Glasgow Math. J. **55** (2013) 695–719. © Glasgow Mathematical Journal Trust 2013.
doi:10.1017/S0017089512000869.

DESCRIPTION OF SIMPLE MODULES FOR SCHUR SUPERALGEBRA $S(2|2)$

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(Received 10 August 2012; revised 10 September 2012; accepted 20 September 2012; first published online 25 February 2013)

Abstract. The goal of this paper is to describe explicitly simple modules for Schur superalgebra $S(2|2)$ over an algebraically closed field K of characteristic zero or positive characteristic $p > 2$.

2000 *Mathematics Subject Classification.* 17A70, 20G05, 15A72, 13A50, 05E15.

1. Notation and outline of the paper.

1.1. Notation. Throughout the paper we shall work over an algebraically closed field K of characteristic $p = 0$ or $p > 2$ and use the basic terminology of bialgebras $A(m|n)$, general linear supergroups $GL(m|n)$, Schur superalgebras $S(m|n)$ and superderivations ${}_{ij}D$ from papers [**6**, **8**]. All modules considered in this paper will be left modules and all superderivations will be right superderivations. In this paper we shall only work with $G = GL(2|2)$ and $S(2|2)$ and can therefore describe them in a down-to-earth fashion as follows.

Start by defining the parity $|i|$ of symbols $i = 1, \ldots, 4$ by $|1| = |2| = 0$ and $|3| = |4| = 1$, and the parity $|c_{ij}|$ of an element c_{ij} by $|c_{ij}| = |i| + |j| \pmod 2$. Elements of parity 0 will be called even and that of parity 1 will be called odd. Let $A = A(2|2)$ be a commutative superalgebra freely generated over K by elements c_{ij} for $1 \leq i, j \leq 4$, where $c_{11}, c_{12}, c_{21}, c_{22}, c_{33}, c_{34}, c_{43}$ and c_{44} are even and $c_{13}, c_{14}, c_{23}, c_{24}, c_{31}, c_{32}, c_{41}$ and c_{42} are odd. The superalgebra A has a structure of a bialgebra given by comultiplication $\delta : A \to A \otimes A$ defined as $\delta(c_{ij}) = \sum_k c_{ik} \otimes c_{kj}$. The superalgebra A has a natural grading given by the total degree and is a direct sum $A = \oplus_{r\geq 0} A(2|2, r)$ of its homogeneous components $A(2|2, r)$. Each component $A(2|2, r)$ is a coalgebra and its dual $A(2|2, r)^*$ is the component $S(2|2, r)$ of degree r of the Schur superalgebra $S(2|2) = \oplus_{r\geq 0} S(2|2, r)$. The localization of $A(2|2)$ by elements $d_{12} = c_{11}c_{22} - c_{12}c_{21}$ and $d_{34} = c_{33}c_{44} - c_{34}c_{43}$ is the coordinate superalgebra $K[G]$ of the general linear supergroup G. The general linear supergroup G is a group functor from the category $SAlg_K$ of commutative superalgebras over K to the category of groups represented by its coordinate ring $K[G]$, that is $G(A) = Hom_{SAlg_K}(K[G], A)$ for $A \in SAlg_K$. Here for

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$g \in G(A)$ and $f \in K[G]$ we define $f(g) = g(f)$. Modules over Schur superalgebra $S(2|2)$ correspond to polynomial representations of G.

In order to study the structure of G-modules, we will use the superalgebra of distributions $Dist(G)$ of G described in Section 3 of [**3**]. Denote $Dist_1(G) = (K[G]/\mathfrak{m}^2)^*$, where $*$ is the duality $Hom_K(-, K)$ and $\mathfrak{m}$ is the kernel of the augmentation map ϵ of the Hopf algebra $K[G]$, and by e_{ij} the elements of $Dist_1(G)$ determined by $e_{ij}(c_{hk}) = \delta_{ih}\delta_{jk}$ and $e_{ij}(1) = 0$. Denote the parity of e_{ij} to be sum of parities $|i|$ of i and $|j|$ of j. Then e_{ij} belongs to the Lie superalgebra $Lie(G) = (\mathfrak{m}/\mathfrak{m}^2)^*$, which is identified with the general linear Lie superalgebra $\mathfrak{gl}(m|n)$. Under this identification e_{ij} corresponds to the matrix unit which has all entries zeroes except the entry at the position (i, j), which is equal to one. The commutation relations for the matrix units e_{ij} are given as

$$[e_{ab}, e_{cd}] = e_{ad}\delta_{bc} + (-1)^{(|a|+|b|)(|c|+|d|)}e_{cb}\delta_{ad}.$$

Let $U_{\mathbb{C}}$ be the universal enveloping algebra of $\mathfrak{gl}(m|n)$ over the field of complex numbers. Then the Kostant $\mathbb{Z}$-form $U_{\mathbb{Z}}$ is generated by elements e_{ij} for odd e_{ij}, $e_{ij}^{(r)} = \frac{e_{ij}^r}{r!}$ for even e_{ij} and $\binom{e_{ii}}{r} = \frac{e_{ii}(e_{ii}-1)\ldots(e_{ii}-r+1)}{r!}$ for all $r > 0$.

We will consider all G-modules as left modules and use the terminology of right superderivations ${}_{ij}D$ of $A(2|2)$ defined on generators c_{kl} as $(c_{kl})_{ij}D = c_{kj}$ and $(c_{kl})_{ij}D = 0$ for $l \neq i$. There is a surjective map $Dist(G) \to S(2|2, r)$ explicitly described in Lemma 4.2 of [**6**]. Composition of this map with the representation $S(2|2, r) \to End_K(A(2|2, r))$ given by a left action of $S(2|2, r)$ on $A(2|2, r)$ gives a left action of $Dist(G)$ on $A(2|2, r)$. Under this action, the generators e_{ij}, $e_{ij}^{(t)}$ and $\binom{e_{ii}}{t}$ of $Dist(G)$ correspond to ${}_{ji}D$, ${}_{ji}D^{(t)}$ and $\binom{{}_{ii}D}{t}$, respectively, where ${}_{ij}D^{(t)} = \frac{{}_{ij}D^t}{t!}$ and $\binom{{}_{ii}D}{t} = \frac{{}_{ii}D({}_{ii}D-1)\ldots({}_{ii}D-t+1)}{t!}$ – for more details, see Section 4 of [**6**]. Therefore, the action of $S(2|2, r)$ on $A(2|2, r)$ is completely determined by right superderivations ${}_{ij}D$. While doing computations, we extend the superderivations ${}_{ij}D$ of $A(2|2)$ to superderivations of $A(2|2)_d = K[G]$. Since the simple module $L_{S(2|2)}(\lambda)$ is included in the costandard module $\nabla_{S(2|2)}(\lambda)$, which in turn is included in $A(2|2, r)$ (by Proposition 3.1 of [**6**]), we conclude that the action of $S(2|2, r)$ on $L_{S(2|2)}(\lambda)$ is completely determined using the action of superderivations ${}_{ij}D$.

Let $G_{ev} \simeq GL(2) \times GL(2)$ be an even supersubgroup of G, $Lie(G_{ev}) \simeq gl(2) \times gl(2)$ be the corresponding Lie algebra of G_{ev} and $S = sl(2) \times sl(2)$. Let B be the lower triangular Borel subsupergroup of G. Fix a dominant weight $\lambda = (\lambda_1, \lambda_2|\lambda_3, \lambda_4)$ of G, that is $\lambda_1 \geq \lambda_2$ and $\lambda_3 \geq \lambda_4$. Following [**8**], we denote by $H^0_G(\lambda)$ the induced G-module $H^0(G/B, K_\lambda)$, where K_λ is the one-dimensional (even) B-supermodule corresponding to the weight λ. Finally, using the restriction of B to G_{ev}, denote by $H^0_{G_{ev}}(\lambda)$ the induced G_{ev}-module corresponding to the weight λ. The induced G_{ev}-module $H^0_{G_{ev}}(\lambda)$, denoted by V, can be identified with the subspace of superalgebra $K[c_{11}, c_{12}, c_{21}, c_{22}, c_{33}, c_{34}, c_{43}, c_{44}]$ generated by polynomials

$$d_{12}^{\lambda_2} c_{11}^a c_{12}^{\lambda_1-\lambda_2-a} d_{34}^{\lambda_4} c_{33}^b c_{34}^{\lambda_3-\lambda_4-b},$$

where $d = d_{12} = c_{11}c_{22} - c_{12}c_{21}$, $d_{34} = c_{33}c_{44} - c_{34}c_{43}$ and $0 \leq a \leq \lambda_1 - \lambda_2$, $0 \leq b \leq \lambda_3 - \lambda_4$. The induced G-supermodule $H^0_G(\lambda)$ can be described explicitly using the isomorphism $\phi : H^0_{G_{ev}}(\lambda) \otimes K[c_{13}, c_{14}, c_{23}, c_{24}] \to H^0_G(\lambda)$ defined in [**8**, Lemma 5.2, and p. 163]. This isomorphism ϕ is given by

$$\phi(d_{12}) = d_{12}, \quad \phi(c_{11}) = c_{11}, \quad \phi(c_{12}) = c_{12},$$

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$$\phi(c_{13}) = \frac{c_{22}c_{13} - c_{12}c_{23}}{d} = y_{13}, \quad \phi(c_{14}) = \frac{c_{22}c_{14} - c_{12}c_{24}}{d} = y_{14},$$

$$\phi(c_{23}) = \frac{-c_{21}c_{13} + c_{11}c_{23}}{d} = y_{23}, \quad \phi(c_{24}) = \frac{-c_{21}c_{14} + c_{11}c_{24}}{d} = y_{24},$$

$$\phi(c_{33}) = c_{33} - c_{31}y_{13} - c_{32}y_{23} = z_1, \quad \phi(c_{34}) = c_{34} - c_{31}y_{14} - c_{32}y_{24} = z_2$$

and

$$\phi(d_{34}) = (c_{33} - c_{31}y_{13} - c_{32}y_{23})(c_{44} - c_{41}y_{14} - c_{42}y_{24})$$

$$-(c_{34} - c_{31}y_{14} - c_{32}y_{24})(c_{43} - c_{41}y_{13} - c_{42}y_{23}) = x.$$

Then the supermodule $H_G^0(\lambda)$ has a basis

$$w(a, b, \epsilon_{13}, \epsilon_{14}, \epsilon_{23}, \epsilon_{24}) = d^{\lambda_2} c_{11}^a c_{12}^{\lambda_1-\lambda_2-a} x^{\lambda_4} z_1^b z_2^{\lambda_3-\lambda_4-b} y_{13}^{\epsilon_{13}} y_{14}^{\epsilon_{14}} y_{23}^{\epsilon_{23}} y_{24}^{\epsilon_{24}}$$

with a, b as before and $\epsilon_{13}, \epsilon_{14}, \epsilon_{23}, \epsilon_{24} \in \{0, 1\}$. The weight of $w(a, b, \epsilon_{13}, \epsilon_{14}, \epsilon_{23}, \epsilon_{24})$ is

$$(\lambda_2 + a - \epsilon_{13} - \epsilon_{14}, \lambda_1 - a - \epsilon_{23} - \epsilon_{24} | \lambda_4 + b + \epsilon_{13} + \epsilon_{23}, \lambda_3 - b + \epsilon_{14} + \epsilon_{24}).$$

We shall write $v_{a,b}$ for $w(a, b, 0, 0, 0, 0)$ and $v = v_{\lambda_1-\lambda_2,\lambda_3-\lambda_4}$. Then v is the highest vector of the simple G-module $L(\lambda)$.

1.2. Outline of the paper. We will now explain the basic approach of the paper. For analogous results for $S(2|1)$ and $S(3|1)$ see [**4**] and [**7**].

The basis of our investigation is the description of the G_{ev}-module structure of $H_G^0(\lambda)$ and its simple submodule $L(\lambda)$. Although it would be natural to describe the G_{ev}-module structure using the Lie algebra $Lie(G_{ev}) \simeq gl(2) \times gl(2)$, it is easier to work with modules over the Lie algebra $S = sl(2) \times sl(2)$, since the S-weights are described by only two parameters (instead of four for $Lie(G_{ev})$). The structure of G_{ev}-modules can be easily retrieved once their S-module structure is known. One advantage of this approach is exhibited in Section 4.1, where we use a certain isomorphism Θ_V of S-modules.

If the characteristic p of the ground field K is bigger than two, then using the Steinberg Theorem (see Theorem 4.4 of [**5**]), it is enough to determine the structure of the simple $S(2|2)$-module $L_{S(2|2)}(\lambda)$ for λ restricted, that is when $\lambda_1 - \lambda_2, \lambda_3 - \lambda_4 < p$. If λ is restricted, then the action of even elements $e_{ij}^{(p^r)} \in Dist(G)$ for $r > 0$ on $H_G^0(\lambda)$ is trivial. Since the G_{ev}-structure of $H_{ev}^0(\lambda)$ is known, the G-structure of $H_G^0(\lambda)$ is then determined completely by the action of superderivations ${}_{ij}D$.

In Section 1 we compute the action of superderivations ${}_{ij}D$ on elements in $H_{G_{ev}}^0(\lambda)$ and on elements $\phi(X_{12})$. Furthermore, we define the concept of atypicality of the weight λ (extending the classical definition of Kac from characteristic zero case).

The module $H_G^0(\lambda)$ decomposes into a direct sum of S-submodules $F_0(\lambda) \oplus F_1(\lambda) \oplus F_2(\lambda) \oplus F_3(\lambda) \oplus F_4(\lambda)$, where the submodule $F_k(\lambda)$, which will be called the k-floor, is given as a K-span of all vectors $w(a, b, \epsilon_{13}, \epsilon_{14}, \epsilon_{23}, \epsilon_{24})$, where $\sum_{i=0}^4 \epsilon_{i4} = k$.

