such that $S := \inf\{t > 0\colon |Z_t - z| = \varepsilon\}$ is finite almost surely. Let $S_0 = 0$ and $S_1 := \inf\{t > S\colon Z_t = z\}$. Then S_1 is a regeneration time. In the same way we define a sequence of regeneration times $\{S_n\}$. Now let $S_+ = \inf\{S_n\colon X_{S_n} < 0\}$. Then $\{(X(S_+ + t) - X(S_+), Z(S_+ + t), t); t \geq 0\}$ follows the same law as Y. But the initial capital is $u + X(S_+) < u$. In principle, it would then follow from the renewal theorem that $f(u)$ is converging as $u \to \infty$. Unfortunately, $T_u < S_+$ is possible, and the equation to consider is

$$f(u) = \mathrm{E_Q}\left[\frac{1}{g(Z_{T_u})}\mathrm{e}^{R(X_{T_u}+u)}; T_u \leq S_+\right] + \mathrm{E_Q}[f(u + X(S_+)); T_u > S_+].$$

Theorem 3.1. *Suppose that M is a martingale and that $\mathrm{Q}\{T_u < \infty\} = 1$. Suppose further that Z is Harris recurrent, that the functions*

$$\mathrm{E_Q}\left[\frac{1}{g(Z_{T_u})}\mathrm{e}^{R(X_{T_u}+u)}; T_u \leq S_+\right] \quad \textit{and} \quad \mathrm{Q}\{T_u \leq S_+\}$$

are directly Riemann integrable, and that $\mathrm{Q}\{T_u \leq S_+ \mid X(S_+) = -y\}$ is continuous in u. Then $\lim_{u\to\infty} \Psi(u)\mathrm{e}^{Ru} = C$ for some $C \in (0, \infty)$.

Proof. The result follows from Theorem 2 of [15].

Remark 3.1. Note that the constant C depends on the initial distribution. In general, if $g(z)$ is chosen such that $\mathrm{E_P}[g(Z_0)] = 1$ under the stationary initial distribution then, for any other initial distribution $\tilde{\mathrm{P}}$, say, we obtain the limit with $\tilde{C} = \tilde{\mathrm{E}}[g(Z_0)]C$.

Now let $0 \leq \underline{y} \leq \overline{y} < \infty$, and consider $\Psi(u, \overline{y}u) - \Psi(u, \underline{y}u)$. With the change of measure we can express this as

$$\Psi(u, \overline{y}u) - \Psi(u, \underline{y}u) = \mathrm{E_Q}\left[\frac{1}{g(Z_{T_u})}\mathrm{e}^{r(X_{T_u}+u)}\mathrm{e}^{\theta(r)T_u}; \underline{y}u < T_u \leq \overline{y}u\right]\mathrm{e}^{-ru}.$$

If $0 < r < R$ then $\theta(r) < 0$ and

$$\begin{aligned}
\mathrm{E_Q}&\left[\frac{1}{g(Z_{T_u})}\mathrm{e}^{r(X_{T_u}+u)}; \underline{y}u < T_u \leq \overline{y}u\right]\mathrm{e}^{-(r-\overline{y}\theta(r))u} \\
&\leq \Psi(u, \overline{y}u) - \Psi(u, \underline{y}u) \\
&\leq \mathrm{E_Q}\left[\frac{1}{g(Z_{T_u})}\mathrm{e}^{r(X_{T_u}+u)}; \underline{y}u < T_u \leq \overline{y}u\right]\mathrm{e}^{-(r-\underline{y}\theta(r))u}.
\end{aligned} \tag{3.2}$$

If $R < r < r_\infty$ then $\theta(r) > 0$ and

$$\begin{aligned}
\mathrm{E_Q}&\left[\frac{1}{g(Z_{T_u})}\mathrm{e}^{r(X_{T_u}+u)}; \underline{y}u < T_u \leq \overline{y}u\right]\mathrm{e}^{-(r-\underline{y}\theta(r))u} \\
&\leq \Psi(u, \overline{y}u) - \Psi(u, \underline{y}u) \\
&\leq \mathrm{E_Q}\left[\frac{1}{g(Z_{T_u})}\mathrm{e}^{r(X_{T_u}+u)}; \underline{y}u < T_u \leq \overline{y}u\right]\mathrm{e}^{-(r-\overline{y}\theta(r))u}.
\end{aligned}$$

