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A CHARACTERIZATION AND A VARIATIONAL INEQUALITY FOR THE MULTIVARIATE NORMAL DISTRIBUTION

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Abstract

Various generalizations of the Maxwell characterization of the multivariate standard normal distribution are derived. For example the following is proved: If for a k-dimensional random vector X there exists an $n \in \{1, \ldots, k-1\}$ such that for each n-dimensional linear subspace $H \subset \mathbf{R}^k$ the projections of X on H and $H^{\perp}$ are independent, X is normal. If X has a rotationally symmetric density and its projection on some H has a density of the same functional form, X is normal. Finally we give a variational inequality for the multivariate normal distribution which resembles the isoperimetric inequality for the surface measure on the sphere.

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I. Introduction

This paper is devoted to the consideration of some "geometric" properties of the k-dimensional normal distribution $N_k(\mu, \sigma^2 I_k)$, where $k \geqslant 2$, $\mu \in \mathbb{R}^k$, $\sigma^2 \geqslant 0$ and I_k is the $(k \times k)$-unit matrix. The starting point is the well known Maxwell characterisation stating that $N_k(0, \sigma^2 I_k)$ is the only rotationally symmetric probability measure on $\mathbb{R}^k$ for which the coordinates are independent random variables ([6], or [4], pages 160–161; for a related result on random matrices see [7]). Rotational symmetry of a distribution P on $\mathbb{R}^k$ is equivalent to the property that the characteristic function (c.f.) ϕ (or the density f, if it exists) depends only on the euclidean length $|\cdot|$ of its argument. Theorem 1 says that if $|\cdot|$ is replaced by other functions, the classical independence condition characterizes other distributions, for example the Cauchy or the double exponential distribution.

In Theorem 3 the Maxwell characterization is refined in another way. Let X be a k-dimensional random vector, $k \geqslant 2$. If there is some $n \in \{1, \ldots, k-1\}$ such that for any n-dimensional linear subspace H of $\mathbb{R}^k$ and its orthogonal complement $H^\perp$ the projected random vectors $p_H(X)$ and $p_{H^\perp}(X)$ are independent, then $X \sim N_k(\mu, \sigma^2 I_k)$ for some $\mu \in \mathbb{R}^k$ and some $\sigma^2 \geqslant 0$. It is not assumed that X is rotationally symmetric.

Furthermore Section II contains the following result. Let $1 \leqslant n < k$. If X has a density $f(x) = \tilde{f}(|x|)$, $x \in \mathbb{R}^k$, and for some n-dimensional subspace H of $\mathbb{R}^k$ $p_H(X)$ has a density f_H of the same functional form as that of X, that is,

$$f_H(x) = C_H \tilde{f}(|x|), \qquad x \in H,$$

(C_H some constant), then X is normal (Theorem 2). All three results generalize the Maxwell characterization in different directions. While Theorem 1 integrates the classical statement into a whole class of characterizations, in the other two theorems the restriction to one-dimensional projections is dropped. In Section III we exhibit an interesting relation between the standard normal distribution $N := N_k(0, \sigma^2 I_k)$ and the geometry of euclidean space indicating that N plays a similar role for $\mathbb{R}^k$ as the surface measure for the unit sphere S^{k-1}. The result is expressed by a "variational inequality" resembling the isoperimetric property of S^{k-1}. This property can be expressed in the following way: Among all compact subsets K of S^{k-1} for which the normalized surface measure $\mu_k(K)$ is equal to some fixed $c \in [0,1]$ the geodesic balls have minimal Minkowski surface. Here the Minkowski surface $O(A)$ of a Borel measurable set $A \subset S^{k-1}$ is defined by

$$(1) \qquad O(A) := \liminf_{\delta \to 0} \left(\mu_k\left(U^{\Delta}_{k,\delta}(A) - \mu_k(A)\right)\right)/\delta,$$

where $U^{\Delta}_{k,\delta}(A)$ denotes the closed geodesic δ-neighbourhood of A with respect to S^{k-1}. The isoperimetric property is a corollary of the following fundamental relation: If H is a geodesic ball and K is compact, then for all $\delta > 0$

$$(2) \qquad \mu_k(H) \leqslant \mu_k(K) \Rightarrow \mu_k\left(U^{\Delta}_{k,\delta}(H)\right) \leqslant \mu_k\left(U^{\Delta}_{k,\delta}(K)\right).$$

The first proof of (2) was given in [8], it has then been simplified in [1]. A very short and elegant more recent proof can be found in [2].