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Equivalently, $F_k(\lambda)$ is spanned by vectors of weights $\mu = (\mu_1, \mu_2|\mu_3, \mu_4)$ such that $k = \lambda_1 + \lambda_2 - \mu_1 - \mu_2 = \mu_3 + \mu_4 - \lambda_3 - \lambda_4$. Clearly, each $F_i(\lambda)$ is a S-module. Denote by Y a four-dimensional S-module spanned by elements y_{13}, y_{14}, y_{23} and y_{24}. Then $F_0(\lambda) = V, F_1(\lambda) = V \otimes Y$, $F_2(\lambda) = V \otimes (Y \wedge Y)$, $F_3(\lambda) = V \otimes (Y \wedge Y \wedge Y)$ and $F_4(\lambda) = V \otimes (Y \wedge Y \wedge Y \wedge Y)$. The complete description of the S-module structure of each floor F_i will be carried out in Sections 2 through 5.

In order to describe $L_{S(2|2)}(\lambda)$, we will use the Poincaré–Birkhoff–Witt (PBW) theorem, and corresponding to our choice of the Borel subsupergroup B, we order the generators of $Dist(G)$ as follows: $e_{ij}^{(r)}$ for $i < j$ first, followed by $\binom{e_{ii}}{r}$ and $e_{ij}^{(r)}$ for $i > j$, and then by odd e_{ij}, where $i > j$. The simple module $L_{S(2|2)}(\lambda)$ is generated by the vector v. The elements $e_{ij}^{(r)} \in Dist(G)$ for $i < j$ act trivially on v and elements $\binom{e_{ii}}{r}$ and $e_{ij}^{(r)}$ for $i > j$ applied to v generate the module V. Using the previously discussed action of $Dist(G)$ on $A(2|2, r)$, we conclude that $L_{S(2|2)}(\lambda)$ is generated by V and its images under compositions of superderivations ${}_{ij}D$ for $i < j$. We can identify these images with elements of various floors F_k. Since the superderivations ${}_{ij}D$ supercommute, the maps $\phi_1 : F_1(\lambda) \to F_1(\lambda)$ given by

$$v_{a,b} \otimes y_{ij} \mapsto (v_{a,b})_{ij}D,$$

$\phi_2 : F_2(\lambda) \to F_2(\lambda)$ given by

$$v_{a,b} \otimes (y_{i_1j_1} \wedge y_{i_2j_2}) \mapsto (v_{a,b})_{i_1j_1}D_{i_2j_2}D,$$

$\phi_3 : F_3(\lambda) \to F_3(\lambda)$ given by

$$v_{a,b} \otimes (y_{i_1j_1} \wedge y_{i_2j_2} \wedge y_{i_3j_3}) \mapsto (v_{a,b})_{i_1j_1}D_{i_2j_2}D_{i_3j_3}D,$$

and $\phi_4 : F_4(\lambda) \to F_4(\lambda)$ given by

$$v_{a,b} \otimes (y_{13} \wedge y_{23} \wedge y_{14} \wedge y_{24}) \mapsto (v_{a,b})_{13}D_{23}D_{14}D_{24}D$$

are well defined. It is easy to check that they are S-morphisms. We will compute images $\phi_1(F_1), \phi_2(F_2), \phi_3(F_3)$ and $\phi_4(F_4)$ in Sections 2 through 5. These images together with V constitute the whole module $L_{S(2|2)}(\lambda)$.

In each Section 2 through 5 we follow this procedure: We first determine primitive vectors in characteristics zero, then we establish the S-module structure of each floor. Special care is taken in the cases when either $\lambda_1 - \lambda_2$ or $\lambda_3 - \lambda_4$ is equal to $p - 2$ or $p - 1$, since in these cases $H^0_G(\lambda)$ is not semi-simple as an S-module. Afterwards we compute the S-module structure of the image $\phi_k(F_k(\lambda))$.

Finally, in Section 6 we combine the results of preceding sections and determine the character and dimension of the simple module $L_{S(2|2)}(\lambda)$.

2. Basic formulas.

2.1. Basic formulas for $S(2|2)$**.** It is clear that $V = H^0_{G_{ev}}(\lambda) = L(\lambda)$ is an irreducible S-module if λ is restricted.

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LEMMA 2.1. *The action of superderivations* ${}_{12}D$, ${}_{21}D$, ${}_{13}D$, ${}_{14}D$, ${}_{23}D$, ${}_{24}D$, ${}_{34}D$ *and* ${}_{43}D$ *on elements* $d, x, c_{11}, c_{12}, y_{13}, y_{23}, y_{14}, y_{24}, z_1$ *and* z_2 *is given in the following table.*

	${}_{12}D$	${}_{21}D$	${}_{13}D$	${}_{14}D$	${}_{23}D$	${}_{24}D$	${}_{34}D$	${}_{43}D$
d	0	0	dy_{13}	dy_{14}	dy_{23}	dy_{24}	0	0
x	0	0	xy_{13}	xy_{14}	xy_{23}	xy_{24}	0	0
c_{11}	c_{12}	0	$c_{11}y_{13}+c_{12}y_{23}$	$c_{11}y_{14}+c_{12}y_{24}$	0	0	0	0
c_{12}	0	c_{11}	0	0	$c_{11}y_{13}+c_{12}y_{23}$	$c_{11}y_{14}+c_{12}y_{24}$	0	0
y_{13}	0	$-y_{23}$	0	$-y_{13}y_{14}$	$y_{13}y_{23}$	$-y_{23}y_{14}$	y_{14}	0
y_{23}	$-y_{13}$	0	$y_{23}y_{13}$	$-y_{13}y_{24}$	0	$-y_{23}y_{24}$	y_{24}	0
y_{14}	0	$-y_{24}$	$-y_{14}y_{13}$	0	$y_{13}y_{24}$	$y_{14}y_{24}$	0	y_{13}
y_{24}	$-y_{14}$	0	$y_{23}y_{14}$	$y_{24}y_{14}$	$-y_{24}y_{23}$	0	0	y_{23}
z_1	0	0	z_1y_{13}	z_2y_{13}	z_1y_{23}	z_2y_{23}	z_2	0
z_2	0	0	z_1y_{14}	z_2y_{14}	z_1y_{24}	z_2y_{24}	0	z_1

Proof. It is straightforward computation using the properties $(c_{kl})_{ij}D = \delta_{li}c_{kj}$, where δ_{li} is the Kronecker delta, and $(ab)_{ij}D = (-1)^{|_{ij}D||b|}(a)_{ij}Db + a(b)_{ij}D$, where the symbol $|\ |$ denotes the parity. □

As a consequence, we obtain the following fundamental formulas.

LEMMA 2.2.

$$\begin{aligned}
&(v_{a,b})_{12}D = av_{a-1,b},\\
&(v_{a,b})_{21}D = (\lambda_1-\lambda_2-a)v_{a+1,b},\\
&(v_{a,b})_{13}D = (\lambda_2+\lambda_4+b+a)v_{a,b}y_{13} + av_{a-1,b}y_{23} + (\lambda_3-\lambda_4-b)v_{a,b+1}y_{14},\\
&(v_{a,b})_{14}D = (\lambda_2+\lambda_3-b+a)v_{a,b}y_{14} + av_{a-1,b}y_{24} + bv_{a,b-1}y_{13},\\
&(v_{a,b})_{23}D = (\lambda_1+\lambda_4+b-a)v_{a,b}y_{23} + (\lambda_1-\lambda_2-a)v_{a+1,b}y_{13} + (\lambda_3-\lambda_4-b)v_{a,b+1}y_{24},\\
&(v_{a,b})_{24}D = (\lambda_1+\lambda_3-b-a)v_{a,b}y_{24} + (\lambda_1-\lambda_2-a)v_{a+1,b}y_{14} + bv_{a,b-1}y_{23},\\
&(v_{a,b})_{34}D = bv_{a,b-1},\\
&(v_{a,b})_{43}D = (\lambda_3-\lambda_4-b)v_{a,b+1}.
\end{aligned}$$

Proof. It follows by repeated applications of Lemma 2.1. □

Other identities of interest are

$$dy_{13}y_{23} = c_{13}c_{23}, \quad dy_{14}y_{24} = c_{14}c_{24}, \quad d(y_{13}y_{24} + y_{14}y_{23}) = c_{13}c_{24} + c_{14}c_{23}.$$

2.2. Further notation. The simple S-module of the highest weight μ and the highest vector w shall be denoted either by $L(\mu)$ or $L(w)$ depending on the circumstances.

Denote $\lambda_1-\lambda_2 = A$ and $\lambda_3-\lambda_4 = B$. Further, denote $\omega_{12} = \lambda_1-\lambda_2$, $\omega_{34} = \lambda_3-\lambda_4$, $\omega_{13} = \lambda_1+\lambda_3+1$, $\omega_{14} = \lambda_1+\lambda_4$, $\omega_{23} = \lambda_2+\lambda_3$ and $\omega_{24} = \lambda_2+\lambda_4-1$.

If $p=0$, then we shall write $\delta_{ij} = 0$ if $\omega_{ij} = 0$ and $\delta_{ij} = 1$ otherwise, and $\delta^1_{ij} = 1$ if $\omega_{ij} = 1$ and $\delta^1_{ij} = 1$ otherwise. If $p>2$, then we denote $\delta_{ij} = 0$ if $\omega_{ij} \equiv 0 \pmod p$ and $\delta_{ij} = 1$ otherwise, and $\delta^1_{ij} = 0$ if $\omega_{ij} \equiv 1 \pmod p$ and $\delta^1_{ij} = 1$ otherwise.

DEFINITION 2.3. A weight λ is called typical if $\delta_{13}\delta_{14}\delta_{23}\delta_{24} = 1$,
λ is called 13-atypical if $\delta_{13} = 0$ but $\delta_{14}\delta_{23}\delta_{24} = 1$,
λ is called 14-atypical if $\delta_{14} = 0$ but $\delta_{13}\delta_{23}\delta_{24} = 1$,

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λ is called 23-atypical if $\delta_{23} = 0$ but $\delta_{13}\delta_{14}\delta_{24} = 1$,
λ is called 24-atypical if $\delta_{24} = 0$ but $\delta_{13}\delta_{14}\delta_{23} = 1$,
λ is called (13,14)-atypical if $\delta_{13} = \delta_{14} = 0$ but $\delta_{23}\delta_{24} = 1$,
λ is called (13,23)-atypical if $\delta_{13} = \delta_{23} = 0$ but $\delta_{14}\delta_{24} = 1$,
λ is called (13,24)-atypical if $\delta_{13} = \delta_{24} = 0$ but $\delta_{14}\delta_{23} = 1$,
λ is called (14,23)-atypical if $\delta_{14} = \delta_{23} = 0$ but $\delta_{13}\delta_{24} = 1$,
λ is called (14,24)-atypical if $\delta_{14} = \delta_{24} = 0$ but $\delta_{13}\delta_{23} = 1$,
λ is called (23,24)-atypical if $\delta_{23} = \delta_{24} = 0$ but $\delta_{13}\delta_{14} = 1$,
λ is called (13,14,23,24)-atypical if $\delta_{13} = \delta_{14} = \delta_{23} = \delta_{24} = 0$.

It is easy to see that if $p = 0$, then every dominant weight λ is either typical, 14-atypical, 23-atypical, 24-atypical or (14,23)-atypical.

If $p > 2$, then weights of all the above atypical types are possible. In this case observe the following. If λ is 13-atypical, then $B \not\equiv p - 1 \pmod p$. If λ is 23-atypical, then $B \not\equiv p - 1 \pmod p$ and $A \not\equiv p - 1 \pmod p$. If λ is 24-atypical, then $A \not\equiv p - 1 \pmod p$. If λ is (13, 14)-atypical, then $B \equiv p - 1 \pmod p$ and $A \not\equiv p - 1 \pmod p$. If λ is (13,23)-atypical, then $A \equiv p - 1 \pmod p$ and $B \not\equiv p - 1 \pmod p$. If λ is (14,24)-atypical, then $A \equiv p - 1 \pmod p$ and $B \not\equiv p - 1 \pmod p$. If λ is (23,24)-atypical, then $B \equiv p - 1 \pmod p$ and $A \not\equiv p - 1 \pmod p$. If λ is (13, 24)-atypical, then $A + B \equiv p - 2 \pmod p$ and $A, B \not\equiv p - 1 \pmod p$. If λ is (14, 23)-atypical, then $A \equiv B \pmod p$ and $A, B \not\equiv p - 1 \pmod p$. If λ is (13,14, 23, 24)-atypical, then $A, B \equiv p - 1 \pmod p$.

Furthermore, denote $v \sim w$ if and only if both $v, w \neq 0$ and one of them is a constant multiple of the other.

3. First floor.

3.1. Characteristic zero. In order to describe $V \otimes Y$ as an S-module, consider the following elements:

$l_1 = v_{AB} \otimes y_{23}$,
$l_2 = v_{AB} \otimes y_{24} - v_{A,B-1} \otimes y_{23}$,
$l_3 = v_{AB} \otimes y_{13} + v_{A-1,B} \otimes y_{23}$,
$l_4 = v_{AB} \otimes y_{14} + v_{A-1,B} \otimes y_{24} - v_{A,B-1} \otimes y_{13} - v_{A-1,B-1} \otimes y_{23}$.

LEMMA 3.1. The module $V \otimes Y$ is isomorphic to the direct sum $L(l_1) \oplus \delta_{34}L(l_2) \oplus \delta_{12}L(l_3) \oplus \delta_{12}\delta_{34}L(l_4)$.

Proof. The vectors l_1, l_2, l_3 and l_4 are primitive vectors. A dimension count completes the argument. □

The image of the first floor under the action of superderivations is given in the following Proposition.

PROPOSITION 3.2. Let $\phi_1 : V \otimes Y \to V \otimes Y$ be a morphism of S-modules given by $v \otimes y_{ij} \mapsto (v)_{ij}D$. Then the image $\phi_1(V \otimes Y) \cong \delta_{23}L(l_1) \oplus \delta_{34}\delta_{24}L(l_2) \oplus \delta_{12}\delta_{13}L(l_3) \oplus \delta_{12}\delta_{34}\delta_{14}L(l_4)$.

