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AN ANALOGUE OF THE HADAMARD CONJECTURE FOR $n \times n$ MATRICES WITH $n \equiv 2 \pmod 4$

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Abstract

It is known that the problem of settling the existence of an $n \times n$ Hadamard matrix, where n is divisible by 4, is equivalent to that of finding the cardinality of a smallest set T of 4-circuits in the complete bipartite graph $K_{n,n}$ such that T contains at least one circuit of each copy of $K_{2,3}$ in $K_{n,n}$. Here we investigate the case where $n \equiv 2 \pmod 4$, and we show that the problem of finding the cardinality of T is equivalent to that of settling the existence of a certain kind of $n \times n$ matrix. Moreover, we show that the case where $n \equiv 2 \pmod 4$ differs from that where $n \equiv 0 \pmod 4$ in that the problem of finding the cardinality of T is not equivalent to that of maximising the determinant of an $n \times n$ $(1, -1)$-matrix.

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1. Introduction

In [5], the following theorem is proved.

THEOREM 1. *Let S be the set of all 4-circuits of $K_{n,n}$ where n is divisible by 4. Let $S_1, S_2, \ldots, S_k$ be the collection of all subsets S_i of S, of cardinality 3, such that the union of the three circuits of S_i is $K_{2,3}$. Let T be a smallest subset of S such that $T \cap S_i \neq \varnothing$ for each i. Then $|T| \geqslant \frac{1}{8}n^2(n-1)(n-2)$, and equality holds if and only if there exists an Hadamard matrix of order n.*

Thus the Hadamard conjecture is equivalent to a problem about the 4-circuits of $K_{n,n}$, where $n \equiv 0 \pmod 4$. It is also well-known to be equivalent to the

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problem of maximising the determinant of an $n \times n$ $(1, -1)$-matrix, where $n \equiv 0 \pmod 4$. In this paper we investigate the corresponding problem about the 4-circuits of $K_{n,n}$ where $n \equiv 2 \pmod 4$. In this case, it transpires that the problem is not equivalent to the maximisation of the determinant of an $n \times n$ $(1, -1)$-matrix, where $n \equiv 2 \pmod 4$, although the two problems are closely related. The maximisation of such determinants has been studied by Ehlich [3] (see also [2]). For each $n \equiv 2 \pmod 4$, let α_n denote the maximum value of the determinant of an $n \times n$ $(1, -1)$-matrix. Then Ehlich's paper shows that $\alpha_n \leqslant 2(n-1)(n-2)^{n/2-1}$ (see also [7]). Moreover for each n let I_n and J_n denote the $n \times n$ identity matrix and the $n \times n$ matrix (1) respectively. Suppose there exists an $n \times n$ $(1, -1)$-matrix A, where $n \equiv 2 \pmod 4$, such that $AA^T = \operatorname{diag}[B, B]$ where $B = (n-2)I_{n/2} + 2J_{n/2}$. Then $\alpha_n = |A| = 2(n-1)(n-2)^{n/2-1}$. A search for such matrices A has been conducted by Ehlich [3] and Yang [7]. We use A as the motivation for the following definition. Let $n \equiv 2 \pmod 4$, and write $n = 2s$. Then an $n \times n$ $(1, -1)$-matrix A is a *generalised Ehlich matrix* if $AA^T = B$, where $B = (b_{ij})$ and, for each i and j, b_{ij} is determined as follows:

$$b_{ij} = \begin{cases} n & \text{if } i = j, \\ \pm 2 & \text{if } i \leqslant s \text{ and } j \leqslant s, \text{ or if } i > s \text{ and } j > s, \\ 0 & \text{otherwise.} \end{cases}$$

We then prove the following theorem.

THEOREM 2. *Let S be the set of all 4-circuits of $K_{n,n}$ where $n \equiv 2 \pmod 4$. Let $S_1, S_2, \ldots, S_k$ be the collection of all subsets S_i of S, of cardinality 3, such that the union of the three circuits of S_i is $K_{2,3}$. Let T be a smallest subset of S such that $T \cap S_i \neq \varnothing$ for each i. Then $|T| \geqslant \frac{1}{8}n(n-2)(n^2-n+2)$, and equality holds if and only if there exists a generalised Ehlich matrix of order n.*

2. Proof of Theorem 2

We begin with a lemma.

