GROUPS WITH FINITE DIMENSIONAL IRREDUCIBLE MULTIPLIER REPRESENTATIONS

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1. Introduction. Let $G$ be a locally compact group and $\omega$ a normalized multiplier on $G$. Denote by $V(G)$ (respectively by $V(G, \omega)$) the von Neumann algebra generated by the regular representation (respectively $\omega$-regular representation) of $G$. Kaniuth [6] and Taylor [14] have characterized those $G$ for which the maximal type $I$ finite central projection in $V(G)$ is non-zero (respectively the identity operator in $V(G)$).

In this paper we determine necessary and sufficient conditions on $G$ and $\omega$ such that the maximal type $I$ finite central projection in $V(G, \omega)$ is non-zero (respectively the identity operator in $V(G, \omega)$) and construct this projection explicitly as a convolution operator on $L^2(G)$. As a consequence we prove the following statements are equivalent,

(i) $V(G, \omega)$ is type $I$ finite,

(ii) all irreducible multiplier representations of $G$ are finite dimensional,

(iii) $G^\omega$ (the central extension of $G$) is a Moore group, that is all its irreducible (ordinary) representations are finite dimensional.

A few interesting corollaries result.

Note that in the case where the multiplier $\omega$ is trivial, these results reduce to results about ordinary representations that are well known.

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2. Notation. For the basic definitions and classification of von Neumann algebras see [12]. Given a locally compact group $G$ with normalized multiplier $\omega$ ([1]) and subgroup $H$, we adopt the following symbols consistently throughout the paper.

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\( H^- \) : closure of \( H \) in \( G \)
\( H' \) : commutator subgroup of \( H \)
\( G^\omega \) : central extension with Weil topology (see [8], p. 218)
\( h \) : canonical projection \( G^\omega \to G^\omega / T = G \)
\( \rho \) : left \( \omega \)-regular representation of \( G \)

\[ ((\rho(g)f)(x) = \omega(g^{-1}, x)f(g^{-1}x), g, x \in G, f \in L^2(G)) \]

\( V(G, \omega) \) : von Neumann algebra generated by \( \rho \)
\( V(G) \) : von Neumann algebra generated by the left regular (ordinary) representation of \( G \)
\( e_G \) : maximal type \( I \) finite central projection in \( V(G) \)
\( d_G \) : maximal type \( I \) finite central projection in \( V(G, \omega) \)
\( G_0 \) : von Neumann kernel = \( \cap \{\ker \pi : \pi \) is a finite dimensional representation of \( G \)\)
\( \Delta_G \) : topological finite class group of \( G = \) union of conjugacy classes whose closure is compact.

In the situation where it will be clear from the context which \( G \) is in question, the subscript \( G \) will be dropped from the symbols \( e_G, d_G \) and \( \Delta_G \).

All isomorphisms between von Neumann algebras mentioned in this paper are spatial.

3. Preliminaries. Let \( G \) be a locally compact group. Kaniuth [6] and Taylor [14] have proved the following theorem about \( e_G \).

**Theorem 3.1.** ([6, 14]). For a locally compact group \( G \),

(i) \( e_G \neq 0 \) implies \( G_0 \) is compact and \( e_G V(G) \) is spatially isomorphic to \( V(G/G_0) \)

(ii) \( e_G \neq 0 \) if and only if \( [G:\Delta_G] < \infty \) and \( (\Delta_G)^{-} \) is compact.

Furthermore, the results of Kaniuth [6] and Robertson [11] combine to yield the following result.

**Theorem 3.2.** ([6, 11]). The following statements are equivalent:

(i) \( e_G = I \) (the identity operator)

(ii) \( [G:\Delta_G] < \infty, (\Delta_G)^{-} \) is compact and \( G_0 = \{e\} \)

(iii) \( G \) is a Moore group, that is all its irreducible (ordinary) representations are finite dimensional.

For the remainder of this section we shall fix a locally compact group \( G \) with normalized multiplier \( \omega \).

