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Singularities of hypergeometric functions in several variables

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Abstract

This paper deals with singularities of nonconfluent hypergeometric functions in several complex variables. Typically such a function is a multi-valued analytic function with singularities along an algebraic hypersurface. We describe such hypersurfaces in terms of the amoebas and the Newton polytopes of their defining polynomials. In particular, we show that the amoebas of classical discriminantal hypersurfaces are solid, that is, they possess the minimal number of complement components.

1. Introduction

There exist several approaches to the notion of hypergeometric series, functions and systems of differential equations. In the present paper we use the definition of these objects that was introduced by Horn at the end of the nineteenth century [Hor89]. His original definition of a hypergeometric series is particularly attractive because of its simplicity. A Laurent series in several variables is said to be *hypergeometric* if the quotient of any two adjacent coefficients is a rational function in the summation indices.

In the present paper we study singularities of hypergeometric functions which are defined by means of analytic continuation of hypergeometric series. A hypergeometric series $y(x)$ satisfies the so-called Horn hypergeometric system

$$x_i P_i(\theta) y(x) = Q_i(\theta) y(x), \quad i = 1, \ldots, n. \tag{1}$$

Here P_i and Q_i are nonzero polynomials depending on the vector differential operator $\theta = (\theta_1, \ldots, \theta_n)$, $\theta_i = x_i \partial/\partial x_i$. The nonconfluency of a hypergeometric series or the system (1) means that the polynomials P_i and Q_i are of the same degree:

$$\deg P_i = \deg Q_i, \quad i = 1, \ldots, n.$$

These conditions can be expressed in terms of the Ore–Sato coefficient of a hypergeometric series satisfying the system (1) (see Equations (4) and (5)). Historically the Gauss hypergeometric differential equation was the first one to be studied in detail due to the remarkable fact that any linear homogeneous differential equation of order two with three regular singularities can be reduced to it. The singularities of the Gauss equation are $0, 1$ and ∞. The generalized ordinary hypergeometric differential equation which is a special case of the nonconfluent system (1) corresponding to $n = 1$ also has three singular points, namely $0, t$ and ∞, where t is the quotient of the coefficients in the leading terms in the polynomials P_1 and Q_1. Thus the singular set of an ordinary hypergeometric differential equation is minimal in the following precise sense: There exist *only two* circular domains, namely $\{0 < |x| < |t|\}$ and $\{|t| < |x| < \infty\}$, in which any solution to the equation can be represented

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as a Laurent series with the center at the origin (in the nonresonant case) or as a linear combination of the products of Laurent series and powers of $\log x$ (in the resonant case).

It turns out that algebraic singularities of the system of partial differential equations (1) enjoy a multi-dimensional analog of this minimality property. This property is most conveniently formulated in the language of so-called amoebas, a terminology introduced by Gelfand, Kapranov and Zelevinsky in [GKZ94]. The *amoeba* of an algebraic set $\mathcal{R} = \{R(x) = 0\}$ is defined to be its image under the mapping $\mathrm{Log} : (x_1, \dots, x_n) \mapsto (\log|x_1|, \dots, \log|x_n|)$. The complement of an amoeba consists of a finite number of convex connected components which correspond to domains of convergence of the Laurent series expansions of rational functions with denominator $\mathcal{R}$. The number of such components cannot be smaller than the number of vertices of the Newton polytope of the polynomial $R(x)$. If these two numbers are equal, then we say that the amoeba is *solid*. In § 5 we prove the following theorem.

Theorem (Theorem 7). *The singular hypersurface of any nonconfluent hypergeometric function has a solid amoeba.*

A hypergeometric function satisfying the Gelfand–Kapranov–Zelevinsky system of equations has singularities along the zero locus of the corresponding principal $\mathcal{A}$-determinant (see [GKZ94]). Using Theorem 7 we arrive at the following corollary.

Corollary (Corollary 8). *The zero set of any principal $\mathcal{A}$-determinant has a solid amoeba.*

This corollary implies in particular that the amoeba of the classical discriminant of a general algebraic equation is solid (Corollary 9).

Let us also mention the following results in this paper. Theorem 12 states that any meromorphic nonconfluent hypergeometric function is rational. In the last section we study the problem of describing the class of rational hypergeometric functions. In the class of hypergeometric functions satisfying the Gelfand–Kapranov–Zelevinsky system of equations, this problem was first considered in [CDD99] and [Cat01]. Theorem 13 gives a necessary condition for the Horn system to possess a rational solution. The statement of Proposition 15 emphasizes the fact that only very few rational functions are hypergeometric. The class of rational hypergeometric functions that is described in this proposition consists of those which are contiguous to Bergman kernels of complex ellipsoidal domains.

The proofs of the main results in the paper use the notions of the support and the fan of a hypergeometric series, some facts from toric geometry and the two-sided Abel lemma, which is proved in § 6. Recall that the usual (one-sided) Abel lemma (see [GKZ89] or [McD95]) gives the following relation between the domain of convergence of a Puiseux series and its support (i.e., the set of summation).

Lemma 1 (Abel's lemma for Puiseux series). *Let $y(x)$ be a Puiseux series with a nonempty domain of convergence D. For any $x^{(0)} \in D$ and any cone C containing the convex hull of the support of $y(x)$, we have $\mathrm{Log}(x^{(0)}) - C^\vee \subset \mathrm{Log}(D)$. Here $C^\vee$ is the dual cone to C.*

The two-sided Abel lemma for hypergeometric Puiseux series states that the domain $\mathrm{Log}\,(D)$ is itself contained in a suitable translation of the cone $-C^\vee$.

2. Some basic notation and definitions

To study the singularities of solutions to the Horn system (1), we consider the characteristic variety of this system. Let $\mathcal{D}$ denote the Weyl algebra of differential operators with polynomial coefficients

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in n variables [Bjö79]. For any differential operator $P \in \mathcal{D}$,

$$P = \sum_{|\alpha| \leqslant m} c_\alpha(x) \left(\frac{\partial}{\partial x}\right)^\alpha,$$

its principal symbol $\sigma(P)(x,z) \in \mathbb{C}[x_1,\dots,x_n,z_1,\dots,z_n]$ is defined by

$$\sigma(P)(x,z) = \sum_{|\alpha|=m} c_\alpha(x) z^\alpha.$$

We denote by G_i the differential operator $x_i P_i(\theta) - Q_i(\theta)$ in the ith equation of the Horn system (1). Let $\mathcal{M} = \mathcal{D}/\sum_{i=1}^n \mathcal{D}G_i$ be the left $\mathcal{D}$-module associated with the system (1) and let $J \subset \mathcal{D}$ denote the left ideal generated by the differential operators $G_1,\dots,G_n$. By definition (see [Bjö79, ch. 5, § 2]) the characteristic variety $\operatorname{char}(\mathcal{M})$ of the Horn system is given by

$$\operatorname{char}(\mathcal{M}) = \{(x,z) \in \mathbb{C}^{2n} : \sigma(P)(x,z) = 0, \text{ for all } P \in J\}.$$

We define the set $U_{\mathcal{M}} \subset \mathbb{C}^n$ by

$$U_{\mathcal{M}} = \{x \in \mathbb{C}^n : \exists z \neq 0 \text{ such that } (x,z) \in \operatorname{char}(\mathcal{M})\}.$$

It follows from Proposition 8.1.3 and Theorem 8.3.1 in [Hör90] and Theorem 7.1 in [Bjö79, ch. 5] that a solution to (1) can only be singular on $U_{\mathcal{M}}$. Since any equation of the form $\sigma(P)(x,z) = 0$ is homogeneous in z, it follows that $U_{\mathcal{M}}$ is the image of $\operatorname{char}(\mathcal{M})$ under the projection of the direct product $\mathbb{C}^n \times \mathbb{P}^{n-1} \to \mathbb{C}^n$ onto its first factor. Using the main theorem of elimination theory (see in [Mum76, § 2C]) one can conclude that this image is an algebraic set, possibly the whole of $\mathbb{C}^n$. In the latter case the singularities of a solution to the Horn system are not necessarily algebraic. For instance, if every differential operator G_i contains the factor $(\theta_1 + \dots + \theta_n)$, then any sufficiently smooth function depending on the quotients $x_1/x_n,\dots,x_{n-1}/x_n$ is a solution to the system (1).

In the present paper we consider systems of the Horn type which satisfy the condition $U_{\mathcal{M}} \neq \mathbb{C}^n$. In this case $U_{\mathcal{M}}$ is a proper algebraic subset of $\mathbb{C}^n$. Its irreducible components of codimension greater than one are removable as long as we are concerned with holomorphic solutions to the Horn system. Thus the singular set of a solution to (1) is algebraic and it is contained in the union of irreducible components of codimension one. We denote this union by $\mathcal{R}$ and call it *the singular set of the Horn system*. Let $R(x)$ be the defining function of the set $\mathcal{R}$. The polynomial $R(x)$ will be referred to as *the resultant of the Horn system (1)*. To find a polynomial whose zero set is $\mathcal{R}$ is a difficult task which requires the full use of elimination theory. There exists, however, a simple special case when the set $\mathcal{R}$ can be embedded into the zero set of some polynomial which one can algorithmically compute. Let $H_i(x,z)$ be the principal symbol of the differential operator G_i in the ith equation of the Horn system (1). Since the polynomials $H_1,\dots,H_n$ are homogeneous in $z_1,\dots,z_n$, they determine the classical resultant $R[H_1,\dots,H_n]$, which is a polynomial in $x_1,\dots,x_n$ (see [GKZ94, ch. 13]), and, unless it is identically zero, this resultant vanishes precisely at those points x for which the homogeneous system $H_1(x,z) = \dots = H_n(x,z) = 0$ has a solution $z \in \mathbb{C}^n \setminus \{0\}$. This implies that we have the following result.

Proposition 2. *The singular set $\mathcal{R}$ of the Horn system (1) lies in the zero set of the classical resultant $R[H_1,\dots,H_n]$ of the principal symbols of the operators in (1).*

3. Puiseux series solutions to the Horn system and their supports

The Horn system (1) as well as the Gelfand–Kapranov–Zelevinsky system (see [GKZ89]) has the remarkable property that under some natural assumptions there exists a basis in the space of its holomorphic solutions consisting of (Puiseux) series with the center at the origin (see [GKZ89] for

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the Gelfand–Kapranov–Zelevinsky system and [Sad02] for the Horn system). In this section we introduce some terminology and present preliminary results which will be used later for describing the singular set of the Horn system.

Suppose that a formal Puiseux series centered at the origin satisfies the Horn system (1). Such a series can be written as a linear combination of formal shifted Laurent series, i.e., series of the form

$$y(x) = x^{\gamma} \sum_{s \in \mathbb{Z}^n} \varphi(s) x^s. \tag{2}$$

Here $x^s = x_1^{s_1} \cdots x_n^{s_n}$, and the shift is determined by the initial exponent $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{C}^n, \operatorname{Re} \gamma_i \in [0,1)$. Suppose that the series (2) is a solution to (1). Computing the action of the operator $x_i P_i(\theta) - Q_i(\theta)$ on this series, we arrive at the system of difference equations

$$\varphi(s + e_i) Q_i(s + \gamma + e_i) = \varphi(s) P_i(s + \gamma), \quad i = 1, \dots, n, \tag{3}$$

where $\{e_i\}_{i=1}^n$ is the standard basis of $\mathbb{Z}^n$. The system (3) is equivalent to (1) as long as we are concerned with those solutions to the Horn system which admit a series expansion of the form (2).

The system of difference equations (3) is in general not solvable without further restrictions on P_i and Q_i. Let $R_i(s)$ denote the rational function $P_i(s)/Q_i(s + e_i)$, $i = 1, \dots, n$. Increasing the argument s in the ith equation of (3) by e_j and multiplying the obtained equality by the jth equation of (3), we arrive at the relation

$$\varphi(s + e_i + e_j)/\varphi(s) = R_i(s + e_j) R_j(s).$$

Similarly

$$\varphi(s + e_i + e_j)/\varphi(s) = R_j(s + e_i) R_i(s).$$

Thus the conditions

$$R_i(s + e_j) R_j(s) = R_j(s + e_i) R_i(s), \quad i, j = 1, \dots, n$$

are in general necessary for (3) to be solvable. Throughout this paper we assume that the polynomials P_i and Q_i defining the Horn system (1) satisfy these relations and that they are representable as products of linear factors.