The implication (2) will be seen to hold for N in an analogous form. If μ_k is replaced by N, the geodesic neighbourhoods by euclidean ones and H now denotes a half space in $\mathbb{R}^k$, then (2) remains valid. It is not known (to the author), whether N is the only probability measure on $\mathbb{R}^k$ with this "isoperimetric" property. This would yield a new type of characterization of N quite different from the classical ones. The result is derived by projection techniques similar to those applied in Section II.

II. A characterization of $N(\mu, \sigma^2 I_k)$

Let $k \geqslant 2$. Our first theorem generalizes the following result: If $X = (X_1, \dots, X_k)'$ has a characteristic function of the form $F(|x|)$ and independent components, then $X \sim N(0, \sigma^2 I_k)$. The role of $|\cdot|$ will be played by an arbitrary function of the type $h(|x_1|) + \cdots + h(|x_k|)$, where h: $[0, \infty) \to [0, \infty)$ is continuous, strictly increasing and $h(0) = 0$.

THEOREM 1. *Let h: $[0, \infty) \to [0, \infty)$ be continuous and bijective and let the k-dimensional random vector $X = (X_1, \dots, X_k)'$ have a density (or characteristic function) f of the form*

$$(3) \qquad f(x) = F\big(h(|x_1|) + \cdots + h(|x_k|)\big), \qquad x = (x_1, \dots, x_k)' \in \mathbb{R}^k,$$

for some measurable function F: $[0, \infty) \to \mathbb{C}$. Then $X_1, \dots, X_k$ are independent, if and only if

$$(4) \qquad f(x) = \beta \exp\left\{-\alpha \sum_{i=1}^{k} h(|x_i|)\right\}$$

for some α, $\beta \geqslant 0$.

PROOF. Let $f_1, \dots, f_k$ be the densities (resp. characteristic functions) of $X_1, \dots, X_k$ and $\tilde{f} := F \circ h$. The independence of $X_1, \dots, X_k$ is equivalent to

$$(5) \qquad f_1(x_1) \cdots f_k(x_k) = \tilde{f}\left(h^{-1}\left(\sum_{i=1}^{k} h(|x_i|)\right)\right).$$

If $f_i(0) = 0$ for some i, (5) and the conditions on h imply $\tilde{f} \equiv 0$ and thus $f \equiv 0$ which is impossible. So we obtain by taking $x_j = 0$ for all $j \neq i$ in (5)

$$(6) \qquad f_i(x_i) = c_i \tilde{f}(|x_i|), \qquad i = 1, \dots, k,$$

for some constants $c_1, \dots, c_k$. Inserting (6) into (5) yields

$$(7) \qquad \tilde{f}(|x_1|) \cdots \tilde{f}(|x_k|) = C\tilde{f}\left(h^{-1}\left(\sum_{i=1}^{k} h(|x_i|)\right)\right).$$

If we take $u_i = h(|x_i|)$, we arrive at

$$(8) \qquad F(u_1) \cdots F(u_k) = CF(u_1 + \cdots + u_k), \qquad u_1, \dots, u_k \in [0, \infty).$$

The only measurable solution of (8) is given by $F(u) = \beta e^{-\alpha u}$, where $\beta = C^{1/(k-1)}$ and α is a complex constant. If f is a characteristic function, it follows from $f(0) = 1$ that $\beta = 1$. Since $f(x) = f(-x)$, f is real-valued, and from $|f| \leqslant 1$ we can conclude that $\alpha \geqslant 0$. If f is a density, α, $\beta > 0$ follows from $f \geqslant 0$ and $\int f\,dx = 1$.