**Lemma 3.3.** Let \( A \) be a subset of \( G^\omega \), then

(i) \( A^- \) is compact if and only if \( h(A)^- \) is compact

(ii) if \( A^- \) is compact, then \( h(A)^- = h(A^-) \).
The proof of this result is straightforward.

**Corollary 3.4.** (i) \((\Delta_G)'' = \Delta_G''\).

(ii) If one of \((\Delta_G)'\) and \((\Delta_G)'\) is compact, then so is the other and \((\Delta_G)'\) = \(h(\Delta_G)'\).

**Proof.** For (i), let \(A\) equal the conjugacy class of some \((\lambda, x) \in G'\omega\) and in (ii), let \(A = (\Delta_G)'\). Now use Lemma 3.3.

Combining 3.1 and 3.4 gives \(e_G \neq 0\) if and only if \(e_G\neq 0\). But this follows already from [14, Proposition 5.2].

Let \(E_n, \ n \in Z\) be the maps on \(L^2(G''\omega)\) given by

\[
E_n f(\mu, x) = \mu^n \int_T \alpha^{-n} f(\alpha, x) \, d\alpha,
\]

for almost all \((\mu, x) \in G''\omega, f \in L^2(G''\omega)\). (Compare this with [7, page 563].) Clearly \(E_n L^2(G''\omega)\) consists of all those functions \(f \in L^2(G''\omega)\) such that

\[
f(\mu, x) = \mu^n f(l, x)
\]

for almost all \((\mu, x) \in G''\omega\).

It follows that the \(E_n, n \in Z\) are mutually orthogonal idempotents. For \(n \in Z, E_n\) is just convolution by the measure obtained, if you multiply the measure on \(G''\omega\) supported on \(T\) which restricts to Haar measure on \(T\), with the character \(\chi_n\) of \(T\) given by

\[
\chi_n(\lambda) = \lambda^n, \ n \in Z, \lambda \in T.
\]

It is easy to check that \(E_n\) commutes with the right and left regular representation of \(G''\omega\), so by [13, Theorem 3] \(E_n\) is in the centre of \(V(G''\omega)\).

**Theorem 3.5.** With the above notation, the \(E_n, n \in Z\) are mutually orthogonal central projections in \(V(G''\omega)\) and

(i) The three von Neumann algebras \(E_n V(G''\omega), V(G, \omega'\omega), V(\chi_{-n} \uparrow T_{\omega}^\omega)\) (the non Neumann algebra generated by the induced representation \(\chi_{-n} \uparrow T_{\omega}^\omega\)) are spatially isomorphic.

(ii) \(\sum_n E_n = I\) (the identity operator).

**Proof.** (i) Let \(\tau\) denote the left regular representation of \(G''\omega\) and \(\rho_n, n \in Z\), the left regular \(\omega'\omega\)-representation of \(G\). Fix \(n\). Observe that the representation space of \(\chi_{-n} \uparrow T_{\omega}^\omega\) is just \(E_n L^2(G''\omega)\) and that

\[
E_n \tau = \chi_{-n} \uparrow T_{\omega}^\omega.
\]

It follows that

\[
E_n V(G''\omega) = V(\chi_{-n} \uparrow T_{\omega}^\omega).
\]

The map

\[
\phi: E_n L^2(G''\omega) \rightarrow L^2(G)
\]
defined by \( \phi(f)(x) = f(1, x) \) is an isometry (see \([3]\)).

The spatial monomorphism
\[
E_n V(G^\omega) \to B(L^2(G)): T \to \phi \cdot T \cdot \phi^{-1}
\]
is weakly continuous and maps \( E_n \tau(1, x)(x \in G) \) to \( \rho_n(x) \). Observe that the von Neumann algebra generated by \( \{E_n \tau(1, x): x \in G\} \) is precisely \( E_n V(G^\omega) \), thus the range of \( \phi \) is contained in \( V(G, \omega^n) \). Furthermore this range is weakly closed (\([12, 1.16.2]\)) and contains the operators \( \rho_n(x), x \in G \) and thus must be equal to \( V(G, \omega^n) \).