The latter assumption together with the Ore–Sato theorem (see [Sat90] and [GGR92], § 1.2) yields that the general solution to the system of difference equations (3) is of the form

$$\varphi(s) = t_1^{s_1} \cdots t_n^{s_n} u(s) \prod_{i=1}^{p} \Gamma(\langle A_i, s + \gamma \rangle - c_i) \phi(s). \tag{4}$$

Here $t_i, c_i \in \mathbb{C}$, $A_i = (A_{i1}, \dots, A_{in}) \in \mathbb{Z}^n$, $p \in \mathbb{N}_0$, $u(s)$ is a rational function whose numerator and denominator are representable as products of linear factors, and $\phi(s)$ is an arbitrary periodic function with period 1 in each variable. The fact that all the Γ-functions in (4) are in the numerator is inessential: using the identity $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ and choosing the periodic function $\phi(s)$ in an appropriate way (see [Sad02]), one can move them into the denominator. A formal series (2) with coefficients given by (4) is called a *formal solution* to the system (1). We will call any expression of the form (4) the *Ore–Sato coefficient* of a hypergeometric series (or of the system (1)).

Remark 1. Conversely, the Ore–Sato coefficient (4) defines the system (1) in the sense that for any $i = 1, \dots, n$ the quotient $\varphi(s + e_i)/\varphi(s)$ equals $P_i(s)/Q_i(s + e_i)$. For instance, the Ore–Sato coefficient (12) in Example 1 (see below) defines the Horn system (10).

The specific form of (4) corresponds to our assumption that the polynomials P_i and Q_i can be represented as products of linear factors. In general an Ore–Sato coefficient can include a rational function that is not factorizable up to linear factors (see [GGR92, § 1.2]). We may without loss of

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generality assume that no linear factor in the rational function $u(s)$ can be normalized so that all of its coefficients become integers. Indeed, any linear factor $a_1 s_1 + \cdots + a_n s_n + \lambda$ with $a_i \in \mathbb{Z}$ can be written in the form $\Gamma(a_1 s_1 + \cdots + a_n s_n + \lambda + 1)/\Gamma(a_1 s_1 + \cdots + a_n s_n + \lambda)$ and hence included into the product of the Γ-functions in (4). Proposition 3 (see below) yields that the other linear factors of $u(s)$ (such as $s_1 + \pi s_2$) are inessential as long as one is concerned with series solutions to (1). Throughout this paper we will assume that $u(s) \equiv 1$.

One can easily check that in terms of the parameters of the Ore–Sato coefficient $\varphi(s)$ the nonconfluency condition $\deg P_i = \deg Q_i$ can be written in the form

$$\sum_{i=1}^{p} A_i = 0. \tag{5}$$

Recall that in this paper we only deal with nonconfluent hypergeometric series.

Any shifted Laurent series solution to (1) (formal as well as convergent) can be written in the form

$$y(x) = x^{\gamma} \sum_{s \in S} \varphi(s) x^{s}, \tag{6}$$

where $\varphi(s)$ is given by (4) and S is a subset of $\mathbb{Z}^n$ on which $\varphi(s) \neq 0$. The set $S + \gamma$ will be called the *support* of the series (6). The support $S + \gamma$ is called *irreducible* if there exists no series solution to (1) supported in a proper nonempty subset of $S + \gamma$. A set $S \subset \mathbb{Z}^n$ is said to be $\mathbb{Z}^n$-*connected* if any two points of S can be connected by a polygonal line with unit sides and vertices in S.

Proposition 3 (see below) describes all possible supports of (formal) series solutions to (1) and Proposition 5 allows one to find those of them which have nonempty domains of convergence. While looking for a solution to (3) that is different from zero on some subset S of $\mathbb{Z}^n$ we will assume that the polynomials $P_i(s)$ and $Q_i(s)$, the set S and the vector γ satisfy the condition

$$|P_i(s + \gamma)| + |Q_i(s + \gamma + e_i)| \neq 0, \tag{7}$$

for any $s \in S$ and for all $i = 1, \ldots, n$. This assumption eliminates the case when a solution to (3) can independently take arbitrary values at two adjacent points in the set S. The following statement (see [Sad02]) gives necessary and sufficient conditions for a solution to the system (3) supported in some set $S \subset \mathbb{Z}^n$ to exist.

Proposition 3 (Sadykov [Sad02]). *For $S \subset \mathbb{Z}^n$ define*

$$S_i' = \{s \in S : s + e_i \notin S\}, \quad S_i'' = \{s \notin S : s + e_i \in S\}, \quad i = 1, \ldots, n.$$

Suppose that the conditions (7) are satisfied on S. Then there exists a solution to the system (3) supported in S if and only if the following conditions are fulfilled:

$$P_i(s + \gamma)|_{S_i'} = 0, \quad Q_i(s + \gamma + e_i)|_{S_i''} = 0, \quad i = 1, \ldots, n, \tag{8}$$

$$P_i(s + \gamma)|_{S \setminus S_i'} \neq 0, \quad Q_i(s + \gamma + e_i)|_{S} \neq 0, \quad i = 1, \ldots, n. \tag{9}$$

By definition, the union of the sets S_i' and S_i'', $i = 1, \ldots, n$, is a discrete analog of the boundary of the set S. Since the polynomials P_i and Q_i are assumed to be representable as products of linear factors, it follows from (8) that S_i' and S_i'' lie on hyperplanes. The conditions (9) yield that these hyperplanes bound the set S. Thus we can formulate the following result.

Proposition 4. *The convex hull of the support of a series solution to the Horn system is a polyhedral set.*

Example 1. Let us consider the following system of partial differential equations of the Horn type.

$$\begin{aligned} x_1(\theta_1 + \theta_2)(\theta_1 - 2)y(x) &= (\theta_1 - 1)(\theta_1 - 4)y(x), \\ x_2(\theta_1 + \theta_2)(\theta_2 - 3)y(x) &= (\theta_2 - 1)(\theta_2 - 5)y(x). \end{aligned} \tag{10}$$

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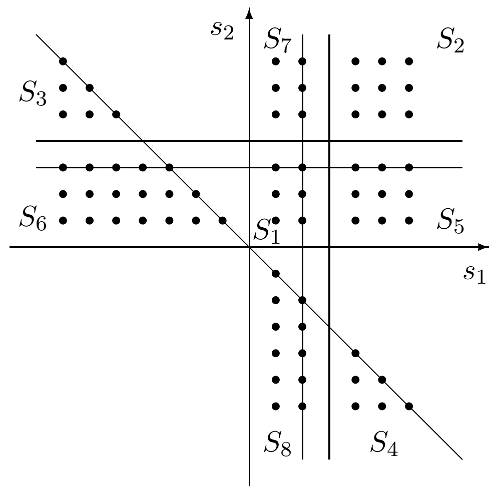

Figure 1. The irreducible supports of the solutions to the Horn system (10).

Assuming that $y(x)$ admits a Laurent series expansion (2) with $\gamma = 0$, we arrive at the following system of difference equations:

$$\begin{aligned} \varphi(s+e_1)s_1(s_1-3) &= \varphi(s)(s_1+s_2)(s_1-2), \\ \varphi(s+e_2)s_2(s_2-4) &= \varphi(s)(s_1+s_2)(s_2-3). \end{aligned} \tag{11}$$

In accordance with the Ore–Sato theorem (see [Sat90] and [GGR92, § 1.2]) the general solution to the system (11) is given by the function

$$\varphi(s) = (s_1-3)(s_2-4)\frac{\Gamma(s_1+s_2)}{\Gamma(s_1)\Gamma(s_2)}\phi(s), \tag{12}$$

where $\phi(s)$ is an arbitrary periodic function with period 1 in s_1 and s_2. There exist eight $\mathbb{Z}^2$-connected subsets of the lattice $\mathbb{Z}^2$ which satisfy the conditions of Proposition 3, namely

$$\begin{aligned} S_1 &= \{(s_1,s_2)\in\mathbb{Z}^2 : 1\leqslant s_1\leqslant 2,\ 1\leqslant s_2\leqslant 3\}, \\ S_2 &= \{(s_1,s_2)\in\mathbb{Z}^2 : 4\leqslant s_1,\ 5\leqslant s_2\}, \\ S_3 &= \{(s_1,s_2)\in\mathbb{Z}^2 : 5\leqslant s_2,\ s_1+s_2\leqslant 0\}, \\ S_4 &= \{(s_1,s_2)\in\mathbb{Z}^2 : 4\leqslant s_1,\ s_1+s_2\leqslant 0\}, \\ S_5 &= \{(s_1,s_2)\in\mathbb{Z}^2 : 4\leqslant s_1,\ 1\leqslant s_2\leqslant 3\}, \\ S_6 &= \{(s_1,s_2)\in\mathbb{Z}^2 : s_1+s_2\leqslant 0,\ 1\leqslant s_2\leqslant 3\}, \\ S_7 &= \{(s_1,s_2)\in\mathbb{Z}^2 : 1\leqslant s_1\leqslant 2,\ 5\leqslant s_2\}, \\ S_8 &= \{(s_1,s_2)\in\mathbb{Z}^2 : 1\leqslant s_1\leqslant 2,\ s_1+s_2\leqslant 0\}. \end{aligned}$$

These irreducible supports of solutions to (11) are displayed in Figure 1.

Using [Sad02, formula (12)] for defining the periodic function $\phi(s)$, one can compute the sums of the corresponding Laurent series. Let $y_i(x)$ denote the series solution to (10) with the support S_i. These functions are defined up to inessential constant factors which we choose in a specific way in

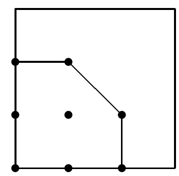

Figure 2. The Newton polytope of the resultant of (10).

order to make the formulas simpler. Computations (which were performed in MAPLE) show that

$$\begin{aligned} y_1(x) &= 3x_1x_2 + 4x_1x_2^2 + 3x_1x_2^3 + 3x_1^2x_2 + 6x_1^2x_2^2 + 6x_1^2x_2^3, \\ y_5(x) &= x_1^4x_2(6x_1^3x_2^2 + 6x_1^3x_2 - 27x_1^2x_2^2 + 3x_1^3 - 26x_1^2x_2 \\ &\qquad + 45x_1x_2^2 - 12x_1^2 + 40x_1x_2 - 30x_2^2 + 15x_1 - 20x_2 - 6)/(1-x_1)^5, \\ y_7(x) &= x_1x_2^5(6x_1x_2^2 - 18x_1x_2 + 3x_2^2 + 15x_1 - 8x_2 + 5)/(1-x_2)^4, \\ y_2(x) &= x_1x_2(6x_1^2 + 14x_1x_2 + 5x_2^2 - 9x_1 - 8x_2 + 3)/(1-x_1-x_2)^4 - y_1(x) - y_5(x) + y_7(x). \end{aligned}$$

The series supported in S_2, S_3 and S_4 represent the same solution to our system since they represent the same rational function in different domains. Finally, $y_6(x) = y_1(x) + y_5(x)$ and $y_8(x) = y_1(x) + y_7(x)$. It follows from [Sad02, Theorem 2.8] that the space of holomorphic solutions to the system (10) has dimension four at any point $x \in \mathbb{C}^2$ such that $(1-x_1)(1-x_2)(1-x_1-x_2) \neq 0$. Hence the rational functions $y_1(x)$, $y_2(x)$, $y_5(x)$ and $y_7(x)$ form a basis in this space. Notice that the resultant of the principal symbols of the operators in the system (10) is given by the polynomial $(x_1x_2)^4(1-x_1)(1-x_2)(1-x_1-x_2)$. The Newton polytope of the resultant of the Horn system (10) is shown in Figure 2.

Recall that a convex cone is called *strongly convex* if it does not contain any lines through the origin. To conclude this section, we formulate one more statement on the properties of supports of hypergeometric series which will be used in the sequel.

Proposition 5. *A nonconfluent hypergeometric series with support S has a nonempty domain of convergence if and only if the convex hull of S is a polyhedral set which is contained in a translation of a strongly convex cone. The domain of convergence of the series (6) is independent on the parameters $c_1, \dots, c_p$ in Equation (4) (we disregard exceptional values of these parameters for which (6) terminates or reduces to a linear combination of hypergeometric series in fewer variables).*

The first conclusion of this proposition follows from Proposition 4, the lemma in [GGR92, § 4.1] and the properties of hypergeometric series in one variable (see [SK85, ch. 1]). The second conclusion of the proposition follows from Theorem 1 in [SK85, § 4.1].