The discussion in [3] then yields (2.4). If $\underline{y} > y_0$ then the exponent $R(\underline{y}, \overline{y})$ does not depend on $\overline{y}$. By letting $\overline{y} \to \underline{y}$ in (3.2) we can show that $R(\underline{y}, \overline{y})$ is in fact the Lundberg exponent,

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i.e. the best possible exponent in an exponential inequality. An analogous argument shows that $R(\underline{y}, \overline{y})$ is the Lundberg exponent in the $\overline{y} < y_0$ case.

4. Ornstein–Uhlenbeck intensities

We now turn to an example, and show how the results of Sections 2 and 3 can be applied to concrete models. Consider an Ornstein–Uhlenbeck process Z. That is, the solution to $\mathrm{d}Z_t = -aZ_t\,\mathrm{d}t + b\,\mathrm{d}W_t$. Let Z_0 be normally distributed with mean 0 and variance $b^2/(2a)$. Then Z is a stationary Gaussian process with $\mathrm{E}[Z_t] = 0$ and $\mathrm{cov}[Z_s, Z_{s+t}] = (b^2/2a)\mathrm{e}^{-a|t|}$; see also [13, p. 562]. The Ornstein–Uhlenbeck process is the only stationary Gaussian Markov process.

In Figure 1 we illustrate a 'standard' Ornstein–Uhlenbeck process ($a = 1, b = \sqrt{2}$) by a randomly generated realization. The illustrations in Sections 4.1 and 4.3 will be based upon this realization.

Assume that $\ell(\cdot)$ is a twice continuously differentiable function. Then it follows from Itô's formula that

$$\mathrm{d}\lambda_t = \big[-a\ell'(Z_t)Z_t + \tfrac{1}{2}b^2\ell''(Z_t)\big]\,\mathrm{d}t + b\ell'(Z_t)\,\mathrm{d}W_t.$$

4.1. Intensity of the form $\lambda_t = Z_t^2$

Consider $\ell(z) = z^2$. In Figure 2 we plot the intensity process. The corresponding Cox process is discussed in [10, pp. 225–228]. In this case we have

$$\mathrm{d}\lambda_t = [-2aZ_t^2 + b^2]\,\mathrm{d}t + 2bZ_t\,\mathrm{d}W_t = [-2a\lambda_t + b^2]\,\mathrm{d}t + 2b\sqrt{\lambda_t}\,\mathrm{sgn}(Z_t)\,\mathrm{d}W_t.$$

It follows that λ is a Markov process. Equation (2.3) reduces to

$$\tfrac{1}{2}b^2g''(z) - azg'(z) - [\theta(r) + cr - z^2h(r)]g(z) = 0. \tag{4.1}$$

We try $g(z) = \kappa\mathrm{e}^{kz^2}$ for some $k < a/b^2$. The restriction is in order to ensure that $\mathrm{E}[g(Z_0)] < \infty$. From $\mathrm{E}[g(Z_0)] = 1$ we find that $\kappa = \sqrt{1 - b^2k/a}$. Since

$$g'(z) = 2zkg(z) \quad \text{and} \quad g''(z) = (4z^2k^2 + 2k)g(z),$$

(4.1) reduces to

$$\tfrac{1}{2}b^2(4z^2k^2 + 2k) - 2az^2k - (\theta(r) + cr) + z^2h(r) = 0,$$

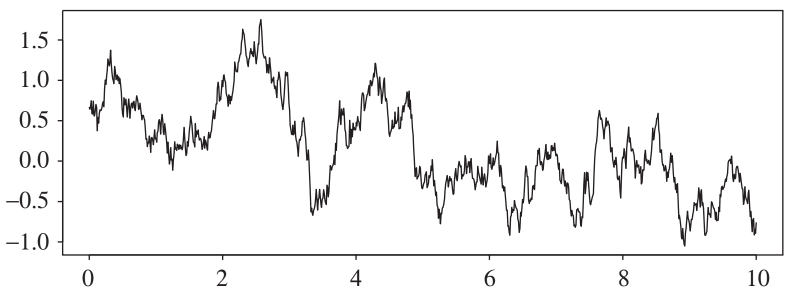

FIGURE 1: Randomly generated Ornstein–Uhlenbeck process with $a = 1$ and $b = \sqrt{2}$.