Proof. We compute $\phi_1(l_1) = \omega_{23}l_1$, $\phi_1(l_2) = \omega_{24}l_2$, $\phi_1(l_3) = \omega_{13}l_3$, $\phi_1(l_4) = \omega_{14}l_4$, and the claim follows. □

3.2. Characteristic p. Assume that the weight λ is restricted.

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The question of describing the S-module structure of $V \otimes Y$ is related to a classical problem of decomposing the tensor product of a simple module and the natural or dual of the natural module over the general linear group $GL(n)$. These questions were studied in [**2**] in relation to a complete reducibility criterion, primitive vectors and socles of the tensor products of the above type. In particular, costandard filtrations of these modules were described in [**2**, p. 88].

We will only need a description of the tensor product of the simple module with the natural module for $G^+ = GL(2)$. Actually we will only determine its explicit structure over $S^+ = sl(2)$. Assign to a G^+-weight $\lambda^+ = (\lambda_1, \lambda_2)$ its corresponding restricted S^+-weight $A = \lambda_1 - \lambda_2 < p$.

Let V^+ be a simple S^+-module generated by an element v_A^+ of the highest S^+-weight A. Then the dimension of V^+ is $A+1$ and V^+ is a span of vectors v_{A-i}^+ for $0 \leq i \leq A$ such that $(v_{A-i}^+)_{12}D = (A-i)v_{A-i-1}^+$, $(v_{A-i}^+)_{21}D = iv_{A-i+1}^+$, and ${}_{12}D^{(p^k)}$ and ${}_{21}D^{(p^k)}$ for $k \geq 1$ act trivially.

Denote by W^+ the two-dimensional S^+-module which is a K-span of elements w_1^+ and w_{-1}^+ for which $(w_1^+)_{21}D = 0, (w_{-1}^+)_{21}D = w_1^+$, $(w_1^+)_{12}D = w_{-1}^+$, $(w_{-1}^+)_{12}D = 0$, and ${}_{12}D^{(p^k)}$ and ${}_{21}D^{(p^k)}$ for $k \geq 1$ act trivially.

The following lemma describes the S^+-module structure of $V^+ \otimes W^+$. Although it is a classical result, we include it here for the convenience of the reader.

LEMMA 3.3. The S^+-module structure of the module $V^+ \otimes W^+$ is given as follows.

If $A = 0$, then $V^+ \otimes W^+ \cong U_1^+$, where $U_1^+ = \langle u_1^+ = v_A^+ \otimes w_1^+ \rangle$, is a simple S^+-module.

If $0 < A < p-1$, then $V^+ \otimes W^+ \cong U_1^+ \oplus U_2^+$, where U_1^+ is as above and $U_2^+ = \langle u_2^+ = v_{A-1}^+ \otimes w_1^+ - v_A^+ \otimes w_{-1}^+ \rangle$, is a simple S^+-module.

If $A = p-1$, then $V^+ \otimes W^+$ has a composition series

$$V^+ \otimes W^+ = \begin{array}{c} U_3^+ \\ | \\ U_1^+ \\ | \\ U_2^+ \end{array},$$

where U_1^+, U_2^+ as above, and $U_3^+ = \langle u_3^+ = v_A^+ \otimes w_{-1}^+ \rangle$ is a simple S^+-module.

Proof. Since $(u_1^+)_{21}D = 0$, u_1^+ is a primitive vector of the highest weight $A+1$. If $A = 0$, then dimensions of both $V^+ \otimes W^+$ and U_1^+ are equal to 2.

If $A > 0$, then $(u_2^+)_{21}D = 0$ shows that u_2^+ is a primitive vector of the highest weight $A-1$ and dimension A. Assume $0 < A < p-1$. Then the dimension of U_1^+ is $A+2$, and the dimensions of U_1^+ and U_2^+ add up to the dimension of $V^+ \otimes W^+$. Since $(u_1^+)_{12}D \not\sim u_2^+$, $Ext^1_{S^+}(U_1^+, U_2^+) = 0$, and the S^+-module structure of $V^+ \otimes W^+$ follows.

Assume now $A = p-1$. Then $(u_3^+)_{21}D = u_1^+$ shows that u_3^+ is a primitive vector of weight $A-1$ and dimension $p-2$. The vector u_1^+ is primitive of weight p and $L(u_1^+)$ has dimension 2 (and is spanned by u_1^+ and $(u_1^+)_{12}D^{(p)}$). Since u_2^+ is a primitive vector of weight $A-1$ and dimension $p-2$, dimensions of U_1^+, U_2^+ and U_3^+ add up to the dimension of $V^+ \otimes W^+$. Finally, $(u_1^+)_{12}D = -u_2^+$ implies that the S^+-module structure of $V^+ \otimes W^+$ is as stated. □

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The even supergroup G_{ev} of G is a product of two copies of $GL(2)$, the first copy (based on letters 1 and 2) can be be identified with G^+ and we can denote the second copy (based on letters 3 and 4) by G^-. Assign to a G^--weight $\lambda^- = (\lambda_3, \lambda_4)$ its corresponding restricted $S^- = sl(2)$-weight $B = \lambda_3 - \lambda_4 < p$. Then we can define V^- and W^- analogously to V^+ and W^+ and obtain the following analogous result for the S^--module structure of $V^- \otimes W^-$.

LEMMA 3.4. The S^--module structure of the module $V^- \otimes W^-$ is given as follows.

If $B = 0$, then $V^- \otimes W^- \cong U_1^-$, where $U_1^- = \langle u_1^- = v_B^- \otimes w_1^- \rangle$, is a simple S^--module.

If $0 < B < p - 1$, then $V^- \otimes W^- \cong U_1^- \oplus U_2^-$, where U_1^- as above and $U_2^- = \langle u_2^- = v_{B-1}^- \otimes w_1^- - v_B^- \otimes w_{-1}^- \rangle$, is a simple S^--module.

If $B = p - 1$, then $V^- \otimes W^-$ has a composition series

$$V^- \otimes W^- = \begin{array}{c} U_3^- \\ | \\ U_1^- \\ | \\ U_2^- \end{array},$$

where U_1^-, U_2^- as above, and $U_3^- = \langle u_3^- = v_B^- \otimes w_{-1}^- \rangle$ is a simple S^--module.

Now we are ready to describe the S-module structure of the first floor.

PROPOSITION 3.5. The S-module $V \otimes Y$ is described as follows.

(1) If $A, B < p - 1$, then $V \otimes Y \cong L(l_1) \oplus \delta_{34} L(l_2) \oplus \delta_{12} L(l_3) \oplus \delta_{12}\delta_{34} L(l_4)$.

(2) If $A = p - 1$ and $B < p - 1$, then $V \otimes Y \cong L_5 \oplus \delta_{34} L_6$. Here the indecomposable module L_5 is given as

$$L_5 = \begin{array}{c} L(l_5) \\ | \\ L(l_1) \\ | \\ L(l_3) \end{array},$$

where $l_5 = v_{A,B} \otimes y_{13}$. The indecomposable module L_6 is given as

$$L_6 = \begin{array}{c} L(l_6) \\ | \\ L(l_2) \\ | \\ L(l_4) \end{array},$$

where $l_6 = -v_{A,B-1} \otimes y_{13} + v_{A,B} \otimes y_{14}$.

(3) If $A < p - 1$ and $B = p - 1$, then $V \otimes Y \cong L_7 \oplus \delta_{12} L_8$. Here the indecomposable module L_7 is given as

$$L_7 = \begin{array}{c} L(l_7) \\ | \\ L(l_1) \\ | \\ L(l_2) \end{array},$$

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where $l_7 = v_{A,B} \otimes y_{24}$. The indecomposable module module L_8 is given as

$$L_8 = \begin{array}{c} L(l_8) \\ | \\ L(l_3) \\ | \\ L(l_4) \end{array},$$

where $l_8 = v_{A-1,B} \otimes y_{24} + v_{A,B} \otimes y_{14}$.

(4) If $A = B = p - 1$, then $V \otimes Y \cong L_9$ has the composition series

$$\begin{array}{ccccccccc}
 & & & & L(l_9) & & & & \\
 & & & / & & \backslash & & & \\
 & & L(l_5) & & & & L(l_7) & & \\
 & / & & \backslash & & / & & \backslash & \\
L(l_6) & & & & L(l_1) & & & & L(l_8) \\
 & \backslash & & / & & \backslash & & / & \\
 & & L(l_2) & & & & L(l_3) & & \\
 & & & \backslash & & / & & & \\
 & & & & L(l_4) & & & &
\end{array}$$

of simple S-modules, where $l_9 = v_{A,B} \otimes y_{14}$.

Proof. There is an isomorphism of S-modules $W^+ \otimes W^-$ and Y which sends $w_1^+ \otimes w_1^- \mapsto y_{23}$, $w_1^+ \otimes w_{-1}^- \mapsto y_{24}$, $w_{-1}^+ \otimes w_1^- \mapsto -y_{13}$ and $w_{-1}^+ \otimes w_{-1}^- \mapsto -y_{14}$.

For an S-module V of the highest weight (A, B) there is an isomorphism $V \cong V^+ \otimes V^-$, where V^+ and V^- are defined as above.

The claim follows from Lemmas 3.3 and 3.4 using the standard properties of tensor products. □

3.3. Image under ϕ_1.

The structure of the S-module $\phi_1(V \otimes Y)$ is given as follows.

Proposition 3.6. The following statements describe S-modules isomorphic to $V_1 = \phi_1(V \otimes Y)$.

If $A, B < p - 1$, then $V_1 \cong \delta_{23} L(l_1) \oplus \delta_{34}\delta_{24} L(l_2) \oplus \delta_{12}\delta_{13} L(l_3) \oplus \delta_{12}\delta_{34}\delta_{14} L(l_4)$.

Assume $A = p - 1$ and $B < p - 1$. If λ is typical, then $V_1 \cong V \otimes Y$. If λ is (13, 23)-atypical, then $V_1 \cong L(l_3) \oplus \delta_{34} L_6$. If λ is (14, 24)-atypical, then $V_1 \cong L_5 \oplus \delta_{34} L(l_4)$.

Assume $A < p - 1$ and $B = p - 1$. If λ is typical, then $V_1 \cong V \otimes Y$. If λ is (13, 14)-atypical, then $V_1 \cong L_7 \oplus \delta_{12} L(l_4)$. If λ is (23, 24)-atypical, then $V_1 \cong L(l_2) \oplus \delta_{12} L_8$.

Assume $A = B = p - 1$. If λ typical, then $V_1 \cong V \otimes Y$. If λ is (13, 14, 23, 24)-atypical, then $V_1 \cong$

$$L_{10} = \begin{array}{ccccc}
 & & L(-l_6 + l_8) & & \\
 & / & & \backslash & \\
L(l_2) & & & & L(l_3) \\
 & \backslash & & / & \\
 & & L(l_4) & &
\end{array}.$$

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Proof. Assume first that $A, B < p - 1$. The images of generators l_1, l_2, l_3 and l_4 of $V \otimes Y$ were determined earlier. The image $V_1 \cong \delta_{23}L(l_1) \oplus \delta_{34}\delta_{24}L(l_2) \oplus \delta_{12}\delta_{13}L(l_3) \oplus \delta_{12}\delta_{34}\delta_{14}L(l_4)$ as in the characteristic zero case.

Next assume that $A = p - 1$ and $B < p - 1$. The images of additional primitive vectors under ϕ_1 equal $\phi_1(l_5) = \omega_{13}l_5 - l_3$ and $\phi_1(l_6) = \omega_{14}l_6 - l_4$. The structure of V_1 follows.

Now assume that $A < p - 1$ and $B = p - 1$. Then the images of additional primitive vectors under ϕ_1 equal $\phi_1(l_7) = \omega_{24}l_7 + l_2$ and $\phi_1(l_8) = \omega_{14}l_8 + l_4$. The structure of V_1 follows.

Finally, assume that $A = B = p - 1$. We compute first the image of the generator l_9 under ϕ_1 as $\phi_1(l_9) = \omega_{14}l_9 - l_8 + l_6$. If λ is typical, then $\phi_1(V \otimes Y) \cong V \otimes Y$. If λ is (13, 14, 23, 24)-atypical, then $\phi_1(l_5) = -l_3, \phi_1(l_7) = l_2, \phi_1(l_6) = -l_4, \phi_1(l_8) = l_4$, and the images of the remaining elements l_i vanish. In this case V_1 is an indecomposable module generated by $-l_6 + l_8 = v_{A,B-1} \otimes y_{13} + v_{A-1,B} \otimes y_{24}$ that has the structure described in the statement of the proposition. □

4. Second floor.

4.1. Characteristic zero. The S-module $Y \wedge Y$ is a direct sum of two irreducible modules $Y_1 = \langle y_{23} \wedge y_{24}\rangle$ and $Y_2 = \langle y_{23} \wedge y_{13}\rangle$.

Consider the following elements:

$m_1 = v_{AB} \otimes y_{23} \wedge y_{24}$,

$m_2 = -2v_{A-1,B} \otimes y_{23} \wedge y_{24} - v_{AB} \otimes y_{13} \wedge y_{24} + v_{AB} \otimes y_{14} \wedge y_{23}$ and

$m_3 = v_{A-2,B} \otimes y_{23} \wedge y_{24} + v_{A-1,B} \otimes y_{13} \wedge y_{24} - v_{A-1,B} \otimes y_{14} \wedge y_{23} + v_{AB} \otimes y_{13} \wedge y_{14}$,

$n_1 = v_{AB} \otimes y_{13} \wedge y_{23}$,

$n_2 = 2v_{A,B-1} \otimes y_{13} \wedge y_{23} - v_{AB} \otimes y_{13} \wedge y_{24} - v_{AB} \otimes y_{14} \wedge y_{23}$ and

$n_3 = v_{A,B-2} \otimes y_{13} \wedge y_{23} - v_{A,B-1} \otimes y_{13} \wedge y_{24} - v_{A,B-1} \otimes y_{14} \wedge y_{23} + v_{AB} \otimes y_{14} \wedge y_{24}$.