EXAMPLES. (i) Let $h(t) := |t|^p$ for some $p > 0$. Then $f(x) = \beta \exp\{-\alpha \sum_{i=1}^{k} |x_i|^p\}$. The normal case comes out for $p = 2$.

(ii) Let $h(t) := \log(1 + |t|^p)$, $p > 0$. Then we get $f(x) = \beta[\prod_{i=1}^{k}(1 + |x_i|^p)]^{-\alpha}$. The Cauchy distribution belongs to this class.

We now prove two theorems on $N_k(\mu, \sigma^2 I_k)$ which generalize the Maxwell characterization in different ways. The following notation is used: For $n \in \mathbb{N}$, λ^n is the n-dimensional Lebesgue measure, $\mathfrak{B}^n$ is the Borel σ-algebra in $\mathbb{R}^n$. p_H denotes the projection on the linear subspace H of $\mathbb{R}^k$, $H^\perp$ the orthogonal complement of H in $\mathbb{R}^k$. For a probability measure P on $(R^k, \mathfrak{B}^k) P_H$ is the projection $P p_H^{-1}$ of P.

THEOREM 2. *Let P be a k-dimensional rotationally symmetric distribution with density $f(x) = \tilde{f}(|x|)$. If a projection P_H of P on some lower-dimensional linear subspace H has a density f_H of the same functional form as P, that is, $f_H(x) = C\tilde{f}(|x|)$ for some normalizing constant C, then P is normal.*

This result is somewhat surprising, if one notes that for *all* rotationally symmetric P the characteristic function ϕ_H of P_H is of the same functional form as that of P ($\phi_H(\lambda) = \tilde{\phi}(|\lambda|)$, where $\zeta \to \tilde{\phi}(|\zeta|)$, $\zeta \in \mathbb{R}^k$, is the characteristic function of P).

PROOF OF THEOREM 2. We can assume that P has a continuous density f, since otherwise we can consider the convolution $P * N_k(0, I_k)$ and then use the proof below to show that $P * N_k(0, I_k) = N_k(0, \sigma^2 I_k)$ for some $\sigma^2 \geqslant 0$ which implies that $\sigma^2 \geqslant 1$ and $P = N_k(0, (\sigma^2 - 1)I_k)$. Thus let f and, consequently, also $\tilde{f}$ be continuous. Let $H_n := \{x \in \mathbb{R}^k | x_j = 0,\ j > n\}$. $p_{H_n}(X)$ has the continuous density f_{H_n} given by

$$f_{H_n}(x) = \frac{2\pi^{(k-n)/2}}{\Gamma((k-n)/2)} \int_{|x|}^{\infty} c\left(c^2 - |x|^2\right)^{(k-n-2)/2} \tilde{f}(c)\, dc \tag{8}$$

(see the remark following this proof). There are continuous functions $\tilde{f}_{H_n}, \tilde{f}_{H_n^\perp}$: $[0, \infty) \to [0, \infty)$ such that $f_{H_n}(x) = \tilde{f}_{H_n}(|x|)$, $f_{H_n^\perp}(x) = \tilde{f}_{H_n^\perp}(|x|)$ for all $x \in \mathbb{R}^k$. Now for $t > 0$ we have $\tilde{f}(t) = \tilde{f}_{H_n}(0)\tilde{f}_{H_n^\perp}(t)$. Further, if we set $m := \tilde{f}_{H_n}(0)$, we obtain

$$\begin{aligned} \tilde{f}(t) &= m\tilde{f}_{H_n^\perp}(t) = m\int_{\mathbb{R}^n} \tilde{f}\left(\sqrt{t^2 + |x|^2}\right)\lambda^n(dx) \\ &= m^2 \int_{\mathbb{R}^n} \tilde{f}_{H_n^\perp}\left(\sqrt{t^2 + |x|^2}\right)\lambda^n(dx) \\ &= m^2 \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \tilde{f}\left(\sqrt{t^2 + |x^{(1)}|^2 + |x^{(2)}|^2}\right)\lambda^n(dx^{(1)})\lambda^n(dx^{(2)}) \\ &= m^2 \int_{\mathbb{R}^{2n}} \tilde{f}\left(\sqrt{t^2 + |x|^2}\right)\lambda^{2n}(dx) \\ &= \cdots = m^j \int_{\mathbb{R}^{jn}} \tilde{f}\left(\sqrt{t^2 + |x|^2}\right)\lambda^{jn}(dx) \end{aligned} \tag{9}$$