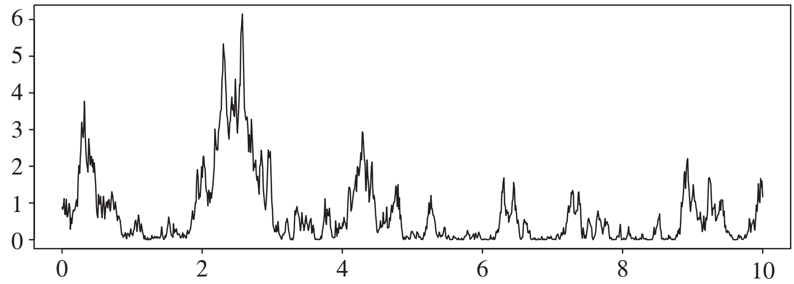

FIGURE 2: Randomly generated Ornstein–Uhlenbeck intensity with $a = 1$ and $b = \sqrt{2}$.

which implies that

$$2b^2k^2 - 2ak + h(r) = 0, \tag{4.2}$$
$$b^2k = \theta(r) + cr.$$

The solutions to (4.2) are $k = a/(2b^2) \pm \sqrt{a^2/(4b^4) - h(r)/(2b^2)}$, corresponding to $\theta(0) = a$ and 0, respectively. Thus, we must have

$$k = \frac{a}{2b^2} - \sqrt{\frac{a^2}{4b^4} - \frac{h(r)}{2b^2}}, \qquad \theta(r) = \frac{a - \sqrt{a^2 - 2b^2h(r)}}{2} - cr,$$

and

$$\kappa = \sqrt{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{b^2h(r)}{2a^2}}}.$$

Standard techniques show that M defined by (2.2) really is a martingale.

Note that k and $\theta(r)$ are real only for $r \leq \tilde{r}$, where $\tilde{r}$ is the solution to $h(r) = a^2/2b^2$. Since $g(z) \geq \kappa$ and $k < a/2b^2$ for all $r < \tilde{r}$, the ruin estimate goes through without any problems. We conclude that R exists if $a > 2c\tilde{r}$.

Let us now consider the process under the measure Q. We have already seen that Z is still a Markov diffusion process. From (3.1) we find that the generator is

$$\tilde{\mathcal{A}} f(z) = \frac{-\kappa e^{kz^2} az + b^2 2kz\kappa e^{kz^2}}{\kappa e^{kz^2}} f'(z) + \frac{1}{2} b^2 f''(z) = -(a - 2kb^2) z f'(z) + \frac{1}{2} b^2 f''(z).$$

Hence, under Q, the process Z is an Ornstein–Uhlenbeck process with the same diffusion coefficient b and drift $-\sqrt{a^2 - 2b^2h(r)}z$. We see that Z will revert to its mean more slowly than under P if $r > 0$. The (stationary under Q) drift of the process X under Q is then

$$c - \frac{b^2}{2\sqrt{a^2 - 2b^2h(r)}} \tilde{h}(r) \frac{h'(r)}{\tilde{h}(r)} = -\theta'(r),$$

proving the result obtained intuitively in Section 3. Suppose now that R exists. Then, by convexity, $\theta'(R) > 0$ and $\mathrm{Q}\{T_u < \infty\} = 1$. Lundberg's inequality then becomes

$$\Psi(u) = \mathrm{E_Q}\bigg[\frac{1}{g(Z_{T_u})} e^{R(u + X_{T_u})}\bigg] e^{-Ru} < \kappa^{-1} e^{-Ru}. \tag{4.3}$$

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In the same way we obtain the two finite-time Lundberg inequalities

$$\Psi(u, yu) < \kappa^{-1}e^{-R(0,y)u} \quad (y < y_0), \tag{4.4}$$

$$\Psi(u) - \Psi(u, yu) < \kappa^{-1}e^{-R(y,\infty)u} \quad (y > y_0). \tag{4.5}$$

Here we write $R(y, \infty)$ to visualise that we let $\overline{y} \to \infty$ because $R(\underline{y}, \overline{y})$ is independent of $\overline{y}$ if $\underline{y} \geq y_0$.