Finally, we remark that there exists a simple relation between the domain of convergence of a nonconfluent hypergeometric series and its support. This relation is described by the two-sided Abel lemma, which will be proved in § 6.

4. The fan of the Horn system

By an *affine convex cone* we mean a set of the form $C + \xi$, where C is a convex cone in $\mathbb{R}^n$ with apex at the origin and $\xi \in \mathbb{R}^n$. Let $C_1 + \xi_1$ and $C_2 + \xi_2$ be affine convex cones with C_1 and C_2 being convex cones and $\xi_1, \xi_2 \in \mathbb{R}^n$. We say that $C_1 + \xi_1$ is *smaller* than $C_2 + \xi_2$ if $C_1 \subset C_2$. If $C_1 = C_2$ then the corresponding affine cones are said to be equal. For a convex set $B \subset \mathbb{R}^n$ its *recession cone* C_B is defined to be $C_B = \{s \in \mathbb{R}^n : u + \lambda s \in B,\ \forall u \in B, \lambda \geqslant 0\}$ (see [Zie95, ch. 1]).

That is, the recession cone of a convex set is the maximal element in the family of those cones whose shifts are contained in this set.

For brevity the recession cone of the convex hull of the support of a Puiseux series solution to the Horn system will be referred to as *the cone of its support.* It has nonempty interior if and only if the corresponding hypergeometric series cannot be represented as a linear combination of hypergeometric series in fewer variables which depend monomially on the original ones. In Example 1 the cone of the irreducible support S_2 is the positive quadrant, the cone of S_5 is $\{(s_1, s_2) : s_1 \geqslant 0, s_2 = 0\}$, and the cone of S_1 is the origin.

Here and later we assume that the rank of the matrix with rows $A_1, \ldots, A_p$ is equal to n, because otherwise the series with the coefficient (4) can be reduced to a hypergeometric series in fewer variables. Let $I = (i_1, \ldots, i_n)$, $i_j \in \{1, \ldots, p\}$, be a multi-index such that the vectors $A_{i_1}, \ldots, A_{i_n}$ are linearly independent. Let γ_I be the solution of the system of linear equations $\langle A_{i_j}, s\rangle - c_{i_j} = 0$, $j = 1, \ldots, n$, and define the set K_I by $K_I = \{s \in \mathbb{Z}^n : \langle A_{i_j}, s + \gamma_I\rangle - c_{i_j} \leqslant 0, j = 1, \ldots, n\}$. Let $\mathbb{Z}^n + \gamma$ denote the shift in $\mathbb{C}^n$ of the lattice $\mathbb{Z}^n$ with respect to the vector γ.

Definition 1. We say that the parameter $c = (c_1, \ldots, c_p) \in \mathbb{C}^p$ is *generic* if for any multi-index I as above none of the hyperplanes $\langle A_j, s + \gamma_I\rangle - c_j = 0$, $j \notin \{i_1, \ldots, i_n\}$, meets the shifted lattice $\mathbb{Z}^n + \gamma_I$.

Proposition 6. *If the vector $c = (c_1, \ldots, c_p)$ is generic then there exists a one-to-one correspondence between the n-dimensional cones of the supports of the convergent series solutions to the Horn system of the form (6) and the multi-indices $I = (i_1, \ldots, i_n)$ such that the vectors $A_{i_1}, \ldots, A_{i_n}$ are linearly independent. The recession cone of the convex hull of the support of any such series is strongly convex and polyhedral.*

Proof. For a multi-index I as above, consider the shifted Laurent series

$$y_I(x) = \sum_{s \in K_I} t^s \prod_{i=1}^{p} \Gamma(\langle A_i, s + \gamma_I\rangle - c_i) x^{s+\gamma_I}. \tag{13}$$

Since the parameter c is assumed to be generic, it follows from Proposition 3 that the coefficient of the series (13) satisfies Equations (3) everywhere on $\mathbb{Z}^n$, i.e., that (13) is at least a formal solution to the Horn system (1). By Proposition 5 the series (13) has a nonempty domain of convergence since its support is contained in a strongly convex (and simplicial) affine cone. Thus with any multi-index I as above one can associate the n-dimensional cone C_I of the support of the series (13).

Since we are interested in n-dimensional cones of the supports of the series solutions to (1), we do not consider polynomial solutions to this system (which may exist even if the parameters are generic). It follows by Proposition 3 that, if the support of a formal series solution to (1) meets at most $n-1$ linearly independent hyperplanes of the form $\langle A_j, s+\gamma\rangle - c_j = 0$ for some $\gamma \in \mathbb{C}^n$, then it cannot be contained in any strongly convex affine cone and by Proposition 5 the series is divergent. By our assumption the parameter c is generic and hence the support of such a series cannot meet more than n hyperplanes of this form. If it meets exactly n hyperplanes with linearly independent normals $A_{i_1}, \ldots, A_{i_n}$ then the cone of the support of this series must coincide with C_I since it is bounded by the same hyperplanes. Thus the correspondence between linearly independent subsets of the set of vectors $\{A_1, \ldots, A_p\}$ and the n-dimensional cones of the supports of shifted Laurent series solutions to (1) is one-to-one. The claim about the recession cone of the convex hull of the support of $y_I(x)$ follows from [Zie95, Proposition 1.12] since the convex hull of K_I is a strongly convex affine polyhedral cone. □

Remark 2. Proposition 6 shows that adding new elements to the family of vectors $\{A_i\}_{i=1}^p$ can only increase the number of series solutions to the Horn system which is defined by the Ore–Sato coefficient (4) as long as the vector c remains generic.

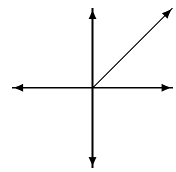

Figure 3. The fan of the Horn system (10).

We now associate with a nonconfluent Horn system a set of strongly convex polyhedral cones which will play an important role in the sequel. Recall that for a cone $C \subset \mathbb{R}^n$ its dual is defined by $C^\vee = \{v \in \mathbb{R}^n : \langle u, v\rangle \geqslant 0, \forall u \in C\}$. For any multi-index $I = (i_1, \ldots, i_n)$ such that the vectors $A_{i_1}, \ldots, A_{i_n}$ are linearly independent we denote by C_I the recession cone of the convex hull of the set K_I whose shift supports the series (13). We partially order the finite family $\{C_I\}$ of strongly convex polyhedral cones with respect to inclusion and denote the maximal elements by $C_{I^{(1)}}, \ldots, C_{I^{(d)}}$. Let us introduce the cones $B_j = -C^\vee_{I^{(j)}}$, $j = 1, \ldots, d$. Since for any I as above the polyhedral cone C_I has a nonempty interior, it follows that B_j is a strongly convex polyhedral cone. The nonconfluency condition (5) implies that $\bigcup_{j=1}^d B_j = \mathbb{R}^n$. If the cones $B_1, \ldots, B_d$ can be identified with the set of the maximal cones of some complete fan then we call it *the fan of the Horn system (1)*. As an example, the fan of the Horn system (10) is shown in Figure 3.

If $n = 2$ then $\{B_j\}_{j=1}^d$ is always the set of the maximal cones of some complete fan. For $n \geqslant 3$ this is not necessarily the case. For instance, let $n = 3$ and let $A_1 = (1, 0, 0)$, $A_2 = (0, 1, 0)$, $A_3 = (0, 0, 2)$, $A_4 = (-1, 0, -1)$, and $A_5 = (0, -1, -1)$. The multi-indices $I^{(1)} = (1, 4, 5)$ and $I^{(2)} = (2, 4, 5)$ define maximal cones but the intersection of their duals has a nonempty interior.

5. Minimality of the singularities of hypergeometric functions and discriminants

Recall (see [GKZ94]) that the *amoeba* $\mathcal{A}_f$ of a Laurent polynomial $f(x)$ (or of the algebraic hypersurface $f(x) = 0$) is defined to be the image of the hypersurface $f^{-1}(0)$ under the map $\mathrm{Log} : (x_1, \ldots, x_n) \mapsto (\log|x_1|, \ldots, \log|x_n|)$. This name is motivated by the typical shape of $\mathcal{A}_f$ with tentacle-like asymptotes going off to infinity (see Figure 5 later in Example 3 in § 7). We quote the following general results on amoebas.

Theorem A (Gelfand, Kapranov and Zelevinsky [GKZ94]). *The connected components of the amoeba complement ${}^c\mathcal{A}_f$ are convex, and they are in bijective correspondence with the different Laurent series expansions centered at the origin of the rational function $1/f$.*

Recall that the Newton polytope $\mathcal{N}_f$ of a Laurent polynomial f is defined to be the convex hull in $\mathbb{R}^n$ of the support of f. The following result shows that the Newton polytope $\mathcal{N}_f$ reflects the structure of the amoeba $\mathcal{A}_f$ (see [FPT00, Theorem 2.8 and Proposition 2.6]).

Theorem B (Forsberg, Passare and Tsikh [FPT00]). *Let f be a Laurent polynomial and let $\{M\}$ denote the family of connected components of the amoeba complement ${}^c\mathcal{A}_f$. There exists an injective function $\nu : \{M\} \to \mathbb{Z}^n \cap \mathcal{N}_f$ such that the cone which is dual to $\mathcal{N}_f$ at the point $\nu(M)$ coincides with the recession cone of M.*

The cited theorems imply that the number of Laurent series expansions of the rational function $1/f$ centered at the origin is at least equal to the number of vertices of the Newton polytope $\mathcal{N}_f$ and at most equal to the number of integer points in $\mathcal{N}_f$. Varying the coefficients of the Laurent polynomial f with fixed Newton polytope $\mathcal{N}_f$, one can attain the upper (see [Mik00]) as well as the lower (see [Rul00]) bounds for the number of connected components of ${}^c\mathcal{A}_f$. Moreover, the vertices of the Newton polytope are always assumed (see [MY82] or [GKZ94]) by the function ν, and by Theorem B the recession cones of those connected components of ${}^c\!\mathcal{A}_f$ which correspond to the vertices of $\mathcal{N}_f$ have nonempty interior. On the other hand, the complement components that correspond to interior points of $\mathcal{N}_f$ are bounded.

Definition 2. The amoeba $\mathcal{A}_f$ of a Laurent polynomial f (or, equivalently, the algebraic hypersurface $f(x)=0$) is called *solid* if the number of connected components of the amoeba complement ${}^c\!\mathcal{A}_f$ equals the number of vertices of the Newton polytope $\mathcal{N}_f$.

The main observation in this section is the following theorem.

Theorem 7. *The singular hypersurface of any nonconfluent hypergeometric function has a solid amoeba.*

Proof. Let $\mathcal{A}$ be the amoeba of the resultant of the Horn system (as defined in § 2) and let $M \subset {}^c\!\mathcal{A}$ be a connected component of its complement. By the remark before Definition 2, it suffices to show that the recession cone C_M of the set M has nonempty interior.

Recall that in this paper we only deal with Horn systems satisfying the assumptions made in § 2. The condition that the projection of the characteristic variety of the Horn system onto the variable space is a proper algebraic subset implies that the Horn system in question is holonomic (see [Bjö79, ch. 3]). Hence it has finitely many analytic solutions in a neighborhood of each nonsingular point.

Our next argument was inspired by the proof of Theorem 2.4.12 in [SST00]. Let $y_1,\dots,y_r$ be a basis in the space of holomorphic solutions to (1) on a simply connected domain in $\mathrm{Log}^{-1} M$. Recall that J denotes the ideal generated by the differential operators in the Horn system. Let $\{1,\partial^{\alpha(1)},\dots,\partial^{\alpha(r-1)}\}$ be a basis of the quotient $\mathbb{C}(x)\langle\partial\rangle/\mathbb{C}(x)\langle\partial\rangle J$, where

$$\partial = (\partial_1,\dots,\partial_n) = \left(\frac{\partial}{\partial x_1},\dots,\frac{\partial}{\partial x_n}\right)$$

and $\mathbb{C}(x)\langle\partial\rangle = \mathbb{C}(x_1,\dots,x_n)\langle\partial_1,\dots,\partial_n\rangle$ is the algebra generated by polynomials in $\partial_1,\dots,\partial_n$ and rational functions in $x_1,\dots,x_n$. Put

$$\Phi(x) = \begin{pmatrix} y_1 & \cdots & y_r \\ \partial^{\alpha(1)}y_1 & \dots & \partial^{\alpha(1)}y_r \\ \vdots & \ddots & \vdots \\ \partial^{\alpha(r-1)}y_1 & \dots & \partial^{\alpha(r-1)}y_r \end{pmatrix}.$$

Since $\{y_i\}$ is a basis, it follows that $\det(\Phi) \not\equiv 0$ and Φ is a (matrix-valued) multi-valued holomorphic function on $\mathrm{Log}^{-1} M$. By Theorem A the set M is convex and therefore the fundamental group $\pi_1(\mathrm{Log}^{-1} M)$ is isomorphic to the direct product of the fundamental groups of at most n punctured disks with center at the origin. Thus $\pi_1(\mathrm{Log}^{-1} M)$ is a free Abelian group generated by the elements η_i which encircle $x_i=0$ (some of these elements might be trivial).