(ii) Since \( \bigoplus_{n \in \mathbb{Z}} \chi_n \) is the regular representation of \( T \), we infer that
\[
\bigoplus_{n \in \mathbb{Z}} (\chi_n \uparrow^T) = \left( \bigoplus_{n \in \mathbb{Z}} \chi_n \right) \uparrow^T G^\omega
\]
is the regular representation of \( G^\omega \), hence
\[
V(G^\omega) = \bigoplus_{n \in \mathbb{Z}} V(\chi_n \uparrow^T)
\]
hence the result.

**Lemma 3.6.** Suppose \( e_G \neq 0 \) (or equivalently \( e_G^\omega \neq 0 \)) and that \( G \) has a finite dimensional \( \omega \)-representation \( \pi \), then \( K = h((G^\omega)_0) \) is compact and \( \omega \) is similar to a multiplier which is lifted from \( G/K \).

**Proof.** That \( K \) is compact follows from 3.1. Suppose \((\lambda, k) \in (G^\omega)_0\), then since \((\mu, g) \to \mu \pi(g), (\mu, g) \in G^\omega \) is a finite dimensional (ordinary) representation of \( G^\omega \), we have \( I = \lambda \pi(k) \), that is \( k \in p\text{-ker } \pi \). The result now follows from \([1, Lemma 1.3]\).

4. The main theorems.

**Theorem 4.1.** Let \( G \) be a locally compact group with normalized Borel multiplier \( \omega \). Adopt the notation of Section 2, then the following three conditions are equivalent.

(i) \( e_G \neq 0 \) (or equivalently \( e_G^\omega \neq 0 \)) and there exists a finite dimensional \( \omega \)-representation \( \pi \) of \( G \).

(ii) \( d_G \neq 0 \).

(iii) \( [\Delta_G: \Delta_G^\pi] < \infty \), \( (\Delta_G)^\pi \) is compact and \( G \) has a finite dimensional \( \omega \)-representation \( \pi \).

Note if \( \omega \) is a trivial multiplier then \( \omega \) is of the form
\[
\omega(x, y) = \gamma(x)\gamma(y)/\gamma(xy)
\]
and thus there automatically exists a finite dimensional \( \omega \)-representation of \( G \). In this case the theorem reduces to 3.1 (ii). See also the comments preceding Example 5.1.
Proof: (i) and (iii) are obviously equivalent using 3.1 (ii). Suppose (ii) is true and let \( \pi' \) be a non-degenerate finite dimensional algebra representation of \( dV(G, \omega) \), then
\[
g \rightarrow \pi'(dp(g^{-1}))^*
\]
is a finite dimensional \( \omega \)-representation of \( G \). By Theorem 3.5, \( e_G \omega \neq 0 \). This shows (ii) implies (i); that (i) implies (ii) will be proved together with Theorem 4.3.

**Lemma 4.2.** Suppose \( K \) is a compact normal subgroup of \( G \) and \( \omega \) is lifted from a multiplier \( \omega' \) on \( G/K \), then \( K \) is also a compact normal subgroup of \( G^\omega \) and \( G^\omega/K \) is topologically isomorphic to \( (G/K)^\omega \).

The proof of this result follows easily from the definition of group extensions.

**Theorem 4.3.** Suppose \( G \) is a locally compact group and \( d_G \) is non-zero. We assume (using Lemma 3.6) that \( \omega \) is lifted from a multiplier \( \omega' \) of \( G/K \), where \( K \) is the compact normal subgroup \( K = h(G^\omega_0) \). Then \( d_G \) is the operator \( L^2(G) \rightarrow L^2(G) \) defined by
\[
d_G f(x) = \int_K f(k^{-1}x) d\lambda(k), \quad \text{almost all } x \in G, f \in L^2(G),
\]
where \( \lambda \) is Haar measure on \( G \) normalized such that \( \lambda(K) = 1 \). Furthermore, for each \( n \in \mathbb{Z} \), \( d_G \) is the maximal type I finite central projection in \( V(G, \omega^n) \) and
\[
d_G V(G, \omega^n) \simeq V(G/K, (\omega')^n).
\]