Let us now turn to the Cramér–Lundberg approximation. For simplicity, we assume that $\theta(r)$ is well defined for some $r > R$.

Proposition 4.1. *Let Z be an Ornstein–Uhlenbeck process, and let $\ell(z) = z^2$. Suppose that $\theta(r)$ is well defined for some $r > R$. There exists a constant $C > 0$ such that, for any initial distribution,*

$$\lim_{u\to\infty} \Psi(u)e^{Ru} = C\,E_P[g(Z_0)],$$

where $g(z)e^{kz^2}$ is normed such that $E_P^s[g(Z_0)] = 1$ for the stationary initial distribution.

Proof. It is well known that the Ornstein–Uhlenbeck process is Harris recurrent. We leave it to the reader to show that the other conditions of Theorem 3.1 are fulfilled.

Since $\alpha = E[Z_t^2] = b^2/(2a)$, it is natural to let $b^2 = 2a\alpha$. Then $r_\lambda(t) = 2\alpha^2e^{-2a|t|}$ and $\sigma_\Lambda^2 = 2\alpha^2/a$. The realization in Figure 2 corresponds to $\alpha = 1$ and $a = 1$.

With this choice of parameters we obtain

$$\theta(r) = \frac{a}{2}\left(1 - \sqrt{1 - \frac{4\alpha h(r)}{a}}\right) - cr,$$

which is a convex function. We have

$$\theta'(r) = \frac{\alpha h'(r)}{\sqrt{1 - (4\alpha h(r))/a}} - c$$

and, thus, $\theta'(0) = \alpha\mu - c$. Since

$$\frac{a}{4\alpha} = h(\tilde{r}) \geq e^{\mu\tilde{r}} - 1 > \mu\tilde{r},$$

we obtain, in the case of a positive safety loading,

$$\theta(\tilde{r}) = \frac{a}{2} - c\tilde{r} > \frac{a}{2} - c\frac{a}{4\alpha\mu} = \frac{a}{4}(1 - \rho).$$

Thus, $\theta(\tilde{r}) > 0$, at least for $\rho \leq 1$, i.e. $\theta(r) = 0$ has a positive solution R and $\theta(r) < \infty$ for some $r > R$.

For exponentially distributed claims, we have $\tilde{r} = a/(a\mu + 4\alpha\mu)$ and $\theta(\tilde{r}) > 0$ for $\rho < 1 + a/(2\alpha)$. By routine calculations we obtain

$$R = \frac{a + \alpha(1+\rho)}{2\alpha\mu(1+\rho)}\left(1 - \sqrt{1 - \frac{4\rho\alpha a}{(a + \alpha(1+\rho))^2}}\right) \quad \text{for } \rho < 1 + \frac{a}{2\alpha}.$$

4.2. Intensity of the form $\lambda_t = m + Z_t^2$

Consider $\ell(z) = m + z^2$. Then (2.3) reduces to

$$\tfrac{1}{2}b^2 g''(z) - azg'(z) - [\theta(r) + cr - mh(r) - z^2 h(r)]g(z) = 0.$$

In comparison with (4.1) this just means that $\theta(r)+cr$ is replaced by $\theta(r)+cr-mh(r)$. Thus, the same function $g(z)$ works, and the only change is that

$$\theta(r) = mh(r) + \frac{a - \sqrt{a^2 - 2b^2 h(r)}}{2} - cr.$$

Remark 4.1. The risk process X can be written as $X_1 + X_2$, where X_1 is the process considered in Section 4.1 and X_2 is an independent classical risk process with intensity m. The martingale M is then of the form $M = M_1 M_2$, with the M_i being obvious martingales. It then immediately follows that M is a martingale, and X under Q is the same type of process. The Lundberg inequalities and the Cramér–Lundberg approximation now readily follow.