LEMMA. *Let S be a set with $|S| = n$ for some $n \equiv 2 \pmod 4$. Suppose there exist subsets $T_1, T_2, \ldots, T_{n-1}$ of S such that*
(i) *$n/2 - 1 \leqslant |T_i| \leqslant n/2 + 1$ for each i,*
(ii) *$|T_i| = n/2$ for exactly $n/2$ values of i, and*
(iii) *$n/2 - 1 \leqslant |T_i - T_j| + |T_j - T_i| \leqslant n/2 + 1$ whenever $i \neq j$.*
Then there exists a generalised Ehlich matrix of order n.

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Proof. Let $S = \{s_1, \dots, s_n\}$. Define $E = (e_{ij})$, where $e_{1j} = 1$ for all $j \in \{1, \dots, n\}$ and, for all $i \in \{2, 3, \dots, n\}$,

$$e_{ij} = \begin{cases} 1 & \text{if } s_j \in T_{i-1}, \\ -1 & \text{otherwise.} \end{cases}$$

For each i, let $\overline{T}_i = S - T_i$. Let $J_1 = \{1, 2, \dots, n/2 - 1\}$ and $J_2 = \{n/2, n/2 + 1, \dots, n - 1\}$. By condition (ii) we may assume without loss of generality that $|T_i| = n/2$ if and only if $i \in J_2$.

Condition (i) shows that the inner product of any row $j > 1$ with row 1 is -2, 0 or 2, and the assumption above shows that this inner product is 0 if and only if $j > n/2$.

Now choose i and j so that $i \geqslant 1$, $j \geqslant 1$ and $i \neq j$. Let $a = |T_i \cap T_j|$, $b = |\overline{T}_i \cap T_j|$, $c = |\overline{T}_i \cap \overline{T}_j|$ and $d = |T_i \cap \overline{T}_j|$. Observe that the inner product of rows $i + 1$ and $j + 1$ is $a + c - b - d$. There are various possibilities.

Case 1. Suppose $|T_i| = |T_j|$. Thus $a + d = a + b$ so that $b = d$. Hence $|T_i - T_j| + |T_j - T_i| = d + b = 2d$. Since n is not divisible by 4, condition (iii) shows that $2d \in \{n/2 - 1, n/2 + 1\}$. If $2d = n/2 - 1$, then $a + c = n/2 + 1$, since $a + b + c + d = n$. If $2d = n/2 + 1$, then $a + c = n/2 - 1$. Hence $a + c - b - d = \pm 2$.

We may now suppose without loss of generality that $|T_i| < |T_j|$.

Case 2. Suppose $|T_i| = n/2 - 1$ and $|T_j| = n/2 + 1$. Then $a + b = a + d + 2$ so that $b = d + 2$; hence $|T_i - T_j| + |T_j - T_i| = 2d + 2$. It follows that $2d + 2 \in \{n/2 - 1, n/2 + 1\}$ and we deduce as before that $a + b - c - d = \pm 2$.

Case 3. We may now assume that $|T_j| = |T_i| + 1$. Now $b = d + 1$. Since $n \equiv 2 \pmod 4$ we deduce that $2d + 1 = n/2$; hence $a + b - c - d = 0$.

In summary, if $i \geqslant 1$ and $j \geqslant 1$ then rows $i + 1$ and $j + 1$ are orthogonal if and only if $|\{i, j\} \cap J_1| = 1$. In all other cases where i and j are distinct and greater than 1, the inner product of rows i and j is ± 2. Hence E is a generalised Ehlich matrix.

The proof of Theorem 2 requires the application of the following special case of a well-known theorem of Turán [6].