Consider the analytic continuation $\eta_i^*\Phi$ of the matrix Φ along the path η_i. Since the first row of $\eta_i^*\Phi$ is again a basis of solutions, there exists an invertible matrix V_i, which is called the *monodromy matrix*, satisfying $\eta_i^*\Phi = \Phi V_i$. Since $\pi_1(\mathrm{Log}^{-1} M)$ is Abelian, the matrices V_i commute with one another. Hence there exists a commutative family of matrices W_i such that $e^{2\pi\sqrt{-1}W_i} = V_i$. Define the matrix

$$\Psi(x) := \Phi(x) x_1^{-W_1}\cdots x_n^{-W_n}.$$

The monodromy of $\Phi(x)$ is killed by $x_1^{-W_1}\cdots x_m^{-W_m}$ since $\eta_i^* x_i^{-W_i} = V_i^{-1}x_i^{-W_i}$. Hence $\Psi(x)$ is a single-valued function on $\mathrm{Log}^{-1} M$. By Lemma 2 in [Bol00, ch. 4] any solution to the Horn system in the domain $\mathrm{Log}^{-1} M$ can be written as a polynomial in Puiseux monomials and $\log x_i$ with

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single-valued coefficients. Here by a Puiseux monomial we mean a monomial with arbitrary (complex) exponent vector.

Let us write such a solution in the form

$$y(x) = \sum_{\alpha,\beta} h_{\alpha\beta}(x) x^{\alpha} (\log x)^{\beta},$$

where $h_{\alpha\beta}(x)$ are single-valued functions in $\operatorname{Log}^{-1} M$, $(\log x)^{\beta} := (\log x_1)^{\beta_1} \cdots (\log x_n)^{\beta_n}$ and the sum is finite. Let β_1' be the highest power of $\log x_1$ appearing in the expression for $y(x)$. Any single-valued function in a Reinhardt domain can be expanded into a Laurent series. Expanding the functions $h_{\alpha\beta}$ into Laurent series and computing the action of the operators in the Horn system on $y(x)$, we conclude that the coefficients of the expansion for $h_{\alpha\beta}$ satisfy difference relations of the form (3). The first of these relations yields an ordinary hypergeometric differential equation for the restriction of $y(x)$ to a suitable line. It is known that no logarithms may appear in a solution to an ordinary generalized hypergeometric differential equation with generic parameters (see [Evg86]). By induction over the highest power of $\log x_1$ appearing in the expression for $y(x)$ we conclude that $\log x_1$ does not appear at all if the parameters of the Horn system are sufficiently general. By the symmetry of the variables it follows that any solution to a Horn system with generic parameters in the domain $\operatorname{Log}^{-1} M$ can be represented as a Puiseux series.

For $\zeta \in \partial M$ let $Y_{\zeta} \subset \mathbb{R}^n$ denote the half-space which is bounded by a supporting hyperplane of M at the point ζ and contains M. There exists a sequence of points $\{\zeta_i\}_{i=1}^{\infty} \subset \partial M$ such that the recession cone of the set $\bigcap_{i=1}^{\infty} Y_{\zeta_i}$ coincides with C_M. Since $\mathcal{A}$ is the logarithmic image of the set of singularities of the function $y(x)$, for any $i \in \mathbb{N}$ there exists a germ $\mathcal{G}_i$ of $y(x)$ which cannot be continued analytically through at least one point in the fiber $\operatorname{Log}^{-1}\zeta_i$. As we have remarked earlier, the analytic continuation of $\mathcal{G}_i$ into the domain $\operatorname{Log}^{-1} M$ can be expanded into a Puiseux series L_i whose domain of convergence contains $\operatorname{Log}^{-1} M$. Let $L^{(k)} = \sum_{i=1}^{k} c_i L_i$ with constant coefficients such that $L^{(k)} \not\equiv 0$. The series $L^{(k)}$ satisfies the same hypergeometric system of equations as $y(x)$ since it is a linear combination of solutions to this system. We denote the domain of convergence of the series $L^{(k)}$ by Ω_k. By the construction, $M \subset \operatorname{Log} \Omega_k$ and the recession cone $C_{\operatorname{Log}\Omega_k}$ is a subset of the recession cone of the finite intersection $\bigcap_{i=1}^{k} Y_{\zeta_i}$.

Suppose that the cone C_M has the empty interior. The two-sided Abel lemma, which will be proved in § 6, states that for a nonconfluent hypergeometric Puiseux series L with the domain of convergence Ω one has $C_{\operatorname{Log}\Omega} = -C_L^{\vee}$, where C_L is the cone of the support of L and $C_{\operatorname{Log}\Omega}$ is the recession cone of the set $\operatorname{Log}\Omega$. Thus we have

$$-C_{L^{(k)}}^{\vee} = C_{\operatorname{Log}\Omega_k} \subset C_{\bigcap_{i=1}^{k} Y_{\zeta_i}}$$

and hence the set $\bigcup_{k=1}^{\infty} C_{L^{(k)}}$ is not strongly convex. By Proposition 4 the cone $C_{L^{(k)}}$ is polyhedral with its boundary being a subset of the union of the zero sets of the polynomials $P_1, \ldots, P_n$ and $Q_1, \ldots, Q_n$. Since this union is a finite arrangement of hyperplanes it follows that the family of cones $\{C_{L^{(k)}}\}_{k=1}^{\infty}$ can only contain a finite number of distinct elements. Therefore there exists $m \in \mathbb{N}$ such that the cone $C_{L^{(m)}}$ is not strongly convex. This contradicts the statement of Proposition 5 and completes the proof. $\square$

Let us recall the definitions of $\mathcal{A}$-discriminants and principal $\mathcal{A}$-determinants which were introduced by Gelfand, Kapranov and Zelevinsky (see [GKZ94]). Let $\mathcal{A}$ be a finite subset of $\mathbb{Z}^n$ and let f be a generic polynomial with support $\mathcal{A}$, i.e.,

$$f = \sum_{\alpha \in \mathcal{A}} c_{\alpha} x^{\alpha}.$$

The corresponding $\mathcal{A}$-discriminant is defined to be the irreducible polynomial in the coefficients c_{α}

which vanishes whenever f together with all of its partial derivatives have a common zero in $(\mathbb{C} \setminus \{0\})^n$. The associated principal $\mathcal{A}$-determinant is then the product of the $\mathcal{A}$-discriminant with (suitable powers of) all the lower-order $\mathcal{A}'$-discriminants, where $A' = A \cap \Gamma$ for each face Γ of the Newton polytope. In particular, the factors corresponding to vertices Γ are simply monomials.

A hypergeometric function satisfying the Gelfand–Kapranov–Zelevinsky system of equations (see [GKZ89]) has singularities along the zero locus of the corresponding principal $\mathcal{A}$-determinant. There always exists a monomial change of variables which transforms an $\mathcal{A}$-hypergeometric series into a Horn series (see [Kap91, § 2]). This monomial change of variables corresponds to a linear transformation of the amoeba space and hence it cannot affect the solidness of an amoeba. (More precisely, the preimage of any point in the amoeba space under this mapping is an affine subspace and hence the preimage of a solid amoeba is also solid.) Using Theorem 7 we arrive at the following corollary.

Corollary 8. *The zero set of any principal $\mathcal{A}$-determinant has a solid amoeba.*

Remark. In the case where the principal $\mathcal{A}$-determinant depends essentially (up to homogeneities) only on two variables, it follows from Corollary 8 that the amoeba of the corresponding $\mathcal{A}$-discriminant $\Delta_{\mathcal{A}}$ is also solid. Indeed, in this case the subset of $\mathcal{A} \subset \mathbb{Z}^n$ consists of at most $n+3$ points. If $n+2$ points lie in a hyperplane then $\Delta_{\mathcal{A}} \equiv 1$, otherwise each discriminant Δ_{Γ}, corresponding to a face Γ of the Newton polytope of $\mathcal{A}$, depends essentially on a single variable and its amoeba will be just a line, so it cannot influence the solidness of the amoeba of $\Delta_{\mathcal{A}}$.

Theorem 7 allows us also to derive the following property of the classical discriminant of the general algebraic equation $y^m + c_1 y^{m_1} + \cdots + c_n y^{m_n} + c_{n+1} = 0$, where $m, m_i \in \mathbb{N}$, $m > m_1 > \cdots > m_n \geqslant 1$, y is the unknown. We provide the following corollary with a proof since the solution to a general algebraic equation satisfies a system of differential equations which is slightly different from (1).

Corollary 9. *The amoeba of the discriminant of a general algebraic equation is solid.*

Proof. By a monomial change of the variable y and the coefficients $c_1, \dots, c_{n+1}$ any algebraic equation can be reduced to an equation of the form

$$y^m + x_1 y^{m_1} + \cdots + x_n y^{m_n} - 1 = 0, \tag{14}$$

where $x = (x_1, \dots, x_n) \in \mathbb{C}^n$. The classical discriminant of this equation is the same as the one-dimensional $\mathcal{A}$-discriminant with $A = \{m, m_1, \dots, m_n, 0\}$. Since the Newton polytope in this case is just the one-dimensional interval $[0, m]$, the principal $\mathcal{A}$-determinant will differ from the discriminant only by a monomial factor, which does not affect the amoeba. The result is thus a direct consequence of Corollary 8. $\square$

The cubic equation is considered in detail in Example 3 (§ 7). The amoeba of the singular locus of a solution to the reduced system is displayed in Figure 5.

Theorem 7 implies in particular that the number of connected components of the complement of the amoeba of the singular hypersurface of a rational hypergeometric function equals the number of vertices of the Newton polytope of its denominator. It turns out that in some cases knowing the hypergeometric system which is satisfied by a given rational function allows one to compute the number of vertices of the Newton polytope of its denominator. We illustrate this fact by means of the following important family of rational hypergeometric functions which are defined as the Bergman kernels of complex ellipsoidal domains (see [FH96] and [Zin74]). This family will be used in § 7 for describing rational hypergeometric functions satisfying some systems of equations of the Horn type.

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Consider the family of *complex ellipsoidal domains* defined by

$$D^{p_1,\dots,p_n} = \{x \in \mathbb{C}^n : |x_1|^{2/p_1} + \dots + |x_n|^{2/p_n} < 1\},$$

where $p_i = 1, 2, 3, \dots,\ i = 1, \dots, n$. The Bergman kernel $K_{p_1,\dots,p_n}(x)$ for this domain can be represented as the hypergeometric series

$$K_{p_1,\dots,p_n}(x) = \frac{1}{\pi^n} \sum_{s\in\mathbb{N}_0^n} \frac{\Gamma(p_1(s_1+1)+\dots+p_n(s_n+1)+1)}{\prod_{i=1}^n p_i\Gamma(p_i(s_i+1))} x^s, \tag{15}$$

but it is in fact also equal (see [Zin74]) to the rational function

$$K_{p_1,\dots,p_n}(x) = \frac{1}{\pi^n}\frac{1}{p_1\cdots p_n}\frac{\partial^n}{\partial x_1\cdots\partial x_n}\sum_{j_1=1}^{p_1}\cdots\sum_{j_n=1}^{p_n}\frac{1}{1-y_{j_1 1}-\dots-y_{j_n n}}, \tag{16}$$

where $y_{j_i i} = x_i^{1/p_i}\varepsilon_{j_i i}$, $\varepsilon_{j_i i}$ are all the p_ith roots of unity, $j_i = 1, \dots, p_i$, $i = 1, \dots, n$. Notice that, except for the factor π^{-n}, the coefficients of this rational function are integers. Let $f_{p_1,\dots,p_n}$ denote the denominator of the rational function (16) (we normalize the denominator so that the greatest common divisor of its coefficients equals 1). Our aim is to find the number of connected components of the amoeba complement ${}^c\mathcal{A}_{f_{p_1,\dots,p_n}}$. For any fixed vector $\gamma \in \mathbb{C}^n$, $\operatorname{Re}\gamma_i \in [0, 1)$, there exist finitely many subsets of the shifted lattice $\mathbb{Z}^n + \gamma$ which satisfy the conditions in Proposition 3 and are contained in some strongly convex affine cone. We call them *γ-admissible sets associated with (1)*. A set is said to be *admissible* if it is γ-admissible for some γ.