LEMMA 4.1. The module $V \otimes (Y \wedge Y)$ is isomorphic to the direct sum $L(m_1) \oplus L(n_1) \oplus \delta_{12}L(m_2) \oplus \delta_{34}L(n_2) \oplus \delta_{12}\delta^1_{12}L(m_3) \oplus \delta_{34}\delta^1_{34}L(n_3)$.

Proof. The action of ${}_{34}D$, and $43D$ on $V \otimes Y_1$ is given by $(v_{a,b} \otimes y_1)_{34}D = bv_{a,b-1} \otimes y_1$ and $(v_{a,b} \otimes y_1)_{43}D = (\lambda_3 - \lambda_4 - b)v_{a,b+1} \otimes y_1$ for any $y_1 \in Y_1$. The action of ${}_{12}D$, and $21D$ on $V \otimes Y_2$ is given by $(v_{a,b} \otimes y_2)_{12}D = av_{a-1,b} \otimes y_2$ and $(v_{a,b} \otimes y_2)_{21}D = (\lambda_1 - \lambda_2 - a)v_{a+1,b} \otimes y_2$ for any $y_2 \in Y_2$.

Since the vectors m_1, m_2 and m_3 are primitive, the dimension count gives that the module $V \otimes Y_1$ is a direct sum of simple modules $L(m_1) \oplus \delta_{12}L(m_2) \oplus \delta_{12}\delta^1_{12}L(m_3)$. Analogously, since the vectors n_1, n_2 and n_3 are primitive, the dimension count gives that the module $V \otimes Y_2$ is a direct sum of simple modules $L(n_1) \oplus \delta_{34}L(n_2) \oplus \delta_{34}\delta^1_{34}L(n_3)$. Therefore, $V \otimes (Y \wedge Y) \cong L(m_1) \oplus L(n_1) \oplus \delta_{12}L(m_2) \oplus \delta_{34}L(n_2) \oplus \delta_{12}\delta^1_{12}L(m_3) \oplus \delta_{34}\delta^1_{34}L(n_3)$. □

If $t, s > 0$, then $L(m_2) \cong L(n_2) \cong L(\lambda_{13,24})$. We shall denote the simple module $L(\lambda_{13,24})$ by L.

LEMMA 4.2. The S-morphism ϕ_2 is described completely by images of generating vectors

$\phi_2(m_1) = \omega_{23}\omega_{24}m_1$,

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$\phi_2(n_1) = \omega_{13}\omega_{23}n_1,$
$\phi_2(m_2) = \left[(\lambda_2 + \lambda_3 + 1)(\lambda_2 + \lambda_4) + \frac{(\lambda_1-\lambda_2)(2\lambda_2+\lambda_3+\lambda_4)}{2}\right]m_2 - \frac{(\lambda_3-\lambda_4)(2+\lambda_1-\lambda_2)}{2}n_2,$
$\phi_2(n_2) = -\frac{(\lambda_1-\lambda_2)(2+\lambda_3-\lambda_4)}{2}m_2 + \left[(\lambda_2 + \lambda_3 - 1)(\lambda_1 + \lambda_3) - \frac{(\lambda_3-\lambda_4)(\lambda_1+\lambda_2+2\lambda_3)}{2}\right]n_2,$
$\phi_2(m_3) = \omega_{13}\omega_{14}m_3,$
$\phi_2(n_3) = \omega_{14}\omega_{24}n_3.$

Proof. It is a straightforward computation. □

LEMMA 4.3. The image $\phi_2(m_2)$ vanishes if and only if $\omega_{23}\omega_{13} = 0$ and $s = 0$. The image $\phi_2(n_2)$ vanishes if and only if $\omega_{23}\omega_{24} = 0$ and $t = 0$.

If $s, t > 0$ and $\omega_{13}\omega_{14}\omega_{23}\omega_{24} = 0$, then $\phi_2(m_2) \sim \phi_2(n_2)$.

Proof. Lemma 4.2 shows that $\phi_2(m_2) = 0$ implies $s = 0$, and $\phi_2(n_2) = 0$ implies $t = 0$. It is easy to verify that $\phi_2(m_2) = \omega_{23}\omega_{13}m_2$ for $s = 0$ and $\phi_2(n_2) = \omega_{23}\omega_{24}m_2$ for $t = 0$.

If $s, t > 0$, then the restriction $\tilde{\phi}_2$ of the map ϕ_2 to the two-dimensional space of primitive vectors $\langle m_2, n_2\rangle$ of weight μ is represented by a matrix

$$\begin{pmatrix} (\lambda_2 + \lambda_3 + 1)(\lambda_2 + \lambda_4) + \frac{(\lambda_1-\lambda_2)(2\lambda_2+\lambda_3+\lambda_4)}{2} & \frac{-(\lambda_1-\lambda_2)(2+\lambda_3-\lambda_4)}{2} \\ \frac{-(\lambda_3-\lambda_4)(2+\lambda_1-\lambda_2)}{2} & (\lambda_2 + \lambda_3 - 1)(\lambda_1 + \lambda_3) - \frac{(\lambda_3-\lambda_4)(\lambda_1+\lambda_2+2\lambda_3)}{2} \end{pmatrix}.$$

The determinant of this matrix is $\omega_{13}\omega_{14}\omega_{23}\omega_{24}$ and our claim follows. □

PROPOSITION 4.4. The image $\phi_2(V \otimes Y \wedge Y) \cong \delta_{23}\delta_{24}L(m_1) \oplus \delta_{13}\delta_{23}L(n_1) \oplus \delta_{12}\delta^1_{12}$ $\delta_{13}\delta_{14}L(m_3) \oplus \delta_{34}\delta^1_{34}\delta_{14}\delta_{24}L(n_3) \oplus Z$, where

$Z \cong L \oplus L$ if $\omega_{13}\omega_{14}\omega_{23}\omega_{24} \neq 0$;

$Z \cong L$ if $\omega_{13}\omega_{14}\omega_{23}\omega_{24} = 0$ and one of the following conditions is satisfied:

- $s, t > 0$,
- $t = 0, s > 0$ and $\omega_{23}\omega_{24} \neq 0$ and
- $s = 0, t > 0$ and $\omega_{23}\omega_{13} \neq 0$;

and $Z \cong 0$ if one of the following conditions is satisfied:

- $t = 0, s > 0$ and $\omega_{23}\omega_{24} = 0$,
- $s = 0, t > 0$ and $\omega_{23}\omega_{13} = 0$ and
- $s = t = 0$.

Proof. It follows from Lemmas 4.2 and 4.3. □

4.2. Characteristic p. Assume that the weight λ is restricted.

As in the case of characteristic zero, we also have that the S-module $Y \wedge Y$ is a direct sum of two irreducible S-modules $Y_1 = \langle y_{23} \wedge y_{24}\rangle$ and $Y_2 = \langle y_{23} \wedge y_{13}\rangle$.

We shall determine the S-module structures of $V \otimes Y_1$ and $V \otimes Y_2$ first and then combine them. Since S^- acts trivially on $V \otimes Y_1$ and S^+ acts trivially on $V \otimes Y_2$, instead of S-modules we shall deal with appropriate $sl(2)$-modules and the computations shall become easier.

4.2.1. Module $V \otimes Y_1$. Define the elements $m_4 = v_{A,B} \otimes (y_{13} \wedge y_{24} - y_{14} \wedge y_{13})$ and $m_5 = -\frac{1}{2}v_{A-2,B} \otimes (y_{23} \wedge y_{24}) + v_{A,B} \otimes (y_{13} \wedge y_{14})$.

LEMMA 4.5. The S-module structure of the module $V \otimes Y_1$ is given

(1) If as follows: $A < p - 2$, then $V \otimes Y_1 \cong L(m_1) \oplus \delta_{12}L(m_2) \oplus \delta_{12}\delta^1_{12}L(m_3)$.

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(2) If $A = p-2$, then $V \otimes Y_1 \cong M_4 \oplus \delta^1_{12} L(m_3)$, where the S-module M_4 has the composition series

$$M_4 = \begin{array}{c} L(m_4) \\ | \\ L(m_1) \\ | \\ L(m_2) \end{array}.$$

(3) If $A = p-1$, then $V \otimes Y_1 \cong M_5 \oplus L(m_2)$, where the S-module M_5 has the composition series

$$M_5 = \begin{array}{c} L(m_5) \\ | \\ L(m_1) \\ | \\ L(m_3) \end{array}.$$

Proof. Since $(m_1)_{21}D = (m_2)_{21}D = (m_3)_{21}D = 0$, we obtain that the vectors m_1, m_2 and m_3 are primitive, of the highest S^+-weights $A+2$,A and $A-2$ respectively.

To find possible extension between $L(m_1)$, $L(m_2)$ and $L(m_3)$, we compute that $(m_2)_{12}D \not\sim m_3$; $(m_1)_{12}D \sim m_2$ if and only if $A = p-2$; and $(m_1)_{12}D^2 \sim m_3$ if and only if $A = p-1$.

If $A < p-2$, the dimensions of modules $L(m_1)$, $L(m_2)$ and $L(m_3)$ are $(A+3)(B+1)$, $(A+1)(B+1)$ and $(A-1)(B+1)$ respectively. Since $(A+3) + \delta_{12}(A+1) + \delta_{12}\delta^1_{12}(A-1) = 3(A+1)$, the dimension of the module $L(m_1) \oplus \delta_{12} L(m_2) \oplus \delta_{12}\delta^1_{12} L(m_3)$ is the same as the dimension of $V \otimes Y_1$. Since there are no extensions between $L(m_1)$, $L(m_2)$ and $L(m_3)$, part (1) follows.

If $A = p-2$, then $(m_4)_{21}D = -2m_1$ shows that the vector m_4 is a primitive vector of the highest S^+-weight A. Since $L(m_1)$ has dimension $2(B+1)$, $L(m_2)$ and $L(m_4)$ have dimensions $(p-1)(B+1)$, and if $A \neq 1$, then $L(m_3)$ has dimension $(p-3)(B+1)$; the dimensions of these modules add up to the dimension of $V \otimes Y_1$. Since $(m_4)_{21}D = -2m_1$ and $(m_1)_{12}D = m_2$, and $(m_4)_{12}D \not\sim m_3$ and $(m_2)_{12}D \not\sim m_3$, we infer the S-module structure of the module $V \otimes Y_1$.

If $A = p-1$, then $(m_5)_{21}D^2 = m_1$ shows that the vector m_5 is a primitive vector of the highest S^+-weight $A-2$. Since the dimensions of the modules $L(m_1)$, $L(m_2)$, $L(m_3)$ and $L(m_5)$ are $4(B+1), p(B+1), (p-2)(B+1)$ and $(p-2)(B+1)$, respectively, they add up to the dimension of $V \otimes Y_1$. Since $(m_5)_{21}D^2 = m_1$, $(m_1)_{12}D^2 = 2m_3$, and $(m_1)_{12}D \not\sim m_2$, we infer the S-submodule structure of the module $V \otimes Y_1$. □

4.2.2. Module $V \otimes Y_2$. Define the elements $n_4 = v_{A,B} \otimes (y_{13} \wedge y_{24} + y_{14} \wedge y_{23})$ and $n_5 = -\frac{1}{2} v_{A,B-2} \otimes y_{13} \wedge y_{23} + v_{A,B} \otimes y_{14} \wedge y_{24}$.

LEMMA 4.6. The S-module structure of the module $V \otimes Y_2$ is given as follows:
(1) If $B < p-2$, then $V \otimes Y_2 \cong L(n_1) \oplus \delta_{34} L(n_2) \oplus \delta_{34}\delta^1_{34} L(n_3)$.

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(2) If $B = p-2$, then $V \otimes Y_2 \cong N_4 \oplus \delta^1_{34} L(n_3)$, where the S-module N_4 has the composition series

$$N_4 = \begin{array}{c} L(n_4) \\ | \\ L(n_1) \\ | \\ L(n_2) \end{array}.$$

(3) If $B = p-1$, then $V \otimes Y_2 \cong N_5 \oplus L(n_2)$, where the S-module N_5 has the composition series

$$N_5 = \begin{array}{c} L(n_5) \\ | \\ L(n_1) \\ | \\ L(n_3) \end{array}.$$

Proof. It is symmetric to the proof of Lemma 4.5. □

4.2.3. $V \otimes (Y \wedge Y)$. Combining the descriptions of $V \otimes Y_1$ and $V \otimes Y_2$ we get the following.

Proposition 4.7. The S-module $V \otimes (Y \wedge Y)$ is isomorphic to
(1) Case $0 < A, B < p-2$
$L(m_1) \oplus L(n_1) \oplus L(m_2) \oplus L(n_2) \oplus \delta^1_{12} L(m_3) \oplus \delta^1_{34} L(n_3)$.
(2) Case $A = 0, 0 < B < p-2$
$L(m_1) \oplus L(n_1) \oplus L(n_2) \oplus \delta^1_{34} L(n_3)$.
(3) Case $0 < A < p-2, B = 0$
$L(m_1) \oplus L(n_1) \oplus L(m_2) \oplus \delta^1_{12} L(m_3)$.
(4) Case $A = B = 0$
$L(m_1) \oplus L(n_1)$.
(5) Case $A = p-2, 0 < B < p-2$
$M_4 \oplus L(n_1) \oplus L(n_2) \oplus \delta^1_{12} L(m_3) \oplus \delta^1_{34} L(n_3)$.
(6) Case $A = p-2, B = 0$
$M_4 \oplus L(n_1) \oplus \delta^1_{12} L(m_3)$.
(7) Case $0 < A < p-2, B = p-2$
$L(m_1) \oplus N_4 \oplus L(m_2) \oplus \delta^1_{12} L(m_3) \oplus \delta^1_{34} L(n_3)$.
(8) Case $A = 0, B = p-2$
$L(m_1) \oplus N_4 \oplus \delta^1_{34} L(n_3)$.
(9) Case $A = B = p-2$
$M_4 \oplus N_4 \oplus \delta^1_{12} L(m_3) \oplus \delta^1_{34} L(n_3)$.
(10) Case $A = p-1, 0 < B < p-2$
$M_5 \oplus L(n_1) \oplus L(m_2) \oplus L(n_2) \oplus \delta^1_{34} L(n_3)$.
(11) Case $A = p-1, B = 0$
$M_5 \oplus L(n_1) \oplus L(m_2)$.
(12) Case $0 < A < p-2, B = p-1$
$L(m_1) \oplus N_5 \oplus L(m_2) \oplus L(n_2) \oplus \delta^1_{12} L(m_3)$.