for all $j \in \mathbb{N}$. Let Q_j be the probability measure on $\mathbb{R}^{(j+1)n}$ with density $y \to C_j \tilde{f}(|y|)$, $y \in \mathbb{R}^{(j+1)n}$, where C_j is some normalizing constant. By (9), P can be expressed as a projection of Q_j. Since this property of P holds for arbitrarily large j, it follows from Theorem 2 of [3], p. 394, that P is a variance mixture of the normal distributions $N_k(0, \sigma^2 I_k)$. Hence f can be written as

$$f(x) = \int_0^\infty (2\pi)^{-k/2}\sigma^{-k}e^{-|x|^2/2\sigma^2}\alpha(d\sigma), \qquad x \in \mathbb{R}^k, \tag{10}$$

for some probability measure α on $(0, \infty)$.

Inserting (10) into the first line of (9) one obtains

$$\begin{aligned} &\int_0^\infty (2\pi)^{-k/2}\sigma^{-k}e^{-|x|^2/2\sigma^2}\alpha(d\sigma) \\ &\quad = m\int_0^\infty \int_{\mathbb{R}^n} (2\pi)^{-k/2}\sigma^{-k}e^{-(|x|^2+|y|^2)/2\sigma^2}\lambda^n(dy)\alpha(d\sigma) \\ &\quad = \int_0^\infty m(2\pi)^{(n-k)/2}\sigma^{n-k}e^{-|x|^2/2\sigma^2}\alpha(d\sigma). \end{aligned} \tag{11}$$

Next we have to note that the mapping $f \to \alpha$ is one-to-one: If one defines the probability measure $\bar{\alpha}(0, t] := \alpha(0, \sqrt{t}]$, the Laplace transform of $\bar{\alpha}$ coincides with the characteristic function ϕ of f, since

$$\phi(u) = \int_0^\infty e^{-|u|^2\sigma^2/2}\alpha(d\sigma) = \int_0^\infty e^{-|u|^2\sigma/2}\bar{\alpha}(d\sigma). \tag{12}$$

But α is determined by $\bar{\alpha}$, and $\bar{\alpha}$ by its Laplace transform which can be computed from f.

Hence the measures $\alpha(d\sigma)$ and $m(2\pi)^{n/2}\sigma^n\alpha(d\sigma)$ are equal which is possible only if α is the point mass in $(2\pi)^{-1/2}m^{-1/n}$. By (10) it is now seen that

$$f(x) = m^{k/n}e^{-|x|^2m^{2/n}\pi}. \tag{13}$$

The proof is now complete.

Remark. Equation (8) follows easily from the results of Eaton [3]. He proved that if X is rotationally symmetric, then $X/|X|$ and $|X|$ are independent, and $p_{H_n}(X/|X|)$ has the density

$$g_{k,n}(u) = \frac{\Gamma\left(\frac{k}{2}\right)}{\pi^{n/2}\Gamma((k-n)/2)}\left(1 - |u|^2\right)^{(k-n-2)/2}1_{(-1,1)}(|u|) \tag{14}$$

for $u \in H_n$. Hence,

$$\begin{aligned} P\left(p_{H_n}(X) \leqslant t\right) &= \int P\left(p_{H_n}(X) \leqslant t \mid |X| = c\right)P(|X| \in dc) \\ &= \int P\left(p_{H_n}(X/|X|) \leqslant \frac{t}{c}\right)\nu(dc), \end{aligned} \tag{15}$$

where $t \in \mathbb{R}^k$ and $\nu(a, b] := P\{x \mid |x| \in (a, b]\}$, $a, b \in \mathbb{R}$. Inserting (15) into (14) shows that $p_{H_n}(X)$ has the density

$$(16) \qquad f_{H_n}(x) = \frac{\Gamma(k/2)}{\pi^{n/2}\Gamma((k-n)/2)} \int_{|x|}^{\infty} c^{-n}\left(1 - \frac{|x|^2}{c^2}\right)^{(k-n-2)/2} \nu(dc).$$