Proposition 10. *The number of connected components of the amoeba complement ${}^c\mathcal{A}_{f_{p_1,\dots,p_n}}$ of the denominator of the Bergman kernel $K_{p_1,\dots,p_n}(x)$ equals $n + 1$.*

Remark 3. The conclusion of Proposition 10 can be deduced from [FPT00, Proposition 4.2] in the following way. Let us introduce new variables $\xi_i = x_i^{1/p_i}$. It follows from [FPT00, Proposition 4.2] that for any choice of the indices $j_1 \in \{1, \dots, p_1\}, \dots, j_n \in \{1, \dots, p_n\}$ the amoeba of the first-order polynomial $1 - \varepsilon_{j_1 1}\xi_1 - \dots - \varepsilon_{j_n n}\xi_n$ is the same. By [FPT00, Corollary 4.5] the number of connected components of its complement equals $n + 1$. Since a monomial change of the variables $x_1, \dots, x_n$ corresponds to a linear transformation of the amoeba space (see [FPT00]), it follows that the number of connected components of the complement of the amoeba of $f_{p_1,\dots,p_n}$ also equals $n+1$. This shows in particular that the amoeba of $f_{p_1,\dots,p_n}$ is solid.

We give here another proof of Proposition 10 which only uses hypergeometric properties of the Bergman kernels and does not use the explicit form of their denominators.

Proof of Proposition 10. The Newton polytope of $f_{p_1,\dots,p_n}$ has nonzero n-dimensional volume. Indeed, the restriction of $K_{p_1,\dots,p_n}(x)$ to the complex line $x_1 = \cdots [i] \cdots = x_n = 0$ is a rational function whose denominator is given by $(1 - x_i)^{k_i}$, $k_i > 0$, $i = 1, \dots, n$. (Here $[i]$ is the sign of omission.) It follows by Theorem B that the number of connected components of the amoeba complement ${}^c\mathcal{A}_{f_{p_1,\dots,p_n}}$ cannot be smaller than $n + 1$.

Let $\varphi(s)$ denote the coefficient of the series (15), i.e.,

$$\varphi(s) = \frac{\Gamma(p_1(s_1+1)+\dots+p_n(s_n+1)+1)}{\prod_{i=1}^n p_i\Gamma(p_i(s_i+1))}.$$

Since for any $i = 1, \dots, n$ the function $\varphi(s)$ satisfies the equation

$$\varphi(s+e_i)\prod_{j=0}^{p_i-1}(p_i(s_i+1)+j) = \varphi(s)\prod_{j=1}^{p_i}(p_1(s_1+1)+\dots+p_n(s_n+1)+j),$$

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it follows that $K_{p_1,\dots,p_n}(x)$ is a solution to the following system of the Horn type:

$$x_i\biggl(\prod_{j=1}^{p_i}(p_1(\theta_1+1)+\cdots+p_n(\theta_n+1)+j)\biggr)K_{p_1,\dots,p_n}(x) \\ =\biggl(\prod_{j=0}^{p_i-1}(p_i\theta_i+j)\biggr)K_{p_1,\dots,p_n}(x),\ i=1,\dots,n. \tag{17}$$

The number of irreducible 0-admissible sets associated with the system (17) equals $n+1$. These sets are

$$S_0=\mathbb{N}_0^n \quad\text{and}\quad S_i=\{s\in\mathbb{Z}^n : p_1(s_1+1)+\cdots+p_n(s_n+1)+1\leqslant 0,\ s_j\geqslant 0,\ j\neq i\},\quad i=1,\dots,n.$$

(Notice that (15) is supported in the 0-admissible set $\mathbb{N}_0^n$.) Since any expansion of a rational solution to a Horn system into a Laurent series with center at the origin is supported in an irreducible 0-admissible set, it follows that the number of connected components of the amoeba complement ${}^c\mathcal{A}_{f_{p_1,\dots,p_n}}$ cannot exceed $n+1$. We have proved earlier that the Newton polytope of $f_{p_1,\dots,p_n}$ has at least $n+1$ vertices. Thus it follows from Theorem B that the number of connected components of ${}^c\mathcal{A}_{f_{p_1,\dots,p_n}}$ cannot be smaller than $n+1$ and hence equals $n+1$. The proof is complete. □

Example 2. Let $n=2$, $p_1=3$ and $p_2=2$. The denominator of the Bergman kernel of the domain $D^{3,2}$ is given by

$$f_{3,2}(x)=(1-2x_1-3x_2+x_1^2-6x_1x_2+3x_2^2-x_2^3)^3.$$

By Proposition 10 the number of connected components of the amoeba complement ${}^c\mathcal{A}_{f_{3,2}}$ equals three.

The Bergman kernel (15) gives an example of a rational hypergeometric function. The problem of describing the class of rational hypergeometric functions was studied in [CDD99] and [Cat01]. Observe, however, that the definition of a hypergeometric function used in these papers is based on the Gelfand–Kapranov–Zelevinsky system of differential equations [GGR92] rather than the Horn system.

6. Meromorphic nonconfluent hypergeometric functions are rational

The aim of this section is to show that a nonconfluent Horn system (1) cannot possess a meromorphic solution different from a rational function (Theorem 12).

The relation between the support of a general Puiseux series and its domain of convergence is described by the Abel lemma (see Introduction and [GKZ89, § 1]). For hypergeometric series the following stronger version of this statement holds.

Lemma 11 (Two-sided Abel lemma). *Suppose that a nonconfluent hypergeometric Puiseux series with the support S has nonempty domain of convergence D. Let C be the cone of S. Then for any $x^{(0)}\in D$ and for some $x^{(1)}\in\mathbb{C}^n\setminus D$,*

$$\operatorname{Log}(x^{(0)})-C^\vee\subset\operatorname{Log}(D)\subset\operatorname{Log}(x^{(1)})-C^\vee.$$

Proof. Let $y(x)=\sum_{s\in S}\varphi(s)x^s$ be a nonconfluent hypergeometric Puiseux series. The first inclusion follows from the general Abel lemma (see Introduction). Let us prove the second inclusion. Let $M\subset\mathbb{R}^n$ be the lattice generated by the elements of the set S. By Proposition 5 the domain D is independent of the parameters $c_1,\dots,c_p$ of the coefficient (4) as long as they remain generic. Thus we may without loss of generality assume that $S=C\cap M$. Since D is nonempty, it follows by Proposition 5 that C is a strongly convex polyhedral cone. Let $u^{(1)},\dots,u^{(N)}\in M$ denote the

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generators of C, i.e., $C = \{\lambda_1 u^{(1)} + \cdots + \lambda_N u^{(N)} : \lambda_j \geqslant 0,\ j = 1,\ldots,N\}$. For each $j = 1,\ldots,N$ we consider the restricted series $y_j(x) = \sum_{k=0}^{\infty} \varphi(ku^{(j)}) x^{ku^{(j)}}$. The nonconfluency condition (5) implies that $\sum_{i=1}^{p} \langle A_i, u^{(j)} \rangle = 0$. By the result on convergence of the generalized hypergeometric series in one variable (see [GGR92, § 1.1]) the domain of convergence of $y_j(x)$ is contained in the set $\{x \in \mathbb{C}^n : |x^{u^{(j)}}| < r_j\}$ for some constant $r_j > 0$. This shows that $\mathrm{Log}(D) \subset \{v \in \mathbb{R}^n : \langle u^{(j)}, v\rangle < \log r_j,\ j = 1,\ldots,N\}$. Since C is strongly convex, we can choose $\xi \in \mathbb{R}^n$ such that $m_j := \langle u^{(j)}, \xi \rangle > 0$. Let

$$x^{(1)} \in \mathrm{Log}^{-1}\left(\xi \max_{j=1,\ldots,N} \frac{\log r_j}{m_j}\right),$$

then $\langle u^{(j)}, \mathrm{Log}\, x^{(1)} \rangle \geqslant \log r_j$, $j = 1,\ldots,N$, and hence

$$\mathrm{Log}(D) \subset \{v \in \mathbb{R}^n : \langle u^{(j)}, v - \mathrm{Log}\, x^{(1)} \rangle \leqslant 0,\ j = 1,\ldots,N\} = \mathrm{Log}\, x^{(1)} - C^{\vee}.$$

The proof is complete. □

The two-sided Abel lemma enables us to prove the following theorem, which is the main result in this section.

Theorem 12. *Any meromorphic nonconfluent hypergeometric function is rational.*

Proof. Let $y(x)$ be a meromorphic nonconfluent hypergeometric function. It follows by Proposition 5 and Lemma 11 that the domain of convergence of any shifted Laurent series representing $y(x)$ is not all of $(\mathbb{C}^*)^n$. Therefore, using the assumption that $y(x)$ is meromorphic, we can write it in the form $h(x)/g(x)$, where $h(x)$ is entire and $g(x)$ is some polynomial which is not a monomial.

Let us first consider the case when $\dim \mathcal{N} = n$. Denote by $C_v^{\vee}$ the cone which is dual to $\mathcal{N}$ at the point v. By the remark after Theorem B, to each vertex v of the polytope $\mathcal{N}$ one can associate a connected component of the amoeba complement ${}^c\mathcal{A}_g$. This component is the image of the domain of convergence of some Laurent series L_v for the function $y(x) = h(x)/g(x)$ under the mapping Log. It contains some translation $w_v + C_v^{\vee}$ of the cone $C_v^{\vee}$. By the two-sided Abel lemma the cone of the support of the series L_v coincides with the cone $-(C_v^{\vee})^{\vee} = -C_v$. The family of the cones $\{C_v^{\vee}\}_{v \in \mathrm{vert}(\mathcal{N})}$ coincides with the set of all maximal cones of the dual fan $\Sigma_{\mathcal{N}}$ of the polytope $\mathcal{N}$. Since for any polytope its dual fan is complete, it follows that the toric variety $\mathbb{X}_{\Sigma_{\mathcal{N}}}$ associated with the fan $\Sigma_{\mathcal{N}}$ is compact (see [Ful93, § 2.4]). This variety can be covered by the affine toric varieties $\{U_{C_v^{\vee}}\}_{v \in \mathrm{vert}(\mathcal{N})}$.

It is known that the monomials $\{x^{\alpha} : \alpha \in -C_v\}$ are holomorphic in $U_{C_v^{\vee}}$ (see [Ful93, § 1.3]). Since the cone of the support of the series L_v coincides with $-C_v$, it follows that for some $w_v \in \mathbb{Z}^n$ the series $x^{w_v} L_v$ contains only those monomials which are holomorphic in $U_{C_v^{\vee}}$. Thus $x^{w_v} y(x)$ is meromorphic in $U_{C_v^{\vee}}$ for all $v \in \mathrm{vert}(\mathcal{N})$. Since the toric variety $\mathbb{X}_{\Sigma_{\mathcal{N}}}$ is projective, it follows by the GAGA-principle that $y(x)$ is rational as claimed.

Now we shall show how to reduce the general case to the already treated case when $\dim \mathcal{N} = n$. Let $T \subset \mathbb{R}^n$ denote the minimal linear subspace whose translation contains the polytope $\mathcal{N}$. Choose a basis $u_1,\ldots,u_n \in \mathbb{Z}^n$ of the lattice $\mathbb{Z}^n$ such that $u_1,\ldots,u_m$ is a basis of the sublattice $T \cap \mathbb{Z}^n$. Let us introduce new variables $\xi_i = x^{u_i} = x_1^{u_{i1}} \cdots x_n^{u_{in}}$, $i = 1,\ldots,n$.