Theorem 3. *The maximum number of edges in a graph with n vertices and no triangles is $[\frac{1}{4}n^2]$. Moreover, the only such graphs with $[\frac{1}{4}n^2]$ edges are $K_{n/2,n/2}$ (if n is even) and $K_{(n+1)/2,(n-1)/2}$ (if n is odd).*

Proof. In outline the proof of Theorem 2 is similar to the proof of Theorem 1 in [5], but we present the whole argument here for the sake of completeness and clarity. Let A be an $n \times n$ $(1, -1)$-matrix (a_{ij}). Let $K_{n,n}$ be the complete

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bipartite graph with vertex set $\{v_1, v_2, \ldots, v_n, w_1, w_2, \ldots, w_n\}$, where v_i and w_j are adjacent for each i and j. Furthermore, for each i and j let the edge joining v_i to w_j be directed from v_i to w_j if $a_{ij} = 1$ and from w_j to v_i otherwise.

Note that a pair of rows and a pair of columns of A corresponds in an obvious way to an undirected 4-circuit in $K_{n,n}$. We say that this 4-circuit is clockwise even if the number of edges directed in the clockwise sense is even, and clockwise odd otherwise. Let C be a 4-circuit of $K_{n,n}$ with vertex set $\{v_h, v_i, w_j, w_k\}$. If $a_{hj} = a_{ij}$, then exactly one of the two edges of C incident on w_j is directed in the clockwise sense. If $a_{hj} \neq a_{ij}$, then those edges are directed in the same sense on C. Analogous results hold for a_{hk} and a_{ik}. It follows that C is clockwise odd if and only if exactly one of the equations $a_{hj} = a_{ij}$ and $a_{hk} = a_{ik}$ holds.

Let X_{hi} be the set of columns j of A for which $a_{hj} = a_{ij}$ and let Y_{hi} be the set of all the remaining columns of A. It follows from the above paragraph that the number of clockwise odd 4-circuits containing v_h and v_i is $|X_{hi}||Y_{hi}|$. This product is a maximum if $|X_{hi}| = |Y_{hi}|$, and this condition holds if and only if rows h and i of A are orthogonal. If rows h and i are not orthogonal, then the product $|X_{hi}||Y_{hi}|$ is maximised if and only if $||X_{hi}| - |Y_{hi}|| = 2$, and this condition holds if and only if the inner product of rows h and i is ± 2. Thus the number of clockwise odd 4-circuits of $K_{n,n}$ is maximised if as many pairs of rows as possible are orthogonal and the remaining pairs have ± 2 as their inner product. Observe that since n is not divisible by 4, no three rows can be mutually orthogonal, and therefore the maximum number of pairs of orthogonal rows is no greater than the maximum number of edges in a simple graph with n vertices and no triangles. By Theorem 3, this number is $\frac{1}{4}n^2$. Let us assume then that this is the number of pairs of orthogonal rows. (Clearly this is the case for a generalised Ehlich matrix.) If rows h and i are orthogonal, then $|X_{hi}| = |Y_{hi}| = n/2$, so that such pairs of rows contribute $\frac{1}{4}n^2$ clockwise odd 4-circuits each, yielding a total of $\frac{1}{16}n^4$ clockwise odd circuits. For rows h and i which are not orthogonal, we have $\{|X_{hi}|, |Y_{hi}|\} = \{n/2 - 1, n/2 + 1\}$, so that such pairs of rows contribute $\frac{1}{4}n^2 - 1$ clockwise odd 4-circuits each, for a total of $2 \cdot \frac{1}{2} \cdot (n/2)(n/2 - 1) \cdot (\frac{1}{4}n^2 - 1) = \frac{1}{16}n^4 - \frac{1}{8}n^3 - \frac{1}{4}n^2 + n/2$ clockwise odd circuits. Therefore the maximum number of clockwise odd circuits is $\frac{1}{8}n^4 - \frac{1}{8}n^3 - \frac{1}{4}n^2 + n/2$. Since there are $\binom{n}{2}^2$ 4-circuits in all, the minimum number of clockwise even circuits is

$$\binom{n}{2}^2 - \left(\frac{n^4}{8} - \frac{n^3}{8} - \frac{n^2}{4} + \frac{n}{2}\right) = \frac{n(n-2)(n^2-n+2)}{8}.$$