(8) now follows, because ν has the density

$$(17) \qquad \left(2\pi^{k/2}/\Gamma(k/2)\right)c^{k-1}\tilde{f}(c)1_{(0,\infty)}(c), \qquad c \in \mathbb{R}.$$

Formula (15) incidentally shows that f_{H_n} is non-increasing, if $k - n \geqslant 2$, as has been shown in [3], Proposition 1. However this is not true for $k - n = 1$ (a case not excluded in [3]), as can be seen by the example of the uniform distribution on S^1. Then

$$(18) \qquad f_{H_1}(u) = 1/\pi(1 - u^2)^{1/2}, \qquad -1 < u < 1.$$

The final theorem of this section shows that a "Maxwell characterization" of $N_k(\mu, \sigma^2 I_k)$ is possible without the assumption of rotational symmetry.

THEOREM 3. *Let X be a k-dimensional random vector for which there exists a $n \in \{1, \ldots, k-1\}$ such that $p_H(X)$ and $p_{H^\perp}(X)$ are independent for each n-dimensional linear subspace H of $\mathbb{R}^k$. Then $X \sim N_k(\mu, \sigma^2 I_k)$ for some $\mu \in \mathbb{R}^k$ and* $\sigma^2 \geqslant 0$.

PROOF. Let P be the distribution of X. We have to show that P is rotationally symmetric, for then Theorem 2 can be applied.

If the result holds for all symmetric probability measures Q (satisfying $Q(B) = Q(-B)$ for all $B \in \mathfrak{B}^k$), it is true in general. For if we define $\overline{P}(B) := P(-B)$, $B \in \mathfrak{B}^k$, the convolution $\tilde{P} = P * \overline{P}$ of P and $\overline{P}$ clearly fulfils the assumptions of the theorem and is symmetric so that we have $\tilde{P} = N_k(0, \sigma^2 I_k)$ for some $\sigma \in [0, \infty)$. Then the multivariate version of Cramèr's theorem ([9], page 46) implies that $P = N_k(\mu, \frac{1}{2}\sigma^2 I_k)$ for some $\mu \in \mathbb{R}^k$.

Hence we can assume that P is symmetric. We show that in this case P is even rotationally symmetric. It suffices to prove that, if λ, $\tilde{\lambda} \in \mathbb{R}^k$ satisfy $|\lambda| = |\tilde{\lambda}|$, then $\phi(\lambda) = \phi(\tilde{\lambda})$, where ϕ is the characteristic function of P. We can choose an orthonormal basis $e^1, \ldots, e^k$ of $\mathbb{R}^k$ such that $\lambda = (\lambda_1, \lambda_2, 0, \ldots, 0)'$ and $\tilde{\lambda} = (-\lambda_1, \lambda_2, 0, \ldots, 0)'$ are the coordinate representations of λ and $\tilde{\lambda}$ with respect to this basis. Let H be the span of $e^2, \ldots, e^{n+1}$. Then clearly $p_H(\lambda) = (0, \lambda_2, 0, \ldots, 0)' = p_H(\tilde{\lambda})$ and $p_{H^\perp}(\lambda) = (\lambda_1, 0, \ldots, 0)' = -p_{H^\perp}(\tilde{\lambda})$. Now it follows from the assumptions that

$$(19) \qquad \begin{aligned} \phi(\lambda) &= \phi(p_H(\lambda))\phi(p_{H^\perp}(\lambda)) = \phi(p_H(\tilde{\lambda}))\phi(-p_{H^\perp}(\tilde{\lambda})) \\ &= \phi(p_H(\tilde{\lambda}))\phi(p_{H^\perp}(\tilde{\lambda})) = \phi(\tilde{\lambda}). \end{aligned}$$

III. A variational inequality for N

Our final aim is to prove the inequality for N announced in the introduction: Among all Borel sets B of given N-measure the half spaces minimize the N-measure of their euclidean ε-neighbourhood for arbitrary $\varepsilon > 0$. This result indicates a similarity between the relation of S^{k-1} to its surface measure and that of $\mathbb{R}^k$ to the standard normal distribution.