**Proof:** First we give a proof, as promised, of the statement ‘(i) implies (ii)’ of Theorem 4.1. Let \( \alpha:L^2(G) \rightarrow L^2(G) \) be defined by
\[
\alpha f(x) = \int_K f(k^{-1}x) d\lambda(k), \quad \text{almost all } x \in G, f \in L^2(G).
\]
The proof that \( \alpha \) is a central idempotent in \( V(G, \omega^n) = E_n V(G^\omega) \) (and hence in \( V(G^\omega_0) \)) and that \( \alpha V(G, \omega^n) \) and \( V(G/K, (\omega')^n) \) are spatially isomorphic is similar to the proof of the corresponding facts about \( E_n \) in Theorem 3.5. Since
\[
V(G, \omega^n) \simeq V(G/K, (\omega')^n) \oplus V(G/K, (\omega')^n)^\perp
\]
(where \( \perp \) denotes orthogonal complement), we have by Theorem 3.5,
\[
V(G^\omega) \simeq V( (G/K)^\omega ) \oplus V( (G/K)^\omega )^\perp
\]
but \( (G/K)^\omega \) and \( G^\omega/(G^\omega)_0 \) are topologically isomorphic and \( V(G^\omega/(G^\omega)_0) \) is isomorphic to the maximal type I finite direct summand in \( V(G^\omega) \) (3.1 (i) ), it follows that \( V(G/K, (\omega')^n) \) is isomorphic to the maximal type I direct summand of \( V(G, \omega^n) \). In particular we have \( d_G = \alpha \neq 0 \).

Now assume \( d_G \neq 0 \), then by ‘(ii) implies (i)’ of Theorem 4.1, \( e_G \omega \neq 0 \), thus by the same argument as above, we reach the desired conclusion.
Corollary 4.4. Suppose $d_G \neq 0$ and let $n \in \mathbb{Z}$, then the following equations obtain

$$G_0 = \mathcal{H}(G_0)^{\omega} = \mathcal{H}(G_0)^{\omega} \cap \{ \ker \pi: \pi \text{ is a finite dimensional representation of } G^{\omega} \text{ such that } \pi|_{\Gamma}(t) = t^n \}$$

$$= \{ g \in G: \text{there exists } \gamma(g) \in \Gamma \text{ such that } \pi(g) = \gamma(g) I \text{ for all finite dimensional } \omega^g \text{-representations of } G \}.$$  

Proof. Let $K = \mathcal{H}(G_0)^{\omega}$ and denote the last two sets in the above equality by $H$ and $L$ respectively. That $K \subseteq H \subseteq L$ is clear from the definitions and the property that an $\omega^g$-representation $\pi$ of $G$ extends to an ordinary representation $\pi'$ of $G^{\omega}$ such that

$$\pi'|_{\Gamma}(t) = t^n.$$  

Using the proof of Lemma 3.6, we assume that $\omega$ is lifted from a multiplier on $G/L$. The non-degenerate finite dimensional representations of $d_G V(G, \omega^n)$ separate the points of $d_G V(G, \omega^n)$, hence $\pi^0(g) = I$ if and only if

$$\pi(d_G \rho(g^{-1} ))^* = \pi(d_G )^*$$

for all such representations $\pi$, where $\pi^0$ denotes the $\omega^n$-representation

$$g \rightarrow \pi(d_G \rho(g^{-1} ))^*$$

and $\rho$ is the left regular $\omega^n$-representation of $G$. This happens if and only if $\rho(g)d_G = d_G$ and using the previous Theorem, this occurs if and only if $g \in K$. This shows

$$K = \cap \{ \ker \pi^0: \pi \text{ is a finite dimensional non-degenerate representation of } d_G V(G, \omega^n) \} \supseteq L.$$  

If we let $n = 0$, we obtain the remainder of the corollary: $G_0 = L$.

Theorem 4.5. Let $G$ be a locally compact group and $\omega$ a normalised Borel multiplier on $G$. The following conditions are equivalent.