Let T_0 be the set of all clockwise even 4-circuits of $K_{n,n}$. If $K_{2,3}$ is oriented so that the vertices of degree 3 are sources or sinks, then all three circuits are clockwise even. Since every edge of $K_{2,3}$ belongs to exactly two circuits of $K_{2,3}$, it follows that for any orientation of $K_{2,3}$ there are an odd number of clockwise even circuits. Hence $T_0 \cap S_i \neq \varnothing$ for all i. Thus we have proved that if there exists a generalised Ehlich matrix of order n, then $|T| \leqslant \frac{1}{8}n(n-2)(n^2-n+2)$. We prove next that in fact $|T| \geqslant \frac{1}{8}n(n-2)(n^2-n+2)$. The existence of an $n \times n$ generalised Ehlich matrix will then imply that $|T| = \frac{1}{8}n(n-2)(n^2-n+2)$. We will then prove the converse.

Suppose therefore that $T \cap S_i \neq \varnothing$ for all i. Consider first those copies of $K_{2,3}$ in $K_{n,n}$ which contain exactly three vertices of $\{v_1, \ldots, v_n\}$. Let C_1 and C_2 be the components of the complement of $K_{n,n}$, where $V(C_1) = \{v_1, \ldots, v_n\}$. The complement (in K_5) of a copy of $K_{2,3}$ containing three vertices of $\{v_1, \ldots, v_n\}$ is $P_1 \cup P_2$, where P_1 is a triangle of C_1 and P_2 an edge of C_2. The complement (in K_4) of a circuit in $K_{2,3}$ is then the union of P_2 with an edge of P_1. If we fix P_2 and let P_1 run through all triangles in C_1, then in order to contain at least one circuit in each of the corresponding copies of $K_{2,3}$, T must contain at least as many circuits as the cardinality of the smallest set of edges whose deletion from K_n yields a graph with no triangles. Moreover each such circuit contains both end-vertices of P_2. By Theorem 3, the largest subgraph of K_n having no triangles is $K_{n/2,n/2}$. Since K_n has $\binom{n}{2}$ edges and $K_{n/2,n/2}$ has $\frac{1}{4}n^2$ edges, T must contain at least $\binom{n}{2} - \frac{1}{4}n^2$ circuits which include the end-vertices of P_2.

Let us suppose that there exists a triangle Q_2 such that, for each choice of P_2 in Q_2, T contains only $\binom{n}{2} - \frac{1}{4}n^4$ circuits that include the end-vertices of P_2. Consider the copies of $K_{2,3}$ in $K_{n,n}$ which contain the three vertices of Q_2 and two vertices of $\{v_1, \ldots, v_n\}$. The complement (in K_5) of such a copy of $K_{2,3}$ is $Q_1 \cup Q_2$ where Q_1 is an edge of C_1. The complement (in K_4) of any circuit in such a copy Z of $K_{2,3}$ is the union of Q_1 with an edge e of Q_2. We have already seen that in order to include at least one circuit of each copy of $K_{2,3}$ that includes the end-vertices of e and three vertices of $\{v_1, \ldots, v_n\}$, T must contain all the 4-circuits whose complements in K_4 are pairs of edges where one edge of the pair is e and the other is chosen from the complement, $2K_{n/2}$, in C_1 of a fixed copy of $K_{n/2,n/2}$. In order to ensure that T contains a circuit of Z, the copies of $K_{n/2,n/2}$ in C_1 corresponding to the edges of Q_2 must be chosen in such a way that the edge Q_1 appears in the complement of at least one of them. Since Q_1 is any edge of C_1, we find that C_1 must be the union of three copies of $2K_{n/2}$, each copy being the complement in C_1 of a copy of $K_{n/2,n/2}$ chosen to correspond to an edge of Q_2. For any edge e of Q_2, let us denote by $V_1(e)$ and $V_2(e)$ the vertex sets of the copies of $K_{n/2}$ in the subgraph $2K_{n/2}$ of C_1 corresponding to e. Thus $|V_1(e)| = |V_2(e)| = n/2$ for each e.