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(13) Case $A = 0$, $B = p - 1$
$L(m_1) \oplus N_5 \oplus L(n_2)$.
(14) Case $A = p - 1$, $B = p - 2$
$M_5 \oplus N_4 \oplus L(m_2) \oplus \delta^1_{34} L(n_3)$.
(15) Case $A = p - 2$, $B = p - 1$
$M_4 \oplus N_5 \oplus L(n_2) \oplus \delta^1_{12} L(m_3)$.
(16) Case $A = B = p - 1$
$M_5 \oplus N_5 \oplus L(m_2) \oplus L(n_2)$.

Proof. Combine Lemmas 4.5 and 4.6. □

The reason why we split the above Proposition into 16 cases instead of nine is to prepare for its application in the next section.

4.3. Image under ϕ_2. We shall analyse the modules $\phi_2(V \otimes Y_1)$ and $\phi_2(V \otimes Y_2)$ first, and then determine $\phi_2(V \otimes (Y \wedge Y))$.

The starting point is Lemma 4.2, which holds in positive characteristic as well. Lemma 4.3 is modified as follows.

LEMMA 4.8. If $\omega_{23}\omega_{13} = 0$ and $B = 0$, then $\phi_2(m_2) = 0$. If $\omega_{23}\omega_{13} = 0$ and $B = p - 2$, then $\phi_2(n_2) = 0$. If $\omega_{23}\omega_{24} = 0$ and $A = 0$, then $\phi_2(n_2) = 0$. If $\omega_{23}\omega_{24} = 0$ and $A = p - 2$, then $\phi_2(m_2) = 0$.

In all other cases, when defined, the images $\phi_2(m_2)$ and $\phi_2(n_2)$ are non-zero, and if $A, B > 0$ and $\omega_{13}\omega_{14}\omega_{23}\omega_{24} = 0$, then $\phi_2(m_2) \sim \phi_2(n_2)$.

Proof. Lemma 4.2 shows that $\phi_2(m_2) = 0$ implies $A = p - 2$ or $B = 0$, and $\phi_2(n_2) = 0$ implies $A = 0$ or $B = p - 2$. It is easy to verify that $\phi_2(m_2) = \omega_{23}\omega_{13}m_2$ for $B = 0$, $\phi_2(n_2) = \omega_{23}\omega_{13}n_2$ for $B = p - 2$, $\phi_2(n_2) = \omega_{23}\omega_{24}m_2$ for $A = 0$, and $\phi_2(m_2) = \omega_{23}\omega_{24}m_2$ for $A = p - 2$.

The remaining arguments are as in Lemma 4.3. □

For computation of $\phi_2(V \otimes Y_1)$ and $\phi_2(V \otimes Y_2)$ we shall use Lemmas 4.2 and 4.8 repeatedly.

To combine both $\phi_2(V \otimes Y_1)$ and $\phi_2(V \otimes Y_2)$ we shall need the following lemma.

Denote a K-span of $\phi_2(m_2)$ and $\phi_2(n_2)$ by X.

LEMMA 4.9. The dimension of the space X is described as follows:
$dimX = 2$ if and only if $\omega_{13}\omega_{14}\omega_{23}\omega_{24} \neq 0$;
$dimX = 1$ if and only if $\omega_{13}\omega_{14}\omega_{23}\omega_{24} = 0$ and one of the following conditions is satisfied:

- $A, B > 0$,
- $A = 0$, $B > 0$ and $\omega_{23}\omega_{24} \neq 0$,
- $A > 0$, $B = 0$ and $\omega_{23}\omega_{13} \neq 0$; and

$dimX = 0$ if and only if one of the following conditions is satisfied:

- $A = 0$, $B > 0$ and $\omega_{23}\omega_{24} = 0$,
- $A > 0$, $B = 0$ and $\omega_{23}\omega_{13} = 0$ and
- $A = B = 0$

Proof. Use Lemmas 4.2 and 4.8. □

If $\phi_2(M) \cong M$ and $L(\phi_2(w)) \cong L(w)$, then we shall denote $\overline{M} = \phi_2(M)$ and $\overline{w} = \phi_2(w)$ respectively.

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4.3.1. Module $\phi_2(V \otimes Y_1)$. We shall need the following lemma, a part of which shall be useful for the determination of $\phi_2(V \otimes (Y \wedge Y))$.

LEMMA 4.10. If $A = p-2$ and $\omega_{24} = 0$, then $\phi_2(m_4) = \phi_2(n_2) \neq 0$.
Assume $A = p-2$ and $\omega_{23} = 0$. If $B \neq 0, p-2$, then $\phi_2(m_4) \sim \phi_2(n_2)$. If $B = 0$, then $\phi_2(m_4) = 0$ and $\phi_2(n_2) \neq 0$. If $B = p-2$, then $\phi_2(m_4) \neq 0$ and $\phi_2(n_2) = 0$.

Proof. If $\omega_{24} = 0$, then $\phi_2(m_4) = (2+B)m_2 - Bn_2 = \phi_2(n_2)$.
If $\omega_{23} = 0$, then $\phi_2(m_4) = -B(m_2+n_2)$ and $\phi_2(n_2) = (B+2)(m_2+n_2)$. □

The structure of the S-module $\phi_2(V \otimes Y_1)$ is given as follows.

PROPOSITION 4.11. The following statements describe S-modules isomorphic to $V_{21} = \phi_2(V \otimes Y_1)$.
Assume $A < p-2$.
If λ is typical, then $V_{21} \cong L(m_1) \oplus \delta_{12}L(\overline{m}_2) \oplus \delta_{12}\delta^1_{12}L(m_3)$.
If λ is 13-atypical, then $V_{21} \cong L(m_1) \oplus \delta_{12}\delta_{34}L(\overline{m}_2)$.
If λ is 14- or (13, 14)-atypical, then $V_{21} \cong L(m_1) \oplus \delta_{12}L(\overline{m}_2)$.
If λ is 23-atypical, then $V_{21} \cong \delta_{12}\delta_{34}L(\overline{m}_2) \oplus \delta_{12}\delta^1_{12}L(m_3)$.
If λ is 24- or (23, 24)-atypical, then $V_{21} \cong \delta_{12}L(\overline{m}_2) \oplus \delta_{12}\delta^1_{12}L(m_3)$.
If λ is (13, 24)-atypical or (14, 23)-atypical, then $V_{21} \cong \delta_{12}\delta_{34}L(\overline{m}_2)$.
Assume now $A = p-2$.
If λ is typical, then $V_{21} \cong \overline{M}_4 \oplus \delta^1_{12}L(m_3)$.
If λ is 13-, 14- or (13, 14)-atypical, then $V_{21} \cong \overline{M}_4$.
If λ is 24- or (23, 24)-atypical, then $V_{21} \cong L((B+2)m_2 - Bn_2) \oplus \delta^1_{12}L(m_3)$.
If λ is (13, 24)-atypical, then $V_{21} \cong L((B+2)m_2 - Bn_2)$.
If λ is 23-atypical, then $V_{21} \cong \delta_{34}L(m_2+n_2) \oplus \delta^1_{12}L(m_3)$.
If λ is (14, 23)-atypical, then $V_{21} \cong L(m_2+n_2)$.
Finally, assume $A = p-1$.
If λ is typical, then $V_{21} \cong \overline{M}_5 \oplus L(\overline{m}_2)$.
If λ is (13, 23)-atypical and $B \neq 0$, then $V_{21} \cong L(\overline{m}_5) \oplus L(\overline{m}_2)$.
If λ is (13, 23)-atypical and $B = 0$, then $V_{21} \cong L(\overline{m}_5)$.
If λ is (14, 24)-atypical, then $V_{21} \cong L(\overline{m}_5) \oplus L(\overline{m}_2)$.
If λ is (13, 14, 23, 24)-atypical, then $V_{21} \cong L(\overline{m}_2)$.

Proof. If $A < p-2$, then $V \otimes Y_1$ is semi-simple by the first part of Proposition 4.5 and the highest weights of primitive vectors are pairwise different. Therefore, it is enough to determine whether the images of these primitive vectors under ϕ_2 vanish or not, and this follows from Lemmas 4.2 and 4.8.

For $A = p-2$, we use the second part of Proposition 4.5. Since $L(m_2)$ is the S-socle of M_4, Lemma 4.8 shows that $\phi_2(M_4) \cong M_4$, provided $\omega_{23}\omega_{24} \neq 0$. If $\omega_{23}\omega_{24} = 0$, then using Lemma 4.2 we infer $\phi_2(m_1) = 0$, and therefore $\phi_2(M_4) \cong \phi_2(L(m_4))$. Lemma 4.10 determines $\phi_2(m_4)$ and Lemma 4.2 concludes the proof in the case $A = p-2$.

Now assume that $A = p-1$. If λ is typical, then Lemma 4.2 shows that $\phi_2(V_{21}) \cong V_{21}$. Assume now λ is not typical, then $\omega_{13} = \omega_{23}$ and $\omega_{14} = \omega_{24}$ which implies $\omega_{23}\omega_{24} = 0$. Since $\phi_2(m_1) = \omega_{23}\omega_{24}m_1$, using the third part of Proposition 4.5 we conclude $\phi_2(M_5) \cong \phi_2(L(m_5))$. We verify that $\omega_{23} = 0$ implies $\phi_2(m_5) = (B+1)m_3$ and $\omega_{24} = 0$ implies $\phi_2(m_5) = -(B+1)m_3$. Therefore, $\phi_2(m_5) = 0$ if and only if $B = p-1$. Lemma 4.2 concludes the proof. □

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4.3.2. Module $\phi_2(V \otimes Y_2)$. We shall need the following lemma, a part of which shall be useful for the determination of $\phi_2(V \otimes (Y \wedge Y))$.

LEMMA 4.12. If $B = p-2$ and $\omega_{13} = 0$, then $\phi_2(n_4) = \phi_2(m_2) \neq 0$.
Assume $B = p-2$ and $\omega_{23} = 0$. If $A \neq 0, p-2$, then $\phi_2(n_4) \sim \phi_2(m_2)$. If $A = 0$, then $\phi_2(n_4) = 0$ and $\phi_2(m_2) \neq 0$. If $A = p-2$, then $\phi_2(n_4) \neq 0$ and $\phi_2(m_2) = 0$.

Proof. If $\omega_{13} = 0$, then $\phi_2(n_4) = -Am_2 + (2+A)n_2 = \phi_2(m_2)$.
If $\omega_{23} = 0$, then $\phi_2(n_4) = -A(m_2 + n_2)$ and $\phi_2(m_2) = (2+A)(m_2+n_2)$. □

The structure of the S-module $\phi_2(V \otimes Y_2)$ is given as follows.

PROPOSITION 4.13. The following statements describe S-modules isomorphic to $V_{22} = \phi_2(V \otimes Y_2)$.
Assume $B < p-2$.
If λ is typical, then $V_{22} \cong L(n_1) \oplus \delta_{34}L(\overline{n}_2) \oplus \delta_{34}\delta^1_{34}L(n_3)$.
If λ is 14- or (14, 24)-atypical, then $V_{22} \cong L(n_1) \oplus \delta_{34}L(\overline{n}_2)$.
If λ is λ is 24-atypical, then $V_{22} \cong L(n_1) \oplus \delta_{12}\delta_{34}L(\overline{n}_2)$.
If λ is 13- or (13, 23)-atypical, then $V_{22} \cong \delta_{34}L(\overline{n}_2) \oplus \delta_{34}\delta^1_{34}L(n_3)$.
If λ is 23-atypical, then $V_{22} \cong \delta_{12}\delta_{34}L(\overline{n}_2) \oplus \delta_{34}\delta^1_{34}L(n_3)$.
If λ is (13, 24)-atypical or (14, 23)-atypical, then $V_{22} \cong \delta_{12}\delta_{34}L(\overline{n}_2)$.
Assume now $B = p-2$.
If λ is typical, then $V_{22} \cong \overline{N}_4 \oplus \delta^1_{34}L(n_3)$.
If λ is 14- or 24- or (14, 24)-atypical, then $V_{22} \cong \overline{N}_4$.
If λ is 13- or (13, 23)-atypical, then $V_{22} \cong L(-Am_2 + (2+A)n_2) \oplus \delta^1_{34}L(n_3)$.
If λ is (13, 24)-atypical, then $V_{22} \cong L(-Am_2 + (2+A)n_2)$.
If λ is 23-atypical, then $V_{22} \cong \delta_{12}L(m_2+n_2) \oplus \delta^1_{34}L(n_3)$.
If λ is (14, 23)-atypical, then $V_{22} \cong L(m_2+n_2)$.
Finally, assume $B = p-1$.
If λ is typical, then $V_{22} \cong \overline{N}_5 \oplus L(\overline{n}_2)$.
If λ is (23, 24)-atypical and $A \neq 0$, then $V_{22} \cong L(\overline{n}_5) \oplus L(\overline{n}_2)$.
If λ is (23, 24)-atypical and $A = 0$, then $V_{22} \cong L(\overline{n}_5)$.
If λ is (13, 14)-atypical, then $V_{22} \cong L(\overline{n}_5) \oplus L(\overline{n}_2)$.
If λ is (13, 14, 23, 24)-atypical, then $V_{22} \cong L(\overline{n}_2)$.