By construction the polynomial $g(\xi)$ is given by the product of a monomial and another polynomial which only depends on the variables $\xi_1,\ldots,\xi_m$. The Newton polytope of $g(\xi)$ has nonzero m-dimensional volume. It follows by the two-sided Abel lemma that the cone of the support of any Laurent series $\sum_{s \in \mathbb{Z}^n} \varphi(s)\xi^s$ representing the function $y(\xi)$ is contained in the linear subspace $s_{m+1} = \cdots = s_n = 0$. Hence $y(\xi)$ depends polynomially on the variables $\xi_{m+1},\ldots,\xi_n$. Let $\xi = (\xi', \xi'')$, where $\xi' = (\xi_1,\ldots,\xi_m)$, $\xi'' = (\xi_{m+1},\ldots,\xi_n)$. With this notation the function $y(\xi)$

can be written in the form

$$y(\xi) = \sum_{\alpha \in W} a_\alpha \xi''^{\alpha} y_\alpha(\xi'),$$

where W is a finite subset of the lattice $\mathbb{Z}^{n-m}$, $y_\alpha(\xi')$ is a meromorphic function depending on the variables $\xi_1, \ldots, \xi_m$ only and $a_\alpha \in \mathbb{C}$. We will prove that $y_\alpha(\xi')$ is a hypergeometric function for any $\alpha \in W$.

Let $E_i^{\lambda_i}$ denote the operator which increases the ith argument of a function depending on n variables by λ_i, i.e., $E_i^{\lambda_i} f(x) = f(x + \lambda_i e_i)$. For $\lambda \in \mathbb{R}^n$ we denote the composition of the operators $E_1^{\lambda_1}, \ldots, E_n^{\lambda_n}$ by E^λ, that is, $E^\lambda f(x) = f(x_1 + \lambda_1, \ldots, x_n + \lambda_n)$. Since the commutator $[\theta_i, x_j^{\lambda_j}]$ equals $\delta_{ij} \lambda_j x_j^{\lambda_j}$, it follows that for any polynomial P in n variables and any $\lambda \in \mathbb{Z}^n$,

$$P(\theta) x^\lambda = x^\lambda (E^\lambda P)(\theta). \tag{18}$$

By definition the function $y(x)$ is hypergeometric and hence satisfies the Horn system (1). Using the relation (18) and the ith equation of (1) we compute

$$\begin{aligned} x_i^2 (E_i^1 P_i)(\theta) P_i(\theta) y(x) &= (x_i P_i(\theta))^2 y(x) = x_i P_i(\theta) Q_i(\theta) y(x) \\ &= (E_i^{-1} Q_i)(\theta) x_i P_i(\theta) y(x) = (E_i^{-1} Q_i)(\theta) Q_i(\theta) y(x). \end{aligned}$$

Repeating this argument λ_i times we arrive at the formula

$$x_i^{\lambda_i} \biggl(\prod_{j=0}^{\lambda_i - 1} (E_i^j P_i)(\theta) \biggr) y(x) = \biggl(\prod_{j=0}^{\lambda_i - 1} (E_i^{-j} Q_i)(\theta) \biggr) y(x), \tag{19}$$

which holds for any $\lambda_i \in \mathbb{N}$. For $u_{ki} \geqslant 0$ define polynomials

$$\rho_{ki}(s) = \prod_{j=0}^{u_{ki}-1} E_i^j P_i(s) \quad \text{and} \quad \tau_{ki}(s) = \prod_{j=0}^{u_{ki}-1} E_i^{-j} Q_i(s)$$

(by definition the empty product equals 1). For $u_{ki} < 0$ define polynomials

$$\rho_{ki}(s) = \prod_{j=0}^{-u_{ki}-1} E_i^{-j} Q_i(s) \quad \text{and} \quad \tau_{ki}(s) = \prod_{j=0}^{-u_{ki}-1} E_i^j P_i(s).$$

It follows from (19) that, for any $k = 1, \ldots, n$,

$$x_i^{u_{ki}} \rho_{ki}(\theta) y(x) = \tau_{ki}(\theta) y(x), \quad i = 1, \ldots, n. \tag{20}$$

Composing the operators in Equations (20) in the same way as we did before in order to obtain the formula (19), we arrive at the system of equations

$$x^{u_k} \biggl(\prod_{j=1}^{n} \biggl(\prod_{l=j+1}^{n} E_l^{u_{kl}} \biggr) \rho_{kj}(\theta) \biggr) y(x) = \biggl(\prod_{j=1}^{n} \biggl(\prod_{l=1}^{j-1} E_l^{-u_{kl}} \biggr) \tau_{kj}(\theta) \biggr) y(x), \quad k = 1, \ldots, n. \tag{21}$$

For instance,

$$\begin{aligned} x_1^{u_{k1}} x_2^{u_{k2}} (E_2^{u_{k2}} \rho_{k1})(\theta) \rho_{k2}(\theta) y(x) &= x_1^{u_{k1}} \rho_{k1}(\theta) x_2^{u_{k2}} \rho_{k2}(\theta) y(x) && [\text{by } (18)] \\ &= x_1^{u_{k1}} \rho_{k1}(\theta) \tau_{k2}(\theta) y(x) && [\text{by the 2nd equation in } (20)] \\ &= (E_1^{-u_{k1}} \tau_{k2})(\theta) x_1^{u_{k1}} \rho_{k1}(\theta) y(x) && [\text{by } (18)] \\ &= (E_1^{-u_{k1}} \tau_{k2})(\theta) \tau_{k1}(\theta) y(x) && [\text{by the 1st equation in } (20)]. \end{aligned}$$

Each equation in (21) is obtained by repeating this argument n times.

Making the change of variables $\xi_i = x^{u_i}$ in (21) and using the equality

$$\theta_i = x_i \frac{\partial}{\partial x_i} = u_{1i}\xi_1 \frac{\partial}{\partial \xi_1} + \cdots + u_{ni}\xi_n \frac{\partial}{\partial \xi_n},$$

we conclude that $y(\xi)$ is a solution to the system of equations

$$\xi_i \rho^{(i)}(\theta_\xi) y(\xi) = \tau^{(i)}(\theta_\xi) y(\xi),\ i = 1, \dots, n, \tag{22}$$

where

$$\theta_\xi = \left(\xi_1 \frac{\partial}{\partial \xi_1}, \dots, \xi_n \frac{\partial}{\partial \xi_n}\right),$$

U is the matrix with the rows $u_1, \dots, u_n$ and

$$\rho^{(i)}(s) = \prod_{j=1}^{n} \left(\prod_{l=j+1}^{n} E_l^{u_{kl}}\right) \rho_{kj}((U^{\mathrm{T}})^{-1}s),$$

$$\tau^{(i)}(s) = \prod_{j=1}^{n} \left(\prod_{l=1}^{j-1} E_l^{-u_{kl}}\right) \tau_{kj}((U^{\mathrm{T}})^{-1}s).$$

Since $y(\xi) = \sum_{\alpha \in W} a_\alpha \xi''^\alpha y_\alpha(\xi')$, it follows from the first m equations of the system (22) that

$$(\xi_i \rho^{(i)}(\theta_\xi) - \tau^{(i)}(\theta_\xi)) y(\xi) = \sum_{\alpha \in W} a_\alpha \xi''^\alpha ((\xi_i \rho^{(i)}(\theta_\xi) - \tau^{(i)}(\theta_\xi)) y_\alpha(\xi')) = 0$$

for $i = 1, \dots, m$. Since $y_\alpha(\xi')$ does not depend on $\xi_{m+1}, \dots, \xi_n$, it follows that for any $\alpha \in W$

$$\xi_i \rho^{(i)}(\theta'_\xi) y_\alpha(\xi') = \tau^{(i)}(\theta'_\xi) y_\alpha(\xi'), \quad i = 1, \dots, m. \tag{23}$$

Here

$$\theta'_\xi = \left(\xi_1 \frac{\partial}{\partial \xi_1}, \dots, \xi_m \frac{\partial}{\partial \xi_m}, 0, \dots, 0\right).$$

The system (23) is a Horn system in m variables, and hence the functions $y_\alpha(\xi')$ are hypergeometric and meromorphic for each $\alpha \in W$. Thus we have arrived at the situation where the Newton polytope of the polynomial which defines the singular set of the given meromorphic hypergeometric function has the maximal possible dimension. This completes the proof. □

Thanks to Theorem 12 we do not need to differentiate between meromorphic and rational nonconfluent hypergeometric functions. From now on we formulate all the results using the term 'rational'.

Remark 4. Let $f(x)$ be a rational function in n variables with singularities (poles) along an algebraic hypersurface $V \subset \mathbb{C}^n$ and let $\mathcal{A}$ be the amoeba of V. By Theorem A the connected components of the amoeba complement ${}^c\!\mathcal{A}$ are in bijective correspondence with the Laurent series expansions (with center at the origin) of $f(x)$. For a multi-valued analytic function $F(x)$ with singularities on the same variety V this correspondence is in general not one-to-one. It may happen that some of the connected components of ${}^c\!\mathcal{A}$ do not correspond to any expansion of $F(x)$ since there is no holomorphic branch of $F(x)$ on the pull-back of this component. It is also possible that several connected components of ${}^c\!\mathcal{A}$ correspond to a single series expansion of $F(x)$. (For instance, let $x \in \mathbb{C}$ and consider the function $F(x) = \sqrt{\sqrt{x+2} + \sqrt{3}}$. There exists a holomorphic branch of $F(x)$ in the disk $\{|x| < 2\}$ although $x = 1$ is a branching point. A similar situation in the two-dimensional case is described in Example 3 in the next section.) However, with each series expansion of $F(x)$ centered at the origin one can associate at least one connected component of ${}^c\!\mathcal{A}$.

7. Rational solutions to the Horn system

Typically a hypergeometric function is a multi-valued analytic function with singularities along an algebraic hypersurface (see § 2). In this section we give a necessary condition for a hypergeometric series to represent a germ of a rational function. This allows one to give an explicit description of the class of rational solutions to (1) in the case when $Q_i(s) = \prod_{k=0}^{p_i-1}(s_i + k/p_i)$ for some positive integers p_i, each linear factor of $P_i(s)$ depends on all the variables and the resultant of (1) is irreducible. We prove that any such rational hypergeometric function is contiguous to the Bergman kernel $K_{p_1,\dots,p_n}$ for some $p_1,\dots,p_n$ (Proposition 15).

Recall that $B_1,\dots,B_d$ are defined to be the duals to the maximal elements (with respect to inclusion) of the finite family $\{-C_I\}$ of strongly convex polyhedral cones. Here C_I is the recession cone of the convex hull of the support of the hypergeometric series (13). Let $X_1,\dots,X_N$ denote the recession cones of the connected components of the amoeba complement ${}^c\mathcal{A}_{R(x)}$ of the resultant of (1). These recession cones are well defined since by Theorem A the connected components of the amoeba complement are convex. The following theorem describes the structure of the amoeba $\mathcal{A}_{R(x)}$.

Theorem 13. *Suppose that a nonconfluent Horn system possesses a rational solution with poles along the zero set of its resultant $R(x)$. Then the fan of this Horn system is well defined and dual to the Newton polytope of $R(x)$.*

Proof. Since there exists a rational solution to (1) with the poles on the zero set of its resultant $R(x)$ it follows by Theorems B and 7 that the cone X_i has nonempty interior for any $i = 1,\dots,N$. Thus by Theorem B the cones $\{X_i\}_{i=1}^N$ can be identified with the maximal cones of the fan which is dual to the Newton polytope of $R(x)$.

It suffices to show that the family $\{B_i\}_{i=1}^d$ consists of the same elements as the family $\{X_i\}_{i=1}^N$. As we have already mentioned in § 4 the nonconfluency condition (5) for the Horn system (1) implies that $\bigcup_{j=1}^d B_j = \mathbb{R}^n$. Hence for any $i = 1,\dots,N$ there exists $k_i \in \{1,\dots,d\}$ such that $\operatorname{int}(X_i \cap B_{k_i}) \neq \emptyset$. Let L_i denote a series solution to (1) whose support S_i defines the cone B_{k_i} in the sense that $B_{k_i} = -C_{S_i}^\vee$. Here C_{S_i} is the cone of S_i (see § 4). Let $\tilde{L}_i$ denote the series expansion of the rational solution to (1) such that the recession cone of the image of its domain of convergence under the mapping Log is X_i. Since $\operatorname{int}(X_i \cap B_{k_i}) \neq \emptyset$ it follows that the series $L + \tilde{L}_i$ has a nonempty domain of convergence Ω_i. By the two-sided Abel lemma the cone of the convex set $\operatorname{Log}\Omega_i$ is $X_i \cap B_{k_i}$.