Let e_1, e_2, e_3 be the edges of Q_2. Since C_1 is the union of the corresponding copies of $2K_{n/2}$, each pair of vertices of C_1 must be contained in at least one of the sets $V_1(e_j)$ where $i \in \{1,2\}$ and $j \in \{1,2,3\}$. It follows that

$$\{V_1(e_3), V_2(e_3)\} = \{[V_1(e_1) \cap V_1(e_2)] \cup [V_2(e_1) \cap V_2(e_2)], \\ [V_1(e_1) \cap V_2(e_2)] \cup [V_2(e_1) \cap V_1(e_2)]\}.$$

Note that

$$|V_1(e_1) \cap V_1(e_2)| = |V_2(e_1) \cap V_2(e_2)|,$$

since $|V_1(e_1)| = |V_2(e_2)|$, $|V_1(e_1)| = |V_1(e_1) \cap V_1(e_2)| + |V_1(e_1) \cap V_2(e_2)|$ and $|V_2(e_2)| = |V_1(e_1) \cap V_2(e_2)| + |V_2(e_1) \cap V_2(e_2)|$. Since $|V_1(e_1) \cap V_1(e_2)| + |V_2(e_1) \cap V_2(e_2)| = |V_1(e_3)| = |V_2(e_3)| = n/2$, it follows that $|V_1(e_1) \cap V_1(e_2)| = n/4$ and so n is divisible by 4.

This contradiction shows that for at least one edge e in each triangle Q_2 of C_2, T contains at least $\binom{n}{2} - \frac{1}{4}n^2 + 1$ circuits that include the end vertices of e. Let R be the set of edges of C_2 with this property. Then by Theorem 3, $|R| \geqslant 2 \cdot \frac{1}{2} \cdot (n/2)(n/2 - 1) = \frac{1}{4}n(n-2)$ since that is the size of the smallest set of edges in K_n which meets every triangle. The remaining edges of C_2 are $\frac{1}{4}n^2$ in number. Therefore

$$|T| \geqslant \frac{n(n-2)}{4}\left[\binom{n}{2} - \frac{n^2}{4} + 1\right] + \frac{n^2}{4}\left[\binom{n}{2} - \frac{n^2}{4}\right] = \frac{n(n-2)(n^2-n+2)}{8}.$$

Let us now assume that $|T| = \frac{1}{8}n(n-2)(n^2-n+2)$ and prove the existence of an $n \times n$ generalised Ehlich matrix. Let e be an edge of C_2. If T contains just $\binom{n}{2} - \frac{1}{4}n^2$ circuits that include the end-vertices of e, then T contains all the 4-circuits whose complements in K_4 are pairs of edges where one edge of the pair is e and the other is chosen from the complement, $2K_{n/2}$, in C_1 of a fixed copy of $K_{n/2,n/2}$. Suppose T has $\binom{n}{2} - \frac{1}{4}n^2 + 1$ circuits that include the end-vertices of e. Then $e \in R$ and T contains all the 4-circuits whose complements in K_4 are pairs of edges where one edge of the pair is e and the other is chosen from the complement in C_1 of a fixed copy of some subgraph X of C_1 that has exactly $\frac{1}{4}n^2 - 1$ edges but no triangles. By a theorem of Erdös [4] (see also p. 109 of [1]), X is degree-majorised by some complete bipartite graph H. Because X has n vertices and $\frac{1}{4}n^2 - 1$ edges, the only candidates for H are $K_{n/2,n/2}$ and $K_{n/2-1,n/2+1}$. Suppose H is isomorphic to $K_{n/2,n/2}$. Because R is the smallest set of edges which meets every triangle of C_2, there must be a triangle Q of C_2 in which e is the only edge that belongs to R. Let $E(Q) = \{e, e_1, e_2\}$.