Proof. The proof is analogous to the proof of Proposition 4.11 and uses the following identities. If $B = p-1$ and $\omega_{13} = 0$, then $\phi_2(n_5) = -(A+1)n_3$. If $B = p-1$ and $\omega_{23} = 0$, then $\phi_2(n_5) = (A+1)n_3$. □

4.3.3. Module $\phi_2(V \otimes (Y \wedge Y))$. Now we shall determine $\phi_2(V \otimes (Y \wedge Y))$.

PROPOSITION 4.14. The S-module $V_2 = \phi_2(V \otimes (Y \wedge Y))$ is isomorphic to the following modules:
(1) Case $0 < A, B < p-2$
- $L(m_1) \oplus L(n_1) \oplus L(\overline{m}_2) \oplus L(\overline{n}_2) \oplus \delta^1_{12}L(m_3) \oplus \delta^1_{34}L(n_3)$, if λ is typical,
- $L(m_1) \oplus L(\overline{m}_2) \oplus \delta^1_{34}L(n_3)$, if λ is 13-atypical,
- $L(m_1) \oplus L(n_1) \oplus L(\overline{m}_2)$, if λ is 14-atypical,
- $L(\overline{m}_2) \oplus \delta^1_{12}L(m_3) \oplus \delta^1_{34}L(n_3)$, if λ is 23-atypical,
- $L(n_1) \oplus L(\overline{m}_2) \oplus \delta^1_{12}L(m_3)$, if λ is 24-atypical,
- $L(\overline{m}_2)$, if λ is (13, 24)-atypical or (14, 23)-atypical.

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(2) Case $A = 0, 0 < B < p - 2$
- $L(m_1) \oplus L(n_1) \oplus L(\overline{n}_2) \oplus \delta^1_{34} L(n_3)$, if λ is typical,
- $L(m_1) \oplus L(\overline{n}_2) \oplus \delta^1_{34} L(n_3)$, if λ is 13-atypical,
- $L(m_1) \oplus L(n_1) \oplus L(\overline{n}_2)$, if λ is 14-atypical,
- $\delta^1_{34} L(n_3)$, if λ is 23-atypical,
- $L(n_1)$, if λ is 24-atypical.

(3) Case $0 < A < p - 2, B = 0$
- $L(m_1) \oplus L(n_1) \oplus L(\overline{m}_2) \oplus \delta^1_{12} L(m_3)$, if λ is typical,
- $L(m_1)$, if λ is 13-atypical,
- $L(m_1) \oplus L(n_1) \oplus L(\overline{m}_2)$, if λ is 14-atypical,
- $\delta^1_{12} L(m_3)$, if λ is 23-atypical,
- $L(\overline{m}_2) \oplus L(n_1) \oplus \delta^1_{12} L(m_3)$, if λ is 24-atypical.

(4) Case $A = B = 0$
- $L(m_1) \oplus L(n_1)$, if λ is typical,
- $L(m_1)$, if λ is 13-atypical,
- $L(n_1)$, if λ is 24-atypical,
- 0, if λ is (14, 23)-atypical.

(5) Case $A = p - 2, 0 < B < p - 2$
- $\overline{M}_4 \oplus L(n_1) \oplus L(\overline{n}_2) \oplus \delta^1_{12} L(m_3) \oplus \delta^1_{34} L(n_3)$, if λ is typical,
- $\overline{M}_4 \oplus \delta^1_{34} L(n_3)$, if λ is 13-atypical,
- $L(n_1) \oplus \overline{M}_4$, if λ is 14-atypical,
- $L(m_2 + n_2) \oplus \delta^1_{12} L(m_3) \oplus \delta^1_{34} L(n_3)$, if λ is 23-atypical,
- $L(n_1) \oplus L((B + 2)m_2 - Bn_2) \oplus \delta^1_{12} L(m_3)$, if λ is 24-atypical.

(6) Case $A = p - 2, B = 0$
- $\overline{M}_4 \oplus L(n_1) \oplus \delta^1_{12} L(m_3)$, if λ is typical,
- $L(n_1) \oplus \overline{M}_4$, if λ is 14-atypical,
- $\delta^1_{12} L(m_3)$, if λ is 23-atypical,
- $L(m_2)$, if λ is (13, 24)-atypical.

(7) Case $0 < A < p - 2, B = p - 2$
- $L(m_1) \oplus \overline{N}_4 \oplus L(\overline{m}_2) \oplus \delta^1_{12} L(m_3) \oplus \delta^1_{34} L(n_3)$, if λ is typical,
- $L(m_1) \oplus \overline{N}_4$, if λ is 14-atypical,
- $\overline{N}_4 \oplus \delta^1_{12} L(m_3)$, if λ is 24-atypical,
- $L(m_1) \oplus L(-Am_2 + (A + 2)n_2) \oplus \delta^1_{34} L(n_3)$, if λ is 13-atypical,
- $L(m_2 + n_2) \oplus \delta^1_{12} L(m_3) \oplus \delta^1_{34} L(n_3)$, if λ is 23-atypical.

(8) Case $A = 0, B = p - 2$
- $L(m_1) \oplus \overline{N}_4 \oplus \delta^1_{34} L(n_3)$, if λ is typical,
- $L(m_1) \oplus \overline{N}_4$, if λ is 14-atypical,
- $\delta^1_{34} L(n_3)$, if λ is 23-atypical,
- $L(n_2)$, if λ is (13, 24)-atypical.

(9) Case $A = B = p - 2$
- $\overline{M}_4 \oplus \overline{N}_4 \oplus \delta^1_{12} L(m_3) \oplus \delta^1_{34} L(n_3)$, if λ is typical,
- $\overline{M}_4 \oplus \delta^1_{34} L(n_3)$, if λ is 13-atypical,
- $\overline{N}_4 \oplus \delta^1_{12} L(m_3)$, if λ is 24-atypical,
- $L(m_2 + n_2)$, if λ is (14, 23)-atypical.

(10) Case $A = p - 1, 0 < B < p - 2$
- $\overline{M}_5 \oplus L(n_1) \oplus L(\overline{m}_2) \oplus L(\overline{n}_2) \oplus L(m_3) \oplus \delta^1_{34} L(n_3)$, if λ is typical,
- $L(m_3) \oplus L(\overline{m}_2) \oplus \delta^1_{34} L(n_3)$, if λ is (13, 23)-atypical,
- $L(m_3) \oplus L(n_1) \oplus L(\overline{m}_2)$, if λ is (14, 24)-atypical.

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(11) Case $A = p-1$, $B = 0$
• $\overline{M}_5 \oplus L(n_1) \oplus L(\overline{m}_2) \oplus L(m_3)$, if λ is typical,
• $L(m_3)$, if λ is (13, 23)-atypical,
• $L(m_3) \oplus L(\overline{m}_2) \oplus L(n_1)$, if λ is (14, 24)-atypical.
(12) Case $0 < A < p-2$, $B = p-1$
• $L(m_1) \oplus \overline{N}_5 \oplus L(\overline{m}_2) \oplus L(\overline{n}_2) \oplus \delta^1_{12} L(m_3) \oplus L(n_3)$, if λ is typical,
• $L(m_1) \oplus L(n_3) \oplus L(\overline{m}_2)$, if λ is (13, 14)-atypical,
• $L(n_3) \oplus L(\overline{m}_2) \oplus \delta^1_{12} L(m_3)$, if λ is (23, 24)-atypical.
(13) Case $A = 0$, $B = p-1$
• $L(m_1) \oplus \overline{N}_5 \oplus L(\overline{n}_2) \oplus L(n_3)$, if λ is typical,
• $L(m_1) \oplus L(n_3) \oplus L(\overline{n}_2)$, if λ is (13, 14)-atypical,
• $L(n_3)$, if λ is (23, 24)-atypical.
(14) Case $A = p-1$, $B = p-2$
• $\overline{M}_5 \oplus \overline{N}_4 \oplus L(\overline{m}_2) \oplus L(m_3) \oplus \delta^1_{34} L(n_3)$, if λ is typical,
• $L(m_3) \oplus L(m_2 + n_2) \oplus \delta^1_{34} L(n_3)$, if λ is (13, 23)-atypical,
• $L(m_3) \oplus \overline{N}_4$, if λ is (14, 24)-atypical.
(15) Case $A = p-2$, $B = p-1$
• $\overline{M}_4 \oplus \overline{N}_5 \oplus L(\overline{n}_2) \oplus \delta^1_{12} L(m_3) \oplus L(n_3)$, if λ is typical,
• $L(n_3) \oplus L(m_2 + n_2) \oplus \delta^1_{12} L(m_3)$, if λ is (23, 24)-atypical,
• $L(n_3) \oplus \overline{M}_4$, if λ is (13, 14)-atypical.
(16) Case $A = B = p-1$
• $\overline{M}_5 \oplus \overline{N}_5 \oplus L(\overline{m}_2) \oplus L(\overline{n}_2) \oplus L(m_3) \oplus L(n_3)$, if λ is typical,
• $L(\overline{m}_2)$, if λ is (13, 14, 23, 24)-atypical.

Proof. It follows from Propositions 4.11, 4.13, and Lemmas 4.2, 4.8, 4.9, 4.10 and 4.12. For the convenience of the reader we shall point out cases when the non-zero images under ϕ_2 of different primitive vectors of the highest weight (A, B) are collinear.

If $A = p-2$, $0 < B < p-2$ and λ is 23-atypical or 24-atypical, then $\phi_2(m_4) \sim \phi_2(n_2)$.

If $0 < A < p-2$, $B = p-2$ and λ is 13-atypical or 23-atypical, then $\phi_2(n_4) \sim \phi_2(m_2)$.

If $A = B = p-2$ and λ is 13-atypical, then $\phi_2(n_4) \sim \phi_2(m_2)$.

If $A = B = p-2$ and λ is 24-atypical, then $\phi_2(m_4) \sim \phi_2(n_2)$.

If $A = B = p-2$ and λ is (14, 23)-atypical, then $\phi_2(m_4) \sim \phi_2(n_4)$ while $\phi_2(m_2) = \phi_2(n_2) = 0$.

If $A = p-1$, $0 < B < p-2$ and λ is (13, 23)- or (14, 24)-atypical, then $\phi_2(n_2) \sim \phi_2(m_2)$.

If $0 < A < p-2$, $B = p-1$ and λ is (13, 14)- or (23, 24)-atypical, then $\phi_2(m_2) \sim \phi_2(n_2)$.

If $A = p-1$, $B = p-2$ and λ is (13, 23)-atypical, then $\phi_2(n_4) \sim \phi_2(m_2)$.

If $A = p-1$, $B = p-2$ and λ is (14, 24)-atypical, then $\phi_2(m_2) \sim \phi_2(n_2)$.

If $A = p-2$, $B = p-1$ and λ is (23, 24)-atypical, then $\phi_2(m_4) \sim \phi_2(n_2)$.

If $A = p-2$, $B = p-1$ and λ is (13, 14)-atypical, then $\phi_2(m_2) \sim \phi_2(n_2)$.

If $A = B = p-1$ and λ is (13, 14, 23, 24)-atypical, then $\phi_2(m_2) \sim \phi_2(n_2)$. □

5. Third floor.

5.1. Characteristic zero. Denote $z_{23} = y_{13} \wedge y_{23} \wedge y_{24}$, $z_{24} = y_{14} \wedge y_{23} \wedge y_{24}$, $z_{13} = -y_{13} \wedge y_{14} \wedge y_{23}$, $z_{14} = -y_{13} \wedge y_{14} \wedge y_{24}$.

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Furthermore, set $k_1 = v_{AB} \otimes z_{23}$, $k_2 = v_{AB} \otimes z_{24} - v_{A,B-1} \otimes z_{23}$, $k_3 = v_{AB} \otimes z_{13} + v_{A-1,B} \otimes z_{23}$ and $k_4 = v_{AB} \otimes z_{14} + v_{A-1,B} \otimes z_{24} - v_{A,B-1} \otimes z_{13} - v_{A-1,B-1} \otimes z_{23}$.

Let $\Theta : Y \to Y \wedge Y \wedge Y$ be a map that sends $y_{23} \mapsto z_{23}$, $y_{24} \mapsto z_{24}$, $y_{13} \mapsto z_{13}$ and $y_{14} \mapsto z_{14}$. Then Θ is an isomorphism of S-modules and it induces an isomorphism of S-modules $\Theta_V : V \otimes Y \to V \otimes (Y \wedge Y \wedge Y)$ via $\Theta_V(v \otimes y_{ij}) = v \otimes \Theta(y_{ij}) = v \otimes z_{ij}$ for appropriate i, j.

Lemma 5.1. The module $V \otimes (Y \wedge Y \wedge Y)$ is isomorphic to the direct sum $L(k_1) \oplus \delta_{34}L(k_2) \oplus \delta_{12}L(k_3) \oplus \delta_{12}\delta_{34}L(k_4)$.

Proof. Since the map $\Theta : Y \to Y \wedge Y \wedge Y$ is an isomorphism of S-modules, the module $Y \wedge Y \wedge Y$ is irreducible of the highest vector z_{23}.

Since the map Θ_V is an isomorphism of S-modules and the S-module structure of $V \otimes Y$ was already determined, we can use Θ_V to describe the S-module structure of $V \otimes (Y \wedge Y \wedge Y)$.

Using Lemma 3.1 we establish $V \otimes (Y \wedge Y \wedge Y) \cong L(k_1) \oplus \delta_{34}L(k_2) \oplus \delta_{12}L(k_3) \oplus \delta_{12}\delta_{34}L(k_4)$. □

Proposition 5.2. The image $\phi_3(V \otimes (Y \wedge Y \wedge Y)) \cong \delta_{13}\delta_{23}\delta_{24}K_1 \oplus \delta_{34}\delta_{14}\delta_{23}\delta_{24}K_2 \oplus \delta_{12}\delta_{13}\delta_{14}\delta_{23}K_3 \oplus \delta_{12}\delta_{34}\delta_{13}\delta_{14}\delta_{24}K_4$.

Proof. The S-morphism ϕ_3 is described completely by images of generating vectors $\phi_3(k_1) = \omega_{13}\omega_{23}\omega_{24}k_1$, $\phi_3(k_2) = \omega_{14}\omega_{23}\omega_{24}k_2$, $\phi_3(k_3) = \omega_{13}\omega_{14}\omega_{23}k_3$, and $\phi_3(k_4) = \omega_{13}\omega_{14}\omega_{24}k_4$. □

5.2. Characteristic p.

Assume that the weight λ is restricted.