(a) $V(G, \omega)$ is type I finite.

(b) All irreducible $\omega$-representations of $G$ are finite dimensional.

(c) The following conditions hold:

(i) $|G: \Delta_G| < \infty$

(ii) $(\Delta_G)^{-}$ is compact

(iii) $G$ admits a finite dimensional $\omega$-representation

(iv) $G_0 = \{e\}$

(d) $G^{\omega}$ is a Moore group, that is all irreducible ordinary representations of $G^{\omega}$ are finite dimensional.

Proof. To obtain (a) is equivalent to (c) combine 4.1, 4.3 and 4.4. To prove (b) implies (a), apply [2, 4.2.1 and 5.5.2] and the proof of Moore [10, P. 125].
Lemma 4.1] to the twisted group $C^*$-algebra $C^*(G, \omega)$. Assume (c). If $\pi$ is a finite dimensional $\omega$-representation of $G$, then the $n$-fold tensor product $\pi \otimes \ldots \otimes \pi$ is a finite dimensional $\omega^n$-representation of $G$, hence $V(G, \omega^n)$ is type I finite and by Theorem 3.5 (ii), $V(G^\omega)$ is type I finite. It follows from Theorem 3.2 that $G^\omega$ is a Moore group.

5. Examples. The condition in 4.1 (iii) that $G$ possess a finite dimensional $\omega$-representation cannot be replaced by the weaker property that $\omega^n$ is trivial for some integer $n$ (even in the case where $G$ is discrete) as the following example shows.

Example. 5.1. Let $G = H \times H'$ with the discrete topology, where

$$H = \prod_{h=1}^{\infty} \mathbb{Z}_2 \quad \text{and} \quad H' = \bigoplus_{h=1}^{\infty} \mathbb{Z}_2$$

and define

$$\omega((a_j, b_j), (a'_j, b'_j)) = \exp \left[ \frac{\pi i}{2} \sum_{j=1}^{\infty} (a_j b'_j - a'_j b_j) \right],$$

$(a_j, b_j)(a'_j, b'_j) \in G$. Since $G$ is abelian, Theorem 1.1 and Theorem 4.3 of [4] show that $V(G, \omega)$ is either type I or type II$_1$. But we know from [4, Example 4.4] that it is not type I, thus $V(G, \omega)$ is type II$_1$. This occurs despite the fact that $(G, \omega)$ satisfies $[G: \Delta_G] < \infty$, $(\Delta_G')^\perp$ is compact and $\omega^2 = 1$. We observe in passing that $G$ has no finite dimensional $\omega$-representation (use Theorem 4.1).

Example 5.2. For each $\lambda = e^{2\pi i \alpha} \in \mathbb{T}$, $(\alpha \in [0, 2\pi[)$, we obtain a multiplier $\omega_\lambda$ on $\mathbb{Z} \times \mathbb{Z}$ defined by

$$\omega_\lambda((m, n), (m', n')) = \lambda^{mn'},$$

$(m, n), (m', n') \in \mathbb{Z} \times \mathbb{Z}$. Mackey [9, Theorem 8.6] shows that (up to similarity) all multipliers on $\mathbb{Z} \times \mathbb{Z}$ are of this form. We say that $\lambda$ is rational (respectively irrational) if $\alpha$ is rational (respectively irrational). It follows from [4] that $V(\mathbb{Z} \times \mathbb{Z}, \omega_\lambda)$ is type II, if $\lambda$ is irrational and if $\lambda$ is rational, say $\lambda = \exp 2\pi ip/q$, where $p$ and $q$ are relatively prime integers, then $V(\mathbb{Z} \times \mathbb{Z}, \omega_\lambda)$ is type I (finite) and according to Hannabuss [5, Theorem 4.1], the irreducible $\omega_\lambda$ representations of $\mathbb{Z} \times \mathbb{Z}$ are all dimension $q$.