Since $\alpha = m + b^2/(2a)$, we set $m = (1-p)\alpha$ and $b^2 = 2pa\alpha$, where $0 \le p \le 1$. Then $r_\lambda(t) = 2p^2\alpha^2 \mathrm{e}^{-2a|t|}$ and $\sigma_\Lambda^2 = 2p^2\alpha^2/a$, and

$$\theta(r) = (1-p)\alpha h(r) + \frac{a}{2}\left(1 - \sqrt{1 - \frac{4p\alpha h(r)}{a}}\right) - cr.$$

We have (cf. Section 4.1) $\theta'(0) = (1-p)\alpha h'(0) + p\alpha h'(0) - c = \alpha\mu - c$. Furthermore, for $h(\tilde{r}) = a/(4p\alpha)$, we obtain

$$\theta(\tilde{r}) = \frac{(1-p)a}{4p} + \frac{a}{2} - c\tilde{r} \ge \frac{(1-p)a}{4p} + \frac{a}{2} - c\frac{a}{4\alpha\mu}\frac{(1-p)a}{4p} + \frac{a}{4}(1-\rho).$$

Thus, $\theta(\tilde{r}) > 0$, at least for $\rho < 1/p$.

4.3. Intensity of the form $\lambda_t = (m + Z_t)^2$

Consider the intensity $\ell(z) = (m+z)^2$. In this case λ is not Markovian. The intensity process is illustrated in Figures 3 and 4.

Equation (2.3) reduces to

$$\tfrac{1}{2}b^2 g''(z) - azg'(z) - [\theta(r) + cr - (m+z)^2 h(r)]g(z) = 0. \tag{4.6}$$

We try $g(z) = \kappa \mathrm{e}^{k_1 z + k_2 z^2}$ for some $k_2 < a/b^2$. Since

$$\begin{aligned}
g'(z) &= (k_1 + 2zk_2)g(z),\\
g''(z) &= [(k_1 + 2k_2 z)^2 + 2k_2]g(z) = (k_1^2 + 2k_2 + 4k_1k_2 z + 4k_2^2 z^2)g(z),
\end{aligned}$$

(4.6) reduces to

$$\begin{aligned}
&\tfrac{1}{2}b^2(k_1^2 + 2k_2 + 4k_1k_2 z + 4k_2^2 z^2) - a(k_1 + 2zk_2)z\\
&- [\theta(r) + cr] + m^2 h(r) + 2mzh(r) + z^2 h(r)\\
&\qquad = [2b^2k_2^2 - 2ak_2 + h(r)]z^2 + [2b^2k_1k_2 - ak_1 + 2mh(r)]z\\
&\qquad\quad + \tfrac{1}{2}b^2k_1^2 + b^2k_2 + m^2h(r) - [\theta(r) + cr]\\
&\qquad = 0.
\end{aligned}$$

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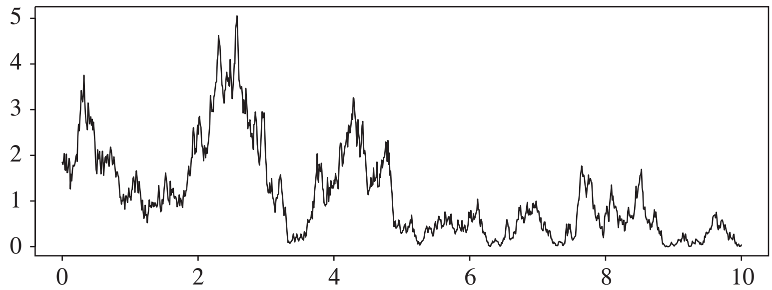

FIGURE 3: Randomly generated Ornstein–Uhlenbeck intensity with $a = 1$ and $m = b = \sqrt{\frac{2}{3}}$.

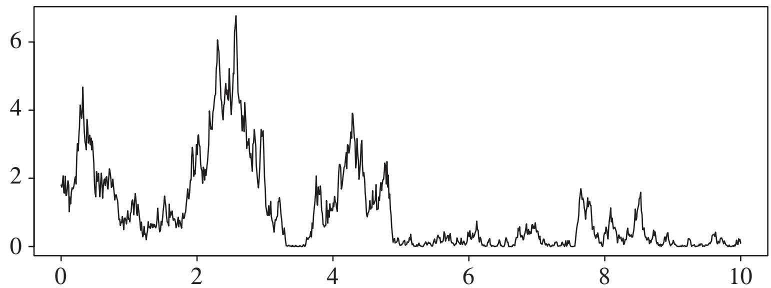

FIGURE 4: Randomly generated Ornstein–Uhlenbeck intensity with $m = \sqrt{\frac{1}{3}}$, $a = 1$, and $b = 2/\sqrt{3}$.