Recall that the map $\Theta_V : V \otimes Y \to V \otimes (Y \wedge Y \wedge Y)$ is an isomorphism of S-modules, and for each $5 \leq i \leq 9$, denote $\Theta_V(l_i) = k_i$ and $K_i = \Theta_V(L_i)$.

Proposition 5.3. The S-module $V \otimes (Y \wedge Y \wedge Y)$ is isomorphic to

(1) Case $A, B < p - 1$

$L(k_1) \oplus \delta_{34}L(k_2) \oplus \delta_{12}L(k_3) \oplus \delta_{12}\delta_{34}L(k_4)$.

(2) Case $A = p - 1, B < p - 1$

$K_5 \oplus \delta_{34}K_6$.

(3) Case $A < p - 1, B = p - 1$

$K_7 \oplus \delta_{12}K_8$.

(4) Case $A = B = p - 1$

K_9.

Here the composition series of the S-module K_i, for every $5 \leq i \leq 9$, is analogous to that of corresponding S-module L_i from Proposition 3.5.

Proof. The map Θ_V is an isomorphism of S-modules. Since the S-module structure of $V \otimes Y$ was already determined, we can use Θ_V to describe the S-module structure of $V \otimes (Y \wedge Y \wedge Y)$. All that is necessary to do this is to replace every appearance of l_j and L_j in Proposition 3.5 with k_j and K_j respectively. □

5.3. Image under ϕ_3.

The structure of the S-module $\phi_3(V \otimes (Y \wedge Y \wedge Y))$ is given as follows.

Proposition 5.4. The following statements describe S-modules isomorphic to $V_3 = \phi_3(V \otimes (Y \wedge Y \wedge Y))$.

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If $A, B < p-1$, then $V_3 \cong \delta_{13}\delta_{23}\delta_{24}K_1 \oplus \delta_{34}\delta_{14}\delta_{23}\delta_{24}K_2 \oplus \delta_{12}\delta_{13}\delta_{14}\delta_{23}K_3 \oplus \delta_{12}\delta_{34}\delta_{13}\delta_{14}\delta_{24}K_4$.

Assume $A = p-1$ and $B < p-1$. If λ is typical, then $V_3 \cong V \otimes (Y \wedge Y \wedge Y)$. If λ is (13, 23)-atypical, then $V_3 \cong \delta_{34}L(k_4)$. If λ is (14, 24)-atypical, then $V_3 \cong L(k_3)$.

Assume $A < p-1$ and $B = p-1$. If λ is typical, then $V_3 \cong V \otimes (Y \wedge Y \wedge Y)$. If λ is (13, 14)-atypical, then $V_3 \cong L(k_2)$. If λ is (23, 24)-atypical, then $V_3 \cong \delta_{12}L(k_4)$.

Assume $A = B = p-1$. If λ is typical, then $V_3 \cong V \otimes (Y \wedge Y \wedge Y)$. If λ is (13, 14, 23, 24)-atypical, then $V_3 \cong 0$.

Proof. If $A, B < p-1$, then $V_3 \cong \delta_{13}\delta_{23}\delta_{24}K_1 \oplus \delta_{34}\delta_{14}\delta_{23}\delta_{24}K_2 \oplus \delta_{12}\delta_{13}\delta_{14}\delta_{23}K_3 \oplus \delta_{12}\delta_{34}\delta_{13}\delta_{14}\delta_{24}K_4$ as in the characteristic zero case.

If $A = p-1$ and $B < p-1$, then the images of generators of K_5 and K_6 are $\phi_3(k_5) = \omega_{13}\omega_{14}\omega_{23}k_5 - \omega_{13}\omega_{23}k_3$ and $\phi_3(k_6) = \omega_{13}\omega_{14}\omega_{24}k_6 - \omega_{14}\omega_{24}k_4$. The structure of V_3 follows.

If $A < p-1$ and $B = p-1$, then the images of generators of K_7 and K_8 are $\phi_3(k_7) = \omega_{14}\omega_{23}\omega_{24}k_7 + \omega_{23}\omega_{24}k_2$ and $\phi_3(k_8) = \omega_{13}\omega_{14}\omega_{24}k_8 - \omega_{13}\omega_{14}k_4$. The structure of V_3 follows.

If $A = B = p-1$, then $\omega_{13} = \omega_{14} = \omega_{23} = \omega_{24} = \omega$. The image of the generator k_9 of $V \otimes (Y \wedge Y \wedge Y)$ is $\phi_3(k_9) = \omega^3 k_9 - \omega^2 k_6 - \omega^2 k_8 - 2\omega k_4$ and the claim follows. □

6. Fourth floor.

6.1. Characteristic zero. The S-module $Y \wedge Y \wedge Y \wedge Y$ is the trivial module of the highest vector $y_{13} \wedge y_{14} \wedge y_{23} \wedge y_{24}$. The S-module $V \otimes (Y \wedge Y \wedge Y \wedge Y)$ is irreducible of the highest vector $l = v_{AB} \otimes y_{13} \wedge y_{14} \wedge y_{23} \wedge y_{24}$ and is isomorphic to V as an S-module.

Proposition 6.1. The image $\phi_4(V \otimes (Y \wedge Y \wedge Y \wedge Y)) = \delta_{13}\delta_{14}\delta_{23}\delta_{24} V \otimes (Y \wedge Y \wedge Y \wedge Y)$.

Proof. The morphism ϕ_4 is given by $\phi_4(l) = \omega_{13}\omega_{14}\omega_{23}\omega_{24}l$. □

6.2. Characteristic p. Assume that the weight λ is restricted.

The S-module $V \otimes (Y \wedge Y \wedge Y \wedge Y) = L(v_{A,B} \otimes (y_{13} \wedge y_{14} \wedge y_{23} \wedge y_{24}))$ is irreducible and isomorphic to V as an S-module.

6.3. Image under ϕ_4. The S-module structure of $\phi_4(V \otimes (Y \wedge Y \wedge Y \wedge Y))$ is given as follows.

Proposition 6.2. The S-module $\phi_4(V \otimes (Y \wedge Y \wedge Y \wedge Y))$ is isomorphic to $V \otimes (Y \wedge Y \wedge Y \wedge Y) \simeq V$ if λ is typical, and is isomorphic to 0 if λ is atypical.

7. Character and dimension of simple module $L_{S(2|2)}(\lambda)$. Combining the previous results, we obtain the following theorem.

Theorem 7.1. The S-module $H^0_G(\lambda)$ is isomorphic to the direct sum $V \oplus (V \otimes Y) \oplus (V \otimes (Y \wedge Y)) \oplus (V \otimes (Y \wedge Y \wedge Y)) \oplus (V \otimes (Y \wedge Y \wedge Y \wedge Y))$, where the middle summands are described in Propositions 3.5, 4.7 and 5.3.

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THEOREM 7.2. The S-module $L_{S(2|2)}(\lambda)$ is isomorphic to

$$V \oplus \phi_1(F_1(\lambda)) \oplus \phi_2(F_2(\lambda)) \oplus \phi_3(F_3(\lambda)) \oplus \phi_4(F_4(\lambda)),$$

where the images $V_1 = \phi_1(F_1(\lambda))$, $V_2 = \phi_2(F_2(\lambda))$, $V_3 = \phi_3(F_3(\lambda))$ and $V_4 = \phi_4(F_4(\lambda))$ are described in Propositions 3.6, 4.14, 5.4 and 6.2 respectively.

7.1. Characteristic zero. Combining previous results, we obtain the following theorem.

THEOREM 7.3. The simple module $L_{S(2|2)}(\lambda)$, viewed as an S-module, is isomorphic to the direct sum $V \oplus \phi_1(F_1(\lambda)) \oplus \phi_2(F_2(\lambda)) \oplus \phi_3(F_3(\lambda)) \oplus \phi_4(F_4(\lambda))$, where the images $\phi_1(F_1(\lambda))$, $\phi_2(F_2(\lambda))$, $\phi_3(F_3(\lambda))$ and $\phi_4(F_4(\lambda))$ are described in Propositions 3.2, 4.4, 5.2 and 6.1 respectively.

COROLLARY 7.4. The induced module $H^0_G(\lambda)$ is isomorphic to $L_{S(2|2)}(\lambda)$ if and only if λ is typical which happens if and only if $\lambda_2 \geq 2$.

In order to relate the last results to Hook Schur functions, we need to explain how a simple module $L_{S(2|2)}\lambda$ corresponds to a (2,2)-hook partition $\gamma = (\gamma_1, \dots, \gamma_k)$. The correspondence is such that $\gamma_1 = \lambda_1$, $\gamma_2 = \lambda_2$ and the partition $(\gamma_3, \dots, \gamma_k)$ is the transpose of (λ_3, λ_4).

The character of the induced module $H^0_G(\lambda)$ is given by the formula $\chi(H^0_G(\lambda)) = (1 + \frac{y_1}{x_1})(1 + \frac{y_2}{x_1})(1 + \frac{y_1}{x_2})(1 + \frac{y_2}{x_2})s_{(\lambda_1,\lambda_2)}(x_1, x_2)s_{(\lambda_3,\lambda_4)}(y_1, y_2)$, where $s_{(\lambda_1,\lambda_2)}(x_1, x_2)$ denotes the Schur function corresponding to the partition (λ_1, λ_2) and $s_{(\lambda_3,\lambda_4)}(y_1, y_2)$ denotes the Schur function corresponding to the transpose of the partition (λ_3, λ_4). The character of $L_{S(2|2)}(\lambda)$ is given by the Hook Schur function $HS_\gamma(x_1, x_2; y_1, y_2)$.

Therefore, we have the following equivalence which strengthens Theorem 6.20 of [**1**] in the case of (2, 2)-hook partitions.

PROPOSITION 7.5. For a (2, 2)-hook partition λ, the following are equivalent:
(1) $\lambda_2 \geq 2$,
(2) $HS_\gamma(x_1, x_2; y_1, y_2) = \chi(H^0_G(\lambda))$,
(3) $H^0_G(\lambda)$ is isomorphic to $L_{S(2|2)}(\lambda)$.

Proof. If $\lambda_2 \geq 2$, then in the notation of Theorem 6.20 of [**1**] we have $\chi(H^0_G(\lambda)) = (x_1 + y_1)(x_1 + y_2)(x_2 + y_1)(x_2 + y_2)s_\mu(x_1, x_2)s_\nu(y_1, y_2)$ and $HS_\gamma(x_1, x_2; y_1, y_2) = \chi(H^0_G(\lambda))$. The remaining statements follow from Corollary 7.4. □

7.2. Characteristic p. We give a compact formula for the character and dimension of a simple $S(2|2)$ module of restricted weight. Using the Steinberg Tensor Product theorem we can then determine the same for an arbitrary highest weight λ.

If $(\mu_1 \geq \mu_2)$ is a dominant weight for the algebra $sl(2)$, then the character of a simple $sl(2)$-module with the highest weight (μ_1, μ_2) is given by the Schur function $S_{(\lambda_1,\lambda_2)}(x_1, x_2)$.

For a dominant weight $\mu = (\mu_1 \geq \mu_2|\mu_3 \geq \mu_4)$ for the algebra S, the character $S(\mu)$ of the simple S-module $L(\mu)$ of the highest weight μ is given by $S_{(\mu_1,\mu_2)}(x_1, x_2)$ $S_{(\mu_3,\mu_4)}(x_3, x_4)$.

Denote by $S(\lambda_1, \lambda_2|\lambda_3, \lambda_4)$ the product $S_{(\lambda_1,\lambda_2)}(x_1, x_2)S_{(\lambda_3,\lambda_4)}(x_3, x_4)$ of two Schur functions if $\lambda_1 \geq \lambda_2$ and $\lambda_3 \geq \lambda_4$, and $S(\lambda_1, \lambda_2|\lambda_3, \lambda_4) = 0$ otherwise. For short, write it as $S(\lambda)$ and call it the Schur function corresponding to λ. Then $S(k, 0|l, 0) = p(k, l)$.

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Denote $\gamma_{13}=(-1,0|1,0)$, $\gamma_{14}=(-1,0|0,1)$, $\gamma_{23}=(0,-1|1,0)$ and $\gamma_{24}=(0,-1|0,1)$ and write the 'decorated' weights derived from λ as follows: $\tilde\lambda_{13}=\lambda+\gamma_{13}$, $\tilde\lambda_{14}=\lambda+\gamma_{14}$, $\tilde\lambda_{23}=\lambda+\gamma_{23}$, $\tilde\lambda_{24}=\lambda+\gamma_{24}$, $\overline{\lambda}_{13,14}=\lambda+\gamma_{13}+\gamma_{14}$, $\overline{\lambda}_{13,23}=\lambda+\gamma_{13}+\gamma_{23}$, $\overline{\lambda}_{13,24}=\lambda+\gamma_{13}+\gamma_{24}=\overline{\lambda}_{14,23}=\lambda+\gamma_{14}+\gamma_{23}$, $\overline{\lambda}_{14,24}=\lambda+\gamma_{14}+\gamma_{24}$, $\overline{\lambda}_{23,24}=\lambda+\gamma_{23}+\gamma_{24}$, $\check\lambda_{14}=\lambda+\gamma_{13}+\gamma_{23}+\gamma_{24}$, $\check\lambda_{13}=\lambda+\gamma_{14}+\gamma_{23}+\gamma_{24}$, $\check\lambda_{23}=\lambda+\gamma_{13}+\gamma_{14}+\gamma_{24}$, $\check\lambda_{24}=\lambda+\gamma_{13}+\gamma_{14}+\gamma_{23}$, $\hat\lambda=\lambda+\gamma_{13}+\gamma_{14}+\gamma_{23}+\gamma_{24}$.

Then the S-weights of these decorated weights are given in the following table.