Let μ_k be the normalized surface area of S^{k-1}. In the following μ_k is considered as a measure on $(\mathbb{R}^k, \mathfrak{B}^k)$ concentrated on S^{k-1} by setting $\mu_k(B) := \mu_k(B \cap S^{k-1})$. We first note that the conclusion (2) in Section I remains valid if K is replaced by an arbitrary $B \in \mathfrak{B}^k$, because for all $\varepsilon' < \varepsilon$

$$\begin{aligned}(20) \qquad \mu_k\big(U^{\Delta}_{k,\varepsilon'}(H)\big) &\leqslant \mu_k\{y \in \mathbb{R}^k \,|\, \exists x \in \mathrm{cl}(B): |x-y| < \varepsilon\} \\ &= \mu_k\{y \in \mathbb{R}^k \,|\, \exists x \in B: |x-y| < \varepsilon\} \\ &\leqslant \mu_k\{U^{\Delta}_{k,\varepsilon}(B \cap S^{k-1})\}.\end{aligned}$$

Let $\varepsilon' \uparrow \varepsilon$.

THEOREM 3. *Let $B \in \mathfrak{B}^k$ and $H = \{x \in \mathbb{R}^k \,|\, x'x^0 \leqslant a\}$, $a \in \mathbb{R}$, $x^0 \in \mathbb{R}^k$. Then we have*

$$(21) \qquad N(H) \leqslant N(B) \Rightarrow N(H^\varepsilon) \leqslant N(B^\varepsilon) \quad \textit{for all } \varepsilon > 0.$$

PROOF. By p_{nk}: $\mathbb{R}^n \to \mathbb{R}^k$ we denote the projection $(x_1, \ldots, x_n)' \to (x_1, \ldots, x_k)'$, $n > k$. Let $\mu_{n,c}$ be the uniform distribution on $cS^{k-1} := \{x \in \mathbb{R}^k \,|\, |x| = c\}$ (also considered as a measure on $\mathbb{R}^k$). By (18) we immediately obtain

$$(22) \qquad \mu_{n,\sqrt{n}}\, p_{nk}^{-1} \to N \quad \text{in total variation, as } n \to \infty,$$

because the corresponding sequence of densities converges pointwise to the density of N. Without loss of generality we assume $x^0 = (1, 0, \ldots, 0)'$ and consequently $H = \{x \in \mathbb{R}^k \,|\, x_1 \leqslant a\}$. Let $\Delta_{n,c}$ be the geodesic metric on cS^{n-1} and $U^{\Delta}_{n,c,\varepsilon}(B)$ be the closed ε-neighbourhood with respect to $\Delta_{n,c}$. Then it is seen that

$$(23) \quad U^{\Delta}_{n,\sqrt{n},\varepsilon}\big(\sqrt{n}\, S^{n-1} \cap p_{nk}^{-1}(H)\big) = p_{nk}^{-1}\big(\{x \in \mathbb{R}^k \,|\, x_1 \leqslant a + \varepsilon_n\}\big) \cap \sqrt{n}\, S^{n-1},$$

where ε and ε_n satisfy the equation

$$(24) \qquad \varepsilon = \sqrt{n}\left(\arccos\left(\frac{a}{\sqrt{n}}\right) - \arccos\left(\frac{a + \varepsilon_n}{\sqrt{n}}\right)\right).$$

The Taylor expansion of arccos yields

$$\varepsilon = \left(1 - \xi_n^2\right)^{-1/2} \varepsilon_n \qquad \text{for some } \xi_n \in \left[a/\sqrt{n}, (a + \varepsilon_n)/\sqrt{n}\right].$$