This implies that

$$\begin{aligned} 2b^2k_2^2 - 2ak_2 + h(r) &= 0, \qquad (4.7)\\ 2b^2k_1k_2 - ak_1 + 2mh(r) &= 0,\\ \tfrac{1}{2}b^2k_1^2 + b^2k_2 + m^2h(r) &= \theta(r) + cr. \end{aligned}$$

Equation (4.7) agrees with (4.2), and we obtain

$$k_2 = \frac{a}{2b^2} - \sqrt{\frac{a^2}{4b^4} - \frac{h(r)}{2b^2}}, \qquad k_1 = \frac{2mh(r)}{\sqrt{a^2 - 2b^2h(r)}},$$

and

$$\theta(r) = m^2h(r) + \frac{a - \sqrt{a^2 - 2b^2h(r)}}{2} + \frac{2b^2m^2h^2(r)}{a^2 - 2b^2h(r)} - cr.$$

Note that $\theta(0) = 0$. It turns out that M defined by (2.2) is a martingale.

Let us now consider the process Z under the measure Q. From the generator (3.1) we find that

$$\tilde{\mathcal{A}}f(z) = [b^2k_1 - (a - 2b^2k_2)z]f'(z) + \tfrac{1}{2}b^2f''(z).$$

Thus, the process $\{Z_t - b^2k_1/(a - 2k_2b^2)\}$ is an Ornstein–Uhlenbeck process. We therefore obtain the same model as before with the same diffusion coefficient b, drift $-(a - 2k_2b^2)z$, and $\tilde{m} = ma^2/[a^2 - 2b^2h(r)]$. Here the point to which λ returns is also larger, i.e. the risk process becomes more dangerous.

It now follows as in Section 4.1 that the Lundberg inequalities (4.3), (4.4), and (4.5) hold. In the same way as in Theorem 4.1 we obtain the Cramér–Lundberg approximation.

Proposition 4.2. *Let Z be an Ornstein–Uhlenbeck process, and let $\ell(z) = (m + z)^2$. Suppose that $\theta(r)$ is well defined for some $r > R$. There exists a constant $C > 0$ such that, for any initial distribution,* $\lim_{u\to\infty} \Psi(u)\mathrm{e}^{Ru} = C\,\mathrm{E}_{\mathrm{P}}[g(Z_0)]$, *where $g(z) = \kappa \mathrm{e}^{k_1 z + k_2 z^2}$ is normed such that $\mathrm{E}^{\mathrm{s}}_{\mathrm{P}}[g(Z_0)] = 1$ for the stationary initial distribution.*

Since $\alpha = m^2 + b^2/(2a)$, we set $m^2 = (1 - p)\alpha$ and $b^2 = 2pa\alpha$. Then we have

$$r_\lambda(t) = 4(1-p)p\alpha^2\mathrm{e}^{-a|t|} + 2p^2\alpha^2\mathrm{e}^{-2a|t|}, \qquad \sigma^2_\Lambda = \frac{8(1-p)p\alpha^2}{a} + \frac{2p^2\alpha^2}{a},$$

and

$$\theta(r) = (1-p)\alpha h(r) + \frac{a}{2}\left(1 - \sqrt{1 - \frac{4p\alpha h(r)}{a}}\right) + \frac{4(1-p)p\alpha^2 h^2(r)}{a - 4p\alpha h(r)} - cr.$$

For given values of α and a, the asymptotic variance σ^2_Λ is maximised by $p = \frac{2}{3}$, while it equals the $m = 0$ (or $p = 1$) case for $p = \frac{1}{3}$. The choice of parameters in Figures 3 and 4 correspond to $\alpha = 1$, $a = 1$, and $p = \frac{1}{3}$ and $\frac{2}{3}$, respectively.

We have (cf. Section 4.2) $\theta'(0) = \alpha\mu + 0 - c$. For $(1 - p)p \neq 0$, it is seen that $\theta(r) \to \infty$ as $r \uparrow \tilde{r}$, and, thus, $\theta(r) = 0$ always has a positive solution. Since $g(z) \geq g(-k_1/(2k_2)) > 0$, the ruin estimates go through without problems.

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