λ (A,B)	$\tilde\lambda_{13}$ $(A-1,B+1)$	$\tilde\lambda_{14}$ $(A-1,B-1)$	$\tilde\lambda_{23}$ $(A+1,B+1)$	$\tilde\lambda_{24}$ $(A+1,B-1)$	
$\overline{\lambda}_{13,14}$ $(A-2,B)$	$\overline{\lambda}_{13,23}$ $(A,B+2)$	$\overline{\lambda}_{13,24}$ (A,B)	$\overline{\lambda}_{14,23}$ (A,B)	$\overline{\lambda}_{14,24}$ $(A,B-2)$	$\overline{\lambda}_{23,24}$ $(A+2,B)$
$\check\lambda_{14}$ $(A+1,B+1)$	$\check\lambda_{13}$ $(A+1,B-1)$	$\check\lambda_{23}$ $(A-1,B-1)$	$\check\lambda_{24}$ $(A-1,B+1)$	$\hat\lambda$ (A,B)	

We have already seen that weight $\overline{\lambda}_{13,24}=\overline{\lambda}_{14,23}$ is of special significance and we shall denote it by $\overline{\lambda}$.

The space spanned by all elements of the simple module $L_{S(2|2)}(\lambda)$ that lie on the ith floor shall be called the sector of that floor corresponding to λ and shall be denoted by $L_i(\lambda)$. Each $L_i(\lambda)$ is an S-module and to each $L_i(\lambda)$ we assign a 'partial' character $\chi_i(\lambda)$ that records the multiplicities of weight spaces of $L_i(\lambda)$. Then the character $\chi(\lambda)$ of the simple module $L_{S(2|2)}(\lambda)$ equals $\chi(\lambda)=\chi_0(\lambda)+\chi_1(\lambda)+\chi_2(\lambda)+\chi_3(\lambda)+\chi_4(\lambda)$.

The character and dimension of a simple $S(2|2)$-module $L_{S(2|2)}(\lambda)$ of restricted weight λ are given below.

THEOREM 7.6. Let $L_{S(2|2)}(\lambda)$ be a simple $S(2|2)$-module of the restricted highest weight λ.

If λ is typical, then $\chi(L_{S(2|2)}(\lambda))=$

$$S(\lambda)+S(\tilde\lambda_{23})+S(\tilde\lambda_{13})+S(\tilde\lambda_{24})+S(\tilde\lambda_{14})+S(\overline{\lambda}_{13,23})+S(\overline{\lambda}_{23,24})+\delta_{12}S(\overline{\lambda})$$
$$+\delta_{34}S(\overline{\lambda})+S(\overline{\lambda}_{13,14})+S(\overline{\lambda}_{14,24})+S(\check\lambda_{14})+S(\check\lambda_{24})+S(\check\lambda_{13})+S(\check\lambda_{23})+S(\hat\lambda)$$

and $dimL(\lambda)=16(A+1)(B+1)$.

If λ is 13-atypical or (13,14)-atypical, then $\chi(L_{S(2|2)}(\lambda))=$

$$S(\lambda)+S(\tilde\lambda_{23})+S(\tilde\lambda_{24})+S(\tilde\lambda_{14})+S(\overline{\lambda}_{23,24})+\delta_{34}S(\overline{\lambda})+S(\overline{\lambda}_{14,24})+S(\check\lambda_{13})$$

and $dimL(\lambda)=8+4A+12B+8AB$.

If λ is 14-atypical, then $\chi(L_{S(2|2)}(\lambda))=$

$$S(\lambda)+S(\tilde\lambda_{23})+S(\tilde\lambda_{13})+S(\tilde\lambda_{24})+S(\overline{\lambda}_{13,23})+S(\overline{\lambda}_{23,24})+S(\overline{\lambda})+S(\check\lambda_{14})$$

and $dimL(\lambda)=16+12A+12B+8AB$.

If λ is 23-atypical, (13,23)-atypical or (23,24)-atypical, then $\chi(L_{S(2|2)}(\lambda))=$

$$S(\lambda)+S(\tilde\lambda_{13})+S(\tilde\lambda_{24})+S(\tilde\lambda_{14})+\delta_{12}\delta_{34}S(\overline{\lambda})+S(\overline{\lambda}_{13,14})+S(\overline{\lambda}_{14,24})+S(\check\lambda_{23})$$

and $dimL(\lambda)=4A+4B+8AB$.

If λ is 24-atypical, (14,24)-atypical, then $\chi(L_{S(2|2)}(\lambda))=$

$$S(\lambda)+S(\tilde\lambda_{23})+S(\tilde\lambda_{13})+S(\tilde\lambda_{14})+S(\overline{\lambda}_{13,23})+\delta_{12}S(\overline{\lambda})+S(\overline{\lambda}_{13,14})+S(\check\lambda_{24})$$

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and $dimL(\lambda) = 8 + 12A + 4B + 8AB$.

If λ is (13,24)-atypical, then

$$\chi(L_{S(2|2)}(\lambda)) = S(\lambda) + S(\tilde{\lambda}_{23}) + S(\tilde{\lambda}_{14}) + S(\overline{\lambda})$$

and $dimL(\lambda) = 6 + 4A + 4B + 4AB$.

If λ is (14,23)-atypical or (13,14,23,24)-atypical, then

$$\chi(L_{S(2|2)}(\lambda)) = S(\lambda) + \delta_{34}S(\tilde{\lambda}_{13}) + \delta_{12}S(\tilde{\lambda}_{24}) + \delta_{12}\delta_{34}S(\overline{\lambda})$$

and $dimL(\lambda) = 2 + 4A + 4B + 4AB$ if $A \neq 0$ or $B \neq 0$; and $dimL(\lambda) = 1$ if $A = B = 0$.

Proof. In the case $A, B < p - 2$, we confirm that these formulas are valid by inspection of Propositions 3.6, 4.14, 5.4 and 6.2, noting that most of δ_{12}, δ_{34} and δ^1_{12}, δ^1_{34} disappear due to the definition of $S(\lambda)$. The only remaining δ_{12} and δ_{34} are coefficients at $S(\overline{\lambda})$. They reflect the intricacies of the S-module structure of $L_{S(2|2)}(\lambda)$. Looking at the S-weights of decorated λs in the case $A, B < p - 2$, we infer that there is a straightforward correspondence between characters of simple modules with the highest weight given as a decorated λ and the product of the Schur function of the corresponding decorated λ. Adding up dimensions of simple S-modules in each case, we arrive at the dimension of $L_{S(2|2)}(\lambda)$.

In the cases when A or B equals $p - 2$ or $p - 1$, certain components of S-weights of decorated λs are equal to p or $p + 1$. Looking at the characters of various S-modules introduced earlier, we determine that

$\chi(L_5) = S(\tilde{\lambda}_{23}) + S(\tilde{\lambda}_{13})$, $\chi(L_6) = S(\tilde{\lambda}_{24}) + S(\tilde{\lambda}_{14})$,
$\chi(L_7) = S(\tilde{\lambda}_{23}) + S(\tilde{\lambda}_{24})$, $\chi(L_8) = S(\tilde{\lambda}_{13}) + S(\tilde{\lambda}_{14})$,
$\chi(L_9) = S(\tilde{\lambda}_{23}) + S(\tilde{\lambda}_{24}) + S(\tilde{\lambda}_{13}) + S(\tilde{\lambda}_{14})$, $\chi(L_{10}) = S(\tilde{\lambda}_{24}) + S(\tilde{\lambda}_{13})$,
$\chi(M_4) = S(\overline{\lambda}_{23,24}) + \delta_{12}S(\overline{\lambda})$, $\chi(M_5) = S(\overline{\lambda}_{23,24}) + S(\overline{\lambda}_{13,14})$,
$\chi(N_4) = S(\overline{\lambda}_{13,23}) + \delta_{34}S(\overline{\lambda})$, $\chi(N_5) = S(\overline{\lambda}_{13,23}) + S(\overline{\lambda}_{14,24})$,
$\chi(K_5) = S(\check{\lambda}_{14}) + S(\check{\lambda}_{24})$, $\chi(K_6) = S(\check{\lambda}_{13}) + S(\check{\lambda}_{23})$,
$\chi(K_7) = S(\check{\lambda}_{14}) + S(\check{\lambda}_{13})$, $\chi(K_8) = S(\check{\lambda}_{24}) + S(\check{\lambda}_{23})$ and
$\chi(K_9) = S(\check{\lambda}_{14}) + S(\check{\lambda}_{24}) + S(\check{\lambda}_{13}) + S(\check{\lambda}_{23})$.

We will verify the equality for $\chi(L_{10})$, since it is perhaps the most interesting, and leave the remaining equalities to the reader.

We start with the following equality:

$$S_{(\lambda_2+p-1,\lambda_2-1)}(x_1, x_2) = S_{(\lambda_2-1,\lambda_2-1)}(x_1, x_2)\,(x_1^p + x_2^p) + S_{(\lambda_2+p-2,\lambda_2)}(x_1, x_2),$$

which immediately implies

$$\chi(L(l_2)) + \chi(L(l_4)) = \chi(L(\tilde{\lambda}_{24})) + \chi(L(\tilde{\lambda}_{14})) = S(\tilde{\lambda}_{24}).$$

Analogously, we can derive

$$\chi(L(l_3)) + \chi(L(-l_6 + l_8)) = \chi(L(\tilde{\lambda}_{13})) + \chi(L(\tilde{\lambda}_{14})) = S(\tilde{\lambda}_{13}).$$

Combination of the last two equalities yields $\chi(L_{10}) = S(\tilde{\lambda}_{24}) + S(\tilde{\lambda}_{13})$.

Using the above equalities we can inspect Propositions 3.6, 4.14, 5.4 and 6.2 in the cases when A or B equals $p - 2$ or $p - 1$ and arrive at the formulas in the statement of this theorem. □

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Let us note that the above formulas for the character of the simple $S(2|2)$-module depend only on the nature of atypicality of the highest weight λ and not on values of A and B.

In order to find the character of the simple module $L_{S(2|2)}(\lambda)$ for general dominant λ, we write $\lambda = \lambda_r + p\lambda_u$, where both λ_r and λ_u are dominant weights and λ_r is restricted. The Steinberg theorem ([**5**, Theorem 4.4]) states that $L_{S(2|2)}(\lambda) \cong L_{S(2|2)}(\lambda_r) \otimes F^*L(\lambda_u)$, where $F^*L(\lambda_u)$ is the Frobenius twist of $L(\lambda_u)$. This gives the character of $L_{S(2|2)}(\lambda)$, since the character of $L_{S(2|2)}(\lambda_r)$ was determined in Theorem 7.6 and the character of $L(\lambda_u)$ is the product of the character of an S^+-irreducible module of the highest weight λ_u^+ and an S^--irreducible module of the highest weight λ_u^-.

8. Concluding remarks. We conclude with an outline of a subsequent paper that extends the computation presented here.

The highest weights of simple $S(m|n)$-modules in characteristic zero correspond to hook weights λ in the sense of [**1**]. The highest weights of simple $S(m|n)$-modules in the case of positive characteristic were determined in [**3**].

The costandard module $\nabla(\lambda)$ for $S(m|n)$ coincides with the polynomial part of the induced module $H^0_{GL(m|n)}(\lambda)$. It can be proved that $\nabla(\lambda) = H^0_{GL(m|n)}(\lambda)$ if and only if λ is a hook weight (it is equivalent to $\lambda_m \geq n$).

For the Schur superalgebra $S(2|2)$, the costandard module coincide with the induced module if and only if $\lambda_2 \geq 2$. Since the character of $H^0_G(\lambda)$ and all simple $S(2|2)$-modules was determined earlier, using purely combinatorial techniques it is possible to compute all simple composition factors in the filtration of costandard modules $\nabla(\lambda)$ for $\lambda_2 \geq 2$. We shall do this computation for the case of the restricted weight λ and that way we determine the decomposition numbers in the process of modular reduction of all simple $S(2|2)$-modules of the restricted highest weight.

In the remaining case, for every restricted highest weight λ corresponding to simple $S(2|2)$-module such that $\lambda_2 \leq 1$, we determine the corresponding costandard module $\nabla(\lambda)$, that is the polynomial part of $H^0_G(\lambda)$. For that purpose we shall utilize the S-module structure of $H^0_G(\lambda)$ determined in Section 3 of this paper. Further, we shall compute the characters of costandard modules $\nabla(\lambda)$ with the restricted highest weight such that $\lambda_2 \leq 1$ and then we shall determine all simple composition factors in the filtration of those costandard modules.

Acknowledgements. Research was partially supported by CAPES, FAPESP and CNPq (Brazil).

REFERENCES

1. A. Berele and A. Regev, Hook Young diagrams with applications to combinatorics and to representations of Lie superalgebras, *Adv. Math.* **64** (1987), 118–175.

2. J. Brundan and A. Kleshchev, On translation functors for general linear and symmetric groups, *Proc. Lond. Math. Soc.* **80**(3) (2000), 75–106.

3. J. Brundan and J. Kujawa, A new proof of the Mullineux conjecture, *J. Algebr. Combin.* **18** (2003), 13–39.

4. A. N. Grishov, F. Marko and A. N. Zubkov, Description of costandard modules for Schur superalgebra $S(2|1)$ in positive characteristic, *Linear Multilinear Algebra* **59**(1) (2011), 57–64.

https://doi.org/10.1017/S0017089512000869 Published online by Cambridge University Press

5. J. Kujawa, The Steinberg tensor product theorem for $GL(m|n)$, *Contemp. Math.* **413** (2006), 123–132.

6. R. La Scala and A. N. Zubkov, Costandard modules over Schur superalgebras in characteristic p, *J. Algebra Appl.* **7**(2) (2008), 147–166.

7. F. Marko, Description of costandard modules of Schur superalgebra $S(3|1)$, *Comm. Algebra* (to appear). doi:10.1080/00927872.2012.660263.

8. A. N. Zubkov, Some properties of general linear supergroups and of Schur superalgebras, *Algebra Logic* **45**(3) (2006), 147–171.

https://doi.org/10.1017/S0017089512000869 Published online by Cambridge University Press