Let $\Delta_p$ ($p$-prime) be the group of $p$-adic integers and $G$ be the group $\mathbb{Z} \times \mathbb{Z} \times \Delta_p$ with multiplication

$$(a, b, x)(a', b', x') = (a + a', b + b', x + x' + \theta(ab')), \quad (a, b, x)(a', b', x') \in G,$$
where $\theta: \mathbb{Z} \to \Delta_p$ is the canonical injection of $\mathbb{Z}$ into a dense subgroup of $\Delta_p$. We topologize $G$ so that $\Delta_p$ becomes a compact open subgroup; with this topology, $G$ becomes a locally compact separable topological group. For each $\lambda \in \mathbb{T}$, we define a multiplier $\sigma_\lambda$ on $G$ as follows,

$$\sigma_\lambda = \omega_\lambda \circ k,$$

where $k: G \to \mathbb{Z} \times \mathbb{Z}$ is the canonical homomorphism $k(m, n, x) = (m, n)$.

Given an irreducible $\omega_\lambda$-representation $\pi$ of $\mathbb{Z} \times \mathbb{Z}$, denote by $\pi'$ the $\sigma_\lambda$ representation of $G$ obtained by composing $\pi$ with $k$.

Since $\Delta_p$ is a compact normal subgroup of $G$, we can apply the Mackey analysis [9] to construct all $\sigma_\lambda$-representations of $G$.

Identify the abelian group dual $\Delta_{\lambda}$ of $\Delta_p$ with the subgroup

$$\Lambda = \{ \chi \in \mathbb{T} : \chi = \exp[2\pi ik/p^n], k, n \in \mathbb{Z} \}$$

of $\mathbb{T}$ using the correspondence

$$\Lambda \times \Delta_{\lambda} \to \mathbb{T} : (\lambda, x) \to \Lambda^x,$$

where $x \to \chi^x, \chi \in \Lambda$ is the continuous extension from $\mathbb{Z}$ to $\Delta_p$ of the homomorphism $\mathbb{Z} \to \Lambda : n \to \chi^n$.

Let $\lambda \in \mathbb{T}, \chi \in \Lambda$. Then the irreducible $\sigma_\lambda$-representations of $G$ which reduce to a multiple of $\chi$ on $\Delta_p$ are of the form $\chi'\pi'$, where $\chi'$ is the extension

$$\chi' : G \to \mathbb{T} : (a, b, x) \to \chi^x$$

of the character $\chi$ of $\Delta_p$ and $\pi$ is a $\omega_{\chi^{-1}}$-representation of $\mathbb{Z} \times \mathbb{Z}$. As $\chi$ ranges through $\Lambda$, we obtain all irreducible $\sigma_\lambda$-representations of $G$.

Note that $\lambda\chi^{-1}$ is rational if and only if $\lambda$ is rational. Hence the irreducible $\sigma_\lambda$-representations of $G, \lambda \in \mathbb{T}$ are all infinite dimensional if $\lambda$ is irrational and are all finite dimensional (but of arbitrarily high dimension) if $\lambda$ is rational. It follows from Theorem 4.1 and 4.5 that $V(G, \sigma_\lambda)$ is type I finite if and only if $\lambda$ is rational and the type I finite part of $V(G, \sigma_\lambda)$ is zero if $\lambda$ is irrational.

**Remark 5.3.** As pointed out in [4], combining Theorem 1.1 of [4] with the proofs of Taylor [14, Theorem 2] and Moore [10, Theorem 1] yields the equivalence of the following statements:

(i) $V(G, \omega)$ is type $I_{\leq k}$, that is non-zero maximal type $I_n$ central projections occur in $V(G, \omega)$ only if $n \leq k$.

(ii) $G$ has an open abelian subgroup $A$ of finite index in $G$ such that $\omega_{|A \times A}$ is trivial.

(iii) The irreducible $\omega$-representations of $G$ are of dimension at most $k$. 

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Thus in Example 5.2 we see that for $\lambda$ rational, $V(G, \sigma_\lambda)$ admits a non-zero maximal type $I_n$ part of arbitrary large $n$. This phenomenon does not occur in the case where $G$ is discrete.

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