Any Puiseux series solution to (1) whose domain of convergence lies entirely in the preimage of a connected component of the amoeba complement ${}^c\mathcal{A}_{R(x)}$ with respect to the mapping Log converges on the whole of this preimage. Using the two-sided Abel lemma we conclude that B_{k_i} cannot be a proper subset of X_i. Thus either $X_i = B_{k_i}$ or $B_{k_i}^\vee$ is a proper subset of $(X_i \cap B_{k_i})^\vee$. The latter is impossible due to the assumption that $B_{k_i}^\vee$ is a maximal element in the family of the cones of the supports of series solutions to (1). Hence $X_i = B_{k_i}$ for any $i = 1,\dots,N$. Since the cones $\{X_i\}_{i=1}^N$ are the maximal cones of a complete fan, it follows that $d = N$ and thus we can identify the families of the cones $\{X_i\}_{i=1}^N$ and $\{B_i\}_{i=1}^d$. The proof is complete. $\square$

The conditions in Theorem 13 are sufficient for the fan of a Horn system to be dual to the Newton polytope of its resultant, but they are not necessary. For instance, the fan of the system (25) in Example 3 below is dual to the Newton polytope of its resultant though the system (25) has no nonzero rational solutions. Yet, the remark at the very end of § 4 shows that the conclusion of Theorem 13 does not hold in the arbitrary case.

Corollary 14. *If a Horn system possesses a rational solution with the poles on the zero set of its resultant then the number of 0-admissible sets associated with this system cannot be smaller than the number of maximal cones in its fan.*

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Proof. By Theorem 13 the fan of the Horn system is well defined. Let $y(x)$ be a rational solution to (1) with the poles on the zero set of the resultant $R(x)$ of (1). By Theorem A the number of Laurent series expansions of $y(x)$ with the center at the origin equals the number of connected components of the set ${}^c\mathcal{A}_R$. By Theorem 7 the amoeba of $R(x)$ is solid and hence by Theorem 13 there exists a one-to-one correspondence between the connected components of ${}^c\mathcal{A}_R$ and the maximal cones of the fan of the system (1). Since any expansion of $y(x)$ is supported in a 0-admissible set it follows that the number of such sets cannot be smaller than the number of maximal cones in the fan of the Horn system. This completes the proof of the corollary. $\square$

As we have seen in § 2 a solution to the Horn system (1) can only be singular on the set on which the resultant $R(x)$ of (1) vanishes. Typically $R(x)$ is divisible by some monomial x^a, $a \in \mathbb{N}^n$. We denote the quotient $R(x)/x^a$ (with the maximal possible $|a| = a_1 + \cdots + a_n$) by $r(x)$ and call it the *essential resultant* of the system (1). The reason for introducing this terminology is the fact that a Laurent monomial has unique Laurent series development with the center at the origin. Therefore such a monomial is an inessential factor as long as one is concerned with the problem of computing the number of connected components of the amoeba complement of a mapping.

The case when the polynomial $Q_i(s)$ depends only on s_i for all $i = 1, \ldots, n$ is particularly important. Under this assumption it is possible to compute the dimension of the space of holomorphic solutions to the Horn system (1) explicitly and construct a basis in this space if the parameters of the system are sufficiently general [Sad02]. (Theorem 9 in [Sad02] assumes that $\deg Q_i > \deg P_i$, $i = 1, \ldots, n$, which is not the case if the nonconfluency relation (5) holds. Yet, by the lemma in [GGR92, § 1.4] each of the basis series which were constructed in [Sad02, § 3] converges in some neighborhood of the origin if the original Horn system is nonconfluent. The multi-valued analytic functions determined by these series give a global basis in the space of holomorphic solutions to (1).) Recall that two Ore–Sato coefficients (and the corresponding hypergeometric series) are called *contiguous* if their quotient can be reduced to the product of a rational function and an exponential term $\tilde{t}_1^{s_1} \cdots \tilde{t}_n^{s_n}$. The next proposition provides an explicit description of the class of rational solutions to such systems of hypergeometric type under some additional assumptions on the parameters.

Proposition 15. *Suppose that the nonconfluent Ore–Sato coefficient*

$$\psi(s) = t_1^{s_1} \cdots t_n^{s_s} \frac{\prod_{i=1}^{p} \Gamma(\langle A_i, s\rangle - c_i)}{\prod_{j=1}^{n} \Gamma(p_j(s_j + 1))}$$

defines the Horn system (1) with the irreducible essential resultant $r(x)$ and satisfies the conditions $A_{ij} > 0$, $i = 1, \ldots, p$, $j = 1, \ldots, n$. Let $y(x) = \sum_{s \in \mathbb{N}^n} \psi(s) x^s$ and let A be the matrix with rows $A_1, \ldots, A_p$. If $\operatorname{rank} A > 1$ then the series $y(x)$ cannot define a rational function. (We disregard exceptional values of the parameters of $\psi(s)$ for which $y(x)$ reduces to a linear combination of hypergeometric series in fewer variables.) If $\operatorname{rank} A = 1$ and $y(x)$ is rational then it is contiguous to the series (15) converging to the Bergman kernel $K_{p_1,\ldots,p_n}(x)$.

Proof. Suppose that $\operatorname{rank} A > 1$ and $y(x)$ is a rational function. We may without loss of generality assume that $A_{11}A_{22} - A_{12}A_{21} \neq 0$. For each $m = 1, \ldots, p$ consider the Ore–Sato coefficient

$$\chi_m(s) = \frac{\prod_{i=1}^{m} \Gamma(\langle A_i, s\rangle - c_i)}{\prod_{j=1}^{n} \Gamma(p_j(s_j + 1))}.$$

Each of these coefficients defines a system of differential equations of the Horn type (see Remark 1). Let $B_{m1}, \ldots, B_{md_m}$ be the maximal elements in the family of the cones of the admissible sets associated with the system defined by $\chi_m(s)$ (see § 4). Arguing as in the proof of Proposition 10 we conclude that $d_1 = n+1$. Let $\tilde{A}$ be the matrix with rows $A_1, A_2, e_3, \ldots, e_n$, $\tilde{c} = (c_1, c_2, 0, \ldots, 0) \in \mathbb{C}^n$ and define γ to be the solution to the system of linear equations $\tilde{A}s = \tilde{c}$. The set $\{s \in \mathbb{Z}^n + \gamma : \tilde{A}s \geqslant 0\}$

satisfies the conditions in Proposition 3 if the parameters $c_1, \ldots, c_p$ are generic. This yields $d_2 \geqslant n+2$. By Remark 2, $d_i \leqslant d_j$ for $i \leqslant j$. Since $\chi_p(s) = \psi(s)$ it follows by Theorem 13 that the number of connected components of the amoeba complement ${}^c\mathcal{A}_{r(x)}$ at least equals $n+2$. By our assumption the series $y(x)$ represents a germ of a rational function. Since $r(x)$ is irreducible, the function $y(x)$ must be singular on the whole of the hypersurface $\{r(x) = 0\}$. Thus it follows from Theorem A that the number of Laurent series expansions (centered at the origin) of this rational function at least equals $n+2$. Yet, the condition $A_{ij} > 0$ and the conditions (8) and (9) in Proposition 3 imply that the number of 0-admissible subsets associated with the Horn system defined by the Ore–Sato coefficient $\psi(s)$ cannot exceed $n+1$. This contradicts the conclusion of Corollary 14 and shows that the function $y(x)$ cannot be rational unless $\operatorname{rank} A = 1$.

Suppose now that $\operatorname{rank} A = 1$ and that the series $y(x)$ converges to a rational function. Let $\delta = \mathrm{GCD}(p_1, \ldots, p_n)$, $\tilde{p}_i = p_i/\delta$, $i = 1, \ldots, n$. It follows from the nonconfluency condition $\sum_{i=1}^{p} A_i = (p_1, \ldots, p_n)$ and the Gauss multiplication formula for the Γ-function that $\psi(s)$ is contiguous to

$$\tilde{\psi}(s) = \frac{\prod_{l=0}^{\delta-1} \Gamma(\tilde{p}_1 s_1 + \cdots + \tilde{p}_n s_n + a_l)}{\prod_{j=1}^{n} \Gamma(p_j(s_j+1))}.$$

Here $a_0, \ldots, a_{\delta-1} \in \mathbb{C}$ are some constants. Moreover the quotient $\psi(s)/\tilde{\psi}(s)$ is given by an exponential term $\tilde{t}_1^{s_1} \cdots \tilde{t}_n^{s_n}$ and hence the series $\tilde{y}(x) = \sum_{s\in\mathbb{N}^n} \tilde{\psi}(s)x^s$ converges to a rational function. By the assumption, $\tilde{p}_i \neq 0$ for any $i = 1, \ldots, n$. The restriction of $\tilde{y}(x)$ to the complex line $x_1 = \cdots [i] \cdots = x_n = 0$ is a rational function (here $[i]$ is the sign of omission). Let $\tilde{\psi}_i(s_i) = \tilde{\psi}(0, \ldots, s_i, \ldots, 0)$ (s_i in the ith position). Using once again the Gauss multiplication formula we conclude that the series

$$\sum_{s_i=0}^{\infty} \frac{\prod_{l=0}^{\delta-1} \prod_{j=0}^{\tilde{p}_i-1} \Gamma(s_i + (a_l + j)/\tilde{p}_i)}{\prod_{k=0}^{p_i-1} \Gamma(s_i + (k/p_i))} x_i^{s_i}$$

represents a rational function. A criterion for a power series in one variable to converge to a rational function (see [Sta86, Theorem 4.1.1]) implies that for any $l = 0, \ldots, \delta-1$, $j = 0, \ldots, \tilde{p}_i - 1$ there exists $k \in \{0, \ldots, p_i - 1\}$ such that $(a_l + j)/\tilde{p}_i - k/p_i \in \mathbb{N}$. Hence for any $l = 0, \ldots, \delta - 1$ one can find $k \in \{0, \ldots, p_i - 1\}$ such that $a_l - k/\delta \in \mathbb{Z}$. Thus $\psi(s)$ is contiguous to the Ore–Sato coefficient

$$\frac{\prod_{l=0}^{\delta-1} \Gamma(\tilde{p}_1 s_1 + \cdots + \tilde{p}_n s_n + l/\delta)}{\prod_{j=1}^{n} \prod_{k=0}^{p_j-1} \Gamma(s_j + (k/p_j) + 1)}.$$

The Gauss multiplication formula shows that the latter coefficient is contiguous to the coefficient of the series (15) which represents the Bergman kernel $K_{p_1,\ldots,p_n}$. The proof is complete. □

Remark 5. There exist rational hypergeometric functions that cannot be described in terms of the Bergman kernels of complex ellipsoidal domains. For instance, the hypergeometric series

$$\sum_{s\in\mathbb{N}_0^n} \frac{\Gamma(s_1 + p(s_2 + \cdots + s_n + 1))\Gamma(s_2 + \cdots + s_n + 1)}{\Gamma(s_1+1)\cdots\Gamma(s_n+1)\Gamma(p(s_2 + \cdots + s_n + 1))} x^s = ((1-x_1)^p - x_2 - \cdots - x_n)^{-1}$$

is not contiguous to such a kernel whenever $n \geqslant 3$ and $p \geqslant 2$.

Let us now consider an example. This example deals with a simplified version of the hypergeometric series which expresses a solution $y(x)$ to the cubic equation

$$y^3 + x_1 y^2 + x_2 y - 1 = 0$$

in terms of the coefficients x_1 and x_2 (see [Mel21, ST00, Stu00] and Corollary 9).

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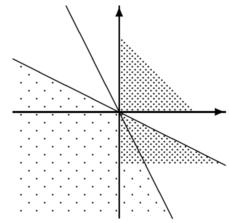

FIGURE 4. The maximal cones of the irreducible supports of solutions to (25).