From this it is seen that $\lim_{n \to \infty} \varepsilon_n = \varepsilon$. By (22) for each $\delta > 0$ there exists a $n_0 \in \mathbb{N}$ such that for $n \geqslant n_0$

$$\begin{aligned}
(25) \quad & \mu_{n,\sqrt{n}}\left(U^{\Delta}_{n,\sqrt{n},\varepsilon}\left(\sqrt{n}\, S^{n-1} \cap p_{nk}^{-1}(H)\right)\right) \\
&= \mu_{n,\sqrt{n}}\left(p_{nk}^{-1}\left(\left\{x \in \mathbb{R}^k \mid x_1 \leqslant a + \varepsilon_n\right\}\right)\right) \\
&\in \left[N\left\{x \in \mathbb{R}^k \mid x_1 \leqslant a + \varepsilon - \delta\right\}, N\left\{x \in \mathbb{R}^k \mid x_1 \leqslant a + \varepsilon + \delta\right\}\right].
\end{aligned}$$

From (25) it follows that

$$(26) \qquad \lim_{n \to \infty} \mu_{n,\sqrt{n}}\left(U^{\Delta}_{n,\sqrt{n},\varepsilon}\left(\sqrt{n}\, S^{n-1} \cap p_{nk}^{-1}(H)\right)\right) = N(H^{\varepsilon}).$$

Further from $|x - y| \leqslant \Delta_{n,\sqrt{n}}(x, y)$ for all $x, y \in \sqrt{n}\, S^{n-1}$ we can conclude that

$$(27) \qquad U^{\Delta}_{n,\sqrt{n},\varepsilon}\left(p_{nk}^{-1}(B)\right) \subset p_{nk}^{-1}(B^{\varepsilon}) \cap \sqrt{n}\, S^{n-1}.$$

Set $H_b := \{x \in \mathbb{R}^k \mid x_1 \leqslant b\}$. Under the supposition of (21) (22) shows that for each $b < a$ there is a $n_1 \in \mathbb{N}$ so that

$$(28) \qquad \mu_{n,\sqrt{n}}\left(p_{nk}^{-1}(H_b)\right) \leqslant \mu_{n,\sqrt{n}}\left(p_{nk}^{-1}(B)\right) \qquad \text{for all } n \geqslant n_1.$$

The isoperimetric inequality (2), when applied to $\sqrt{n}\, S^{n-1}$, yields for all $\varepsilon > 0$ and $n \geqslant n_1$

$$(29) \quad \mu_{n,\sqrt{n}}\left(U^{\Delta}_{n,\sqrt{n},\varepsilon}\left(p_{nk}^{-1}(H_b) \cap \sqrt{n}\, S^{n-1}\right)\right) \leqslant \mu_{n,\sqrt{n}}\left(U^{\Delta}_{n,\sqrt{n},\varepsilon}\left(p_{nk}^{-1}(B) \cap \sqrt{n}\, S^{n-1}\right)\right).$$

Further we have (again by (22))

$$(30) \qquad \lim_{n \to \infty} \mu_{n,\sqrt{n}}\left(p_{nk}^{-1}(B^{\varepsilon})\right) = N(B^{\varepsilon}).$$

Finally combining (26) (applied to H_b), (24), (27) and (30) we obtain

$$\begin{aligned}
(31) \qquad N(H_b^{\varepsilon}) &= \lim_{n \to \infty} \mu_{n,\sqrt{n}}\left(U^{\Delta}_{n,\sqrt{n},\varepsilon}\left(p_{nk}^{-1}(H_b) \cap \sqrt{n}\, S^{n-1}\right)\right) \\
&\leqslant \liminf_{n \to \infty} \mu_{n,\sqrt{n}}\left(U^{\Delta}_{n,\sqrt{n},\varepsilon}\left(p_{nk}^{-1}(B) \cap \sqrt{n}\, S^{n-1}\right)\right) \\
&\leqslant \liminf_{n \to \infty} \mu_{n,\sqrt{n}}\left(p_{nk}^{-1}(B^{\varepsilon})\right) = N(B^{\varepsilon}).
\end{aligned}$$

$b \uparrow a$ yields the conclusion of (21).

Whether N is the only distribution on $\mathbb{R}^k$ which satisfies (21) is an open question.

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