For each $i \in \{1,2\}$, T contains all the 4-circuits whose complements in K_4 are pairs of edges where one edge of the pair is e_i and the other is chosen from the complement, Y_i, in C_1 of a fixed copy of $K_{n/2,n/2}$. It also contains all the 4-circuits whose complements in K_4 are pairs of edges where one edge of the pair is e and the other is chosen from the complement, Y, in C_1 of a fixed copy of $K_{n/2,n/2} - x$ where x is an edge. In order to ensure that T contains a 4-circuit of each copy of $K_{2,3}$, we must choose Y, Y_1, Y_2 so that their union is C_1 itself. For each $i \in \{1,2\}$, let us denote by $V_1(e_i)$ and $V_2(e_i)$ the vertex sets of the copies of $K_{n/2}$ in the subgraph $2K_{n/2}$ of C_1 corresponding to e_i. Thus $|V_1(e_i)| = |V_2(e_i)| = n/2$.

Next, let $A = V_1(e_1) \cap V_2(e_2)$, $B = V_1(e_1) \cap V_1(e_2)$, $C = V_2(e_1) \cap V_1(e_2)$, $D = V_2(e_1) \cap V_2(e_2)$, $a = |A|$, $b = |B|$, $c = |C|$, $d = |D|$. Note that $a + b = c + d = b + c = a + d = n/2$, so that $a = c$ and $b = d$. We shall show that the graph Y must contain all the edges which join two vertices of $A \cup C$ or two vertices of $B \cup D$. Suppose not. Without loss of generality, let u, v be distinct vertices of $A \cup C$ such that the edge y joining them is not in Y. If $u \in A$ and $v \in C$ or vice versa, then we have the contradiction that $y \notin E(Y) \cup E(Y_1) \cup E(Y_2)$. Suppose therefore without loss of generality that u, $v \in A$. Let J, K be complementary subsets of C_1 such that every edge in the complement of Y joins a vertex of J to a vertex of K. Without loss of generality, let $u \in J$ and $v \in K$. Since $a = c$, there must exist distinct vertices u', $v' \in C$. As Y contains only one edge joining a vertex in J to a vertex in K, there must be an edge of C_1 joining a vertex in $\{u, v\}$ to a vertex in $\{u', v'\}$ which is not in $E(Y)$ and hence not in $E(Y) \cup E(Y_1) \cup E(Y_2)$. This contradiction establishes the aforementioned property of Y.

Since the graph Y must contain all the edges which join two vertices of $A \cup C$ or two vertices of $B \cup D$, we have

$$\begin{aligned}\frac{n^2}{4} - \frac{n}{2} + 1 = |E(Y)| &\geqslant \binom{a+c}{2} + \binom{b+d}{2} \\ &= \binom{2a}{2} + \binom{2\left(\frac{n}{2} - a\right)}{2} \\ &= 4a^2 - 2na + \frac{n^2}{2} - \frac{n}{2}.\end{aligned}$$

This function is minimised when $n = 4a$, but n is not divisible by 4. Therefore let $a = n/4 + z$, so that $c = n/4 + z$ and $b = d = n/4 - z$. Then

$$\begin{aligned}\frac{n^2}{4} - \frac{n}{2} + 1 &\geqslant \binom{\frac{n}{2} + 2z}{2} + \binom{\frac{n}{2} - 2z}{2} \\ &= \frac{n^2}{4} - \frac{n}{2} + 4z^2\end{aligned}$$

and we see that $|z| \leqslant \frac{1}{2}$. Since a must be an integer, we have $|z| = \frac{1}{2}$. Hence Y must be isomorphic to $K_{n/2+1} \cup K_{n/2-1}$. This result shows that H, and therefore X, is isomorphic to $K_{n/2-1,n/2+1}$.

In summary, for every edge e in C_2, T contains all the 4-circuits whose complements in K_4 are pairs of edges where one edge of the pair is e and the other is chosen from the complement W in C_1 of a fixed copy of $K_{n/2,n/2}$ or $K_{n/2-1,n/2+1}$. Let $V_1(e)$ and $V_2(e)$ be the vertex sets of the two components of W.