Example 3. Consider the hypergeometric series

$$y(x_1,x_2)=\sum_{s_1,s_2\geqslant 0}\frac{\Gamma(2s_1+s_2+\alpha)\Gamma(s_1+2s_2+\beta)}{\Gamma(3s_1+3)\Gamma(3s_2+3)}x_1^{s_1}x_2^{s_2}, \tag{24}$$

where α,β are arbitrary parameters such that the coefficient of the series is well defined and different from zero on $\mathbb{N}_0^2$. By the lemma in [GGR92, § 1.4] the series (24) converges in some neighborhood of the origin. This series satisfies the system of equations of hypergeometric type

$$\begin{aligned}x_1(2\theta_1+\theta_2+\alpha)(2\theta_1+\theta_2+\alpha+1)(\theta_1+2\theta_2+\beta)y(x)&=3\theta_1(3\theta_1+1)(3\theta_1+2)y(x),\\ x_2(2\theta_1+\theta_2+\alpha)(\theta_1+2\theta_2+\beta)(\theta_1+2\theta_2+\beta+1)y(x)&=3\theta_2(3\theta_2+1)(3\theta_2+2)y(x).\end{aligned} \tag{25}$$

The principal symbols of the operators in (25) are

$$\begin{aligned}H_1(x,z)&=x_1(2x_1z_1+x_2z_2)^2(x_1z_1+2x_2z_2)-27(x_1z_1)^3,\\ H_2(x,z)&=x_2(2x_1z_1+x_2z_2)(x_1z_1+2x_2z_2)^2-27(x_2z_2)^3\,,\end{aligned}$$

and their classical resultant is given by

$$\begin{aligned}R(x_1,x_2)=x_1^9x_2^9(&x_1^2x_2^2+64x_1^3-24x_1^2x_2-24x_1x_2^2+64x_2^3\\ &-1296x_1^2+4698x_1x_2-1296x_2^2+8748x_1+8748x_2-19683).\end{aligned} \tag{26}$$

The essential resultant $r(x_1,x_2)=R(x_1,x_2)/(x_1x_2)^9$ is an irreducible polynomial, so by Proposition 2 it is the essential resultant of the system (25). The vectors $(2,1)$ and $(1,2)$ of the coefficients of the linear factors in the arguments of the Γ-functions in the numerator of the coefficient of (24) are linearly independent. By Proposition 15 the series (24) cannot converge to a rational function.

The fact that the series (24) cannot define a germ of a rational function can be seen without appealing to Proposition 15 since we have the explicit expression (26) for the resultant of the principal symbols of the differential operators in (25). If the sum $y(x)$ of the series (24) was rational, then by Theorem 7 the number of expansions of $y(x)$ into a Laurent series with the center at the origin would be equal to four since the Newton polytope of the essential resultant of (25) has four vertices (see Figure 6). However, Proposition 3 shows that for any choice of the parameters α and β at most three of the admissible subsets can belong to $\mathbb{Z}^2$ (see Figure 4). Thus the sum of the series (24) is not a rational function.

To determine the resultant of a general Horn system is a problem of great computational complexity. Theorem 13 and Corollary 14 allow one to describe the amoeba of the resultant of a Horn system (as for example in Figure 5) and draw consequences on its solvability in the class of rational functions without performing this computation.

Finally we give an example which illustrates how Theorem 7 (or its Corollaries 8 and 9) can be applied to the problem of constructing polyhedral decompositions of the Newton polytopes of discriminants.

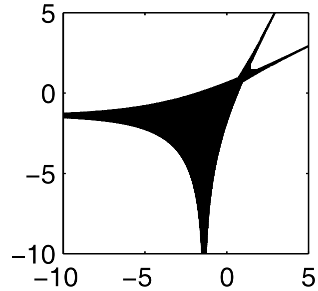

Figure 5. The amoeba of the essential resultant of the Horn system (25).

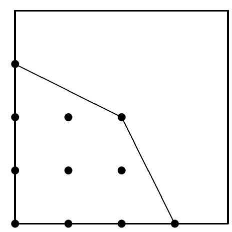

Figure 6. The Newton polytope of the essential resultant of the system (25).

In [PR04] a natural polyhedral decomposition of the Newton polytope of a Laurent polynomial f is given. This decomposition is determined by the piecewise linear convex function constructed from the so-called Ronkin function $N_f(t)$ which is a convex function in $t \in \mathbb{R}^n$. The function N_f is affine-linear on each connected component of ${}^c\mathcal{A}_f$. If such a component M corresponds to a vertex $\nu = \nu(M)$ (see Theorem B) of the Newton polytope of f, then the Ronkin function N_f is given, for $t \in M$, by $N_f(t) = \log|c_\nu| + \langle t, \nu \rangle$, where c_ν denotes the coefficient of x^ν in f. (See [PR04, Theorem 2] for an explanation of this.)

Example 4. Consider the quartic equation

$$y^4 + x_1 y^3 + x_2 y^2 + x_3 y - 1 = 0. \tag{27}$$

The discriminant of (27) is given by the polynomial

$$\begin{aligned} &x_1^2x_2^2x_3^2 - 4x_1^3x_3^3 + 4x_1^2x_2^3 - 4x_2^3x_3^2 - 18x_1^3x_2x_3 + 18x_1x_2x_3^3 - 27x_1^4 - 16x_2^4 - 27x_3^4 \\ &\quad + 80x_1x_2^2x_3 + 6x_1^2x_3^2 + 144x_1^2x_2 - 144x_2x_3^2 - 192x_1x_3 - 128x_2^2 - 256. \end{aligned} \tag{28}$$

By Corollary 9 the zero locus of the polynomial (28) has a solid amoeba. The Newton polytope of (28) is displayed in Figure 7.

From the solidness of the amoeba of the discriminant (28) we conclude that any affine linear part of the function N_f corresponds to one of the eight vertices of the Newton polytope of (28). Taking the maximum of these eight affine linear functions, we obtain the piecewise linear convex function

$$\begin{aligned} \max(&8\log 2,\ 3\log 3 + 4t_1,\ 4\log 2 + 4t_2,\ 3\log 3 + 4t_3,\ 2\log 2 + 2t_1 + 3t_2, \\ &2\log 2 + 3t_1 + 3t_3,\ 2\log 2 + 3t_2 + 2t_3,\ 2t_1 + 2t_2 + 2t_3). \end{aligned} \tag{29}$$

The set of all points t at which the convex function (29) is not smooth is a two-dimensional polyhedral complex called the *spine* of the amoeba, and the Legendre transform of (29) similarly gives rise to a dual polyhedral subdivision of the polytope in Figure 7. It deserves to be mentioned that in this example the polyhedral decomposition of the polytope is not simplicial, for it contains a

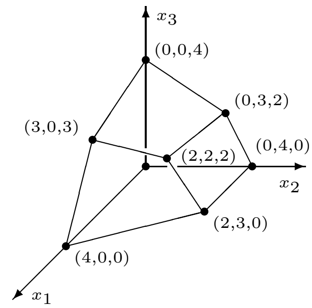

Figure 7. The Newton polytope of the discriminant of Equation (27).

polytope with five vertices, namely the convex hull of the points $(0,4,0), (2,3,0), (3,0,3), (0,3,2)$ and $(2,2,2)$. This is because there is a point, $t = (3\log 2, 4\log 2, 3\log 2)$, at which the maximum in (29) is attained simultaneously by the five functions $4\log 2 + 4t_2$, $2\log 2 + 2t_1 + 3t_2$, $2\log 2 + 3t_1 + 3t_3$, $2\log 2 + 3t_2 + 2t_3$, and $2t_1 + 2t_2 + 2t_3$.

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References

Bjö79 J.-E. Björk, *Rings of differential operators*, North-Holland Mathematical Library, vol. 21 (North-Holland, Amsterdam, 1979).

Bol00 A. A. Bolibrukh, *Fuchsian differential equations and holomorphic bundles* (Moscow Center for Continuous Mathematical Education, Moscow, 2000).

Cat01 E. Cattani, A. Dickenstein and B. Sturmfels, *Rational hypergeometric functions*, Compositio Math. **128** (2001), 217–240.

CDD99 E. Cattani, C. D'Andrea and A. Dickenstein, *The $\mathcal{A}$-hypergeometric system associated with a monomial curve*, Duke Math. J. **99** (1999), 179–207.

Evg86 M. A. Evgrafov, *Series and integral representations*, in *Current problems of mathematics. Fundamental directions*, Itogi Nauki i Tekhniki, vol. 13 (Akad. Nauk SSSR, Moscow, 1986), 5–92.

FH96 G. Francsics and N. Hanges, *The Bergman kernel of complex ovals and multivariable hypergeometric functions*, J. Funct. Anal. **142** (1996), 494–510.

FPT00 M. Forsberg, M. Passare and A. Tsikh, *Laurent determinants and arrangements of hyperplane amoebas*, Adv. Math. **151** (2000), 45–70.

Ful93 W. Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131 (Princeton University Press, Princeton, NJ, 1993).

GGR92 I. M. Gelfand, M. I. Graev and V. S. Retach, *General hypergeometric systems of equations and series of hypergeometric type*, Russian Math. Surveys **47** (1992), 1–88.

GKZ89 I. M. Gelfand, M. M. Kapranov and A. V. Zelevinsky, *Hypergeometric functions and toric varieties*, Funct. Anal. Appl. **23** (1989), 94–106.

GKZ94 I. M. Gelfand, M. M. Kapranov and A. V. Zelevinsky, *Discriminants, resultants and multidimensional determinants* (Birkhäuser, Boston 1994).

Hor89 J. Horn, *Über die Konvergenz der hypergeometrischen Reihen zweier und dreier Veränderlichen*, Math. Ann. **34** (1889), 544–600.

Hör90 L. Hörmander, *The analysis of linear partial differential operators I* (Springer, Berlin, 1990).

Kap91 M. M. Kapranov, *A characterization of $\mathcal{A}$-discriminantal hypersurfaces in terms of the logarithmic Gauss map*, Math. Ann. **290** (1991), 277–285.

McD95 J. McDonald, *Fiber polytopes and fractional power series*, J. Pure Appl. Algebra **104** (1995), 213–233.

Mel21 Hj. Mellin, *Résolution de l'équation algébrique générale à l'aide de la fonction* Γ, C. R. Acad. Sci. **172** (1921), 658–661.

Mik00 G. Mikhalkin, *Real algebraic curves, the moment map and amoebas*, Ann. of Math. (2) **151** (2000), 309–326.

Mum76 D. Mumford, *Algebraic geometry I. Complex projective varieties* (Springer, Berlin, 1976).

MY82 M. A. Mkrtchyan and A. P. Yuzhakov, *The Newton polyhedron and the Laurent series of rational functions of n variables* (in Russian), Izv. Akad. Nauk Armyan. SSR Ser. Mat. **17** (1982), 99–105.

PR04 M. Passare and H. Rullgård, *Amoebas, Monge–Ampère measures and triangulations of the Newton polytope*, Duke Math. J. **121** (2004), 481–507.

Rul00 H. Rullgård, *Stratification des espaces de polynômes de Laurent et la structure de leurs amibes*, C. R. Acad. Sci. Paris Sér. I Math. **331** (2000), 355–358.

Sad02 T. M. Sadykov, *On the Horn system of partial differential equations and series of hypergeometric type*, Math. Scand. **91** (2002), 127–149.

Sat90 M. Sato, *Theory of prehomogeneous vector spaces (algebraic part)*, Nagoya Math. J. **120** (1990), 1–34.

SK85 H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian hypergeometric series* (Ellis Horwood, Chichester, 1985).

SST00 M. Saito, B. Sturmfels and N. Takayama, *Gröbner deformations of hypergeometric differential equations* (Springer, Berlin, 2000).

ST00 A. Yu. Semusheva and A. K. Tsikh, *Continuation of Mellin's studies of solutions to algebraic equations* (in Russian), in *Complex analysis and differential operators* (Krasnoyarsk State University, Krasnoyarsk, 2000), 134–146.

Sta86 R. P. Stanley, *Enumerative combinatorics*, Cole Mathematics Series, vol. 1 (Wadsworth & Brooks, Monterey, CA, 1986).

Stu00 B. Sturmfels, *Solving algebraic equations in terms of $\mathcal{A}$-hypergeometric series*, Discrete Math. **210** (2000), 171–181.

Zie95 G. M. Ziegler, *Lectures on polytopes* (Springer, Berlin, 1995).

Zin74 B. S. Zinov'ev, *On reproducing kernels for multicircular domains of holomorphy*, Siberian Math. J. **15** (1974), 24–33.

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