Finally we consider a subgraph $K_{1,n-1}$ of C_2. Any pair of the $n-1$ edges $f_1, \dots, f_{n-1}$ in this subgraph form two sides of a triangle in C_2. Note that the set U of all edges e for which $|V_1(e)| = n/2$ is a largest set of edges of C_2 which does not include a triangle. By Theorem 3, U is therefore of the form $E(K_{n/2,n/2})$. Hence U includes exactly $n/2$ edges of $\{f_1, \dots, f_{n-1}\}$, and so condition (ii) of Lemma 1 is satisfied if we choose $T_i = V_1(f_i)$ for each $i \in \{1, 2, \dots, n-1\}$. For the edges $f_i \notin U$ we have $\{|V_1(f_i)|, |V_2(f_i)|\} = \{n/2 - 1, n/2 + 1\}$, so that condition (i) is satisfied. To establish condition (iii), choose distinct numbers $i, j \in \{1, 2, \dots, n-1\}$. Since f_i and f_j form two sides of a triangle in C_2, we may define e to be the third edge of that triangle. Certainly for each $i \in \{1, 2\}$, we have $n/2 - 1 \leqslant |V_i(e)| \leqslant n/2 + 1$. Moreover, since f_i, f_j and e are the three sides of a triangle in C_2, the union of the complete graphs induced by the vertex sets $V_1(f_i)$, $V_2(f_i)$, $V_1(f_j)$, $V_2(f_j)$, $V_1(e)$, $V_2(e)$ must be C_1. This observation shows that

$$\{V_1(e), V_2(e)\} = \{(V_1(f_i) \cap V_2(f_j)) \cup (V_2(f_i) \cap V_1(f_j)),\ (V_1(f_i) \cap V_1(f_j)) \cup (V_2(f_i) \cap V_2(f_j))\}.$$

Since $(V_1(f_i) \cap V_2(f_j)) \cup (V_2(f_i) \cap V_1(f_j)) = (T_i - T_j) \cup (T_j - T_i)$, condition (iii) follows. Hence there exists a generalised Ehlich matrix of order n, and the proof of Theorem 2 is complete.

It is interesting to note that although the problem of minimising $|T|$ is equivalent to the problem of maximising the determinant of an $n \times n$ $(1, -1)$-matrix if $n \equiv 0 \pmod 4$, the two problems are not equivalent if $n \equiv 2 \pmod 4$. This point is easily checked by noting that

$$\begin{pmatrix} 1 & 1 & - & 1 & 1 & 1 \\ - & 1 & 1 & 1 & 1 & 1 \\ 1 & - & 1 & 1 & 1 & 1 \\ - & - & - & 1 & - & 1 \\ - & - & - & 1 & 1 & - \\ - & - & - & - & 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & - & - & 1 & 1 \\ - & 1 & 1 & 1 & - & 1 \\ 1 & - & 1 & 1 & 1 & - \\ 1 & - & - & 1 & - & 1 \\ - & 1 & - & 1 & 1 & - \\ - & - & 1 & - & 1 & 1 \end{pmatrix},$$

for example, are 6×6 generalised Ehlich matrices with distinct determinants.

Methods similar to those employed in the proof of Theorem 2 can be used to investigate the case where n is odd. We simply quote the result.

THEOREM 4. *Let S be the set of all 4-circuits of $K_{n,n}$ where n is odd. Let $S_1, S_2, \dots, S_k$ be the collection of all subsets S_i of S, of cardinality 3, such that the union of the three circuits of S_i is $K_{2,3}$. Let T be a smallest subset of S such that $T \cap S_i \neq \varnothing$ for each i. Then $|T| \geqslant \frac{1}{8}n(n-1)^3$, and equality holds if and only if there exists an $n \times n$ $(1, -1)$-matrix A in which the dot product of any pair of distinct rows is ± 1.*

It is known, however (see [3]), that there are odd integers n for which no such matrix A exists.

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