MODULI SPACES OF VECTOR BUNDLES OVER RULED SURFACES

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Abstract. We study moduli spaces $M(c_1, c_2, d, r)$ of isomorphism classes of algebraic 2-vector bundles with fixed numerical invariants $c_1, c_2, d, r$ over a ruled surface. These moduli spaces are independent of any ample line bundle on the surface. The main result gives necessary and sufficient conditions for the non-emptiness of the space $M(c_1, c_2, d, r)$ and we apply this result to the moduli spaces $M_L(c_1, c_2)$ of stable bundles, where $L$ is an ample line bundle on the ruled surface.

Introduction

Let $\pi : X \rightarrow C$ be a ruled surface over a smooth algebraic curve $C$, defined over the complex number field $\mathbb{C}$. Let $f$ be a fibre of $\pi$. Let $c_1 \in \text{Num}(X)$ and $c_2 \in H^4(X, \mathbb{Z}) \cong \mathbb{Z}$ be fixed. For any polarization $L$, denote the moduli space of rank-2 vector bundles stable with respect to $L$ in the sense of Mumford-Takemoto by $\mathcal{M}_L(c_1, c_2)$. Stable 2-vector bundles over a ruled surface have been investigated by many authors; see, for example [T1], [T2], [H-S], [Q1]. Let us mention that Takemoto [T1] showed that there is no rank-2 vector bundle (having $c_1, f$ even) stable with respect to every polarization $L$. In this paper we shall study algebraic 2-vector bundles over ruled surfaces, but we adopt another point of view: we shall study moduli spaces of (algebraic) 2-vector bundles over a ruled surface $X$, which are defined independent of any ample divisor (line bundle) on $X$, by taking into account the special geometry of a ruled surface (see [B], [B-St1], [B-St2] and also [Br1], [Br2], [W]).

In Section 1 (put for the convenience of the reader) we present (see [B]) two numerical invariants $d$ and $r$ for a 2-vector bundle with fixed Chern classes $c_1$ and $c_2$ and we define the set $M(c_1, c_2, d, r)$ of isomorphism classes of bundles with fixed invariants $c_1$, $c_2$, $d$, $r$. The integer $d$ is given by the splitting of the bundle on the general fibre and the integer $r$ is given by some normalization of the bundle. Recall that the set $M(c_1, c_2, d, r)$ carries

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a natural structure of an algebraic variety (see [B], [B-St1], [B-St2]). In Section 2 we study uniform vector bundles and we prove the existence of algebraic vector bundles given by extensions of line bundles and which are not uniform. In Section 3 the main result gives necessary and sufficient conditions for the non-emptiness of the space $M(c_1, c_2, d, r)$ and we apply this result to the moduli space of stable bundles $M_L(c_1, c_2)$.

§1. Moduli spaces of rank-2 vector bundles

In this section we shall recall from ([B], [B-St1], [B-St2]) some basic notions and facts.

The notations and the terminology are those of Hartshorne’s book [Ha]. Let $C$ be a nonsingular curve of genus $g$ over the complex number field and let $\pi : X \to C$ be a ruled surface over $C$. We shall write $X = \mathbb{P}(E)$ where $E$ is normalized. Let us denote by $e$ the divisor on $C$ corresponding to $\mathbb{V}^2(E)$ and by $e = -\deg(e)$. We fix a point $p_0 \in C$ and a fibre $f_0 = \pi^{-1}(p_0)$ of $X$. Let $C_0$ be a section of $\pi$ such that $\mathcal{O}_X(C_0) \cong \mathcal{O}_{\mathbb{P}(E)}(1)$.

Any element of $\text{Num}(X) \cong H^2(X, \mathbb{Z})$ can be written $aC_0 + bf_0$ with $a, b \in \mathbb{Z}$. We shall denote by $\mathcal{O}_C(1)$ the invertible sheaf associated to the divisor $p_0$ on $C$. If $L$ is an element of $\text{Pic}(C)$ we shall write $L = \mathcal{O}_C(k) \otimes L_0$, where $L_0 \in \text{Pic}_0(C)$ and $k = \deg(L)$. We also denote by $F(aC_0 + bf_0) = F \otimes \mathcal{O}_X(a) \otimes \pi^*\mathcal{O}_C(b)$ for any sheaf $F$ on $X$ and any $a, b \in \mathbb{Z}$.

Let $E$ be an algebraic rank-2 vector bundle on $X$ with fixed numerical Chern classes $c_1 = (\alpha, \beta) \in H^2(X, \mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z}$, $c_2 = \gamma \in H^4(X, \mathbb{Z}) \cong \mathbb{Z}$, where $\alpha, \beta, \gamma \in \mathbb{Z}$.

Since the fibres of $\pi$ are isomorphic to $\mathbb{P}^1$ we can speak about the generic splitting type of $E$ and we have $E|_f \cong \mathcal{O}_f(d) \oplus \mathcal{O}_f(d')$ for a general fibre $f$, where $d' \leq d$, $d + d' = \alpha$. The integer $d$ is the first numerical invariant of $E$.

The second numerical invariant is obtained by the following normalization:

$$-r = \inf\{l \mid \exists L \in \text{Pic}(C), \deg(L) = l, \text{ s.t. } H^0(X, E(-dC_0) \otimes \pi^*L) \neq 0\}.$$ 

We shall denote by $M(\alpha, \beta, \gamma, d, r)$ or $M(c_1, c_2, d, r)$ or $M$ the set of isomorphism classes of algebraic rank-2 vector bundles on $X$ with fixed Chern classes $c_1$, $c_2$ and invariants $d$ and $r$.

With these notations we have the following result (see [B]):
Theorem 1. For every vector bundle $E \in M(c_1, c_2, d, r)$ there exist $L_1, L_2 \in \text{Pic}_0(C)$ and $Y \subset X$ a locally complete intersection of codimension 2 in $X$, or the empty set, such that $E$ is given by an extension

\[(1) \quad 0 \to \mathcal{O}_X(dC_0 + rf_0) \otimes \pi^* L_2 \to E \to \mathcal{O}_X(d'C_0 + sf_0) \otimes \pi^* L_1 \otimes I_Y \to 0,\]

where $c_1 = (\alpha, \beta) \in \mathbb{Z} \times \mathbb{Z}$, $c_2 = \gamma \in \mathbb{Z}$, $d + d' = \alpha$, $d \geq d'$, $r + s = \beta$, $l(c_1, c_2, d, r) := \gamma + \alpha(de - r) - \beta d + 2dr - d^2e = \deg(Y) \geq 0$.

Remark. By applying Theorem 1 we can obtain the canonical extensions used in [Br1], [Br2].
Indeed, let us suppose first that $d > d'$. From the exact sequence (1) it follows that

\[\mathcal{O}_C(r) \otimes L_2 \cong \pi_* E(-dC_0)\]

so

\[\mathcal{O}_X(rf_0) \otimes \pi^* L_2 \cong \pi^* \pi_* E(-dC_0)\]

and

\[\mathcal{O}_X(dC_0 + rf_0) \otimes \pi^* L_2 \cong (\pi^* \pi_* E(-dC_0))(dC_0).\]

If $d = d'$ then, by applying $\pi_*$ to the short exact sequence

\[0 \to \mathcal{O}_X(rf_0) \otimes \pi^* L_2 \to E(-dC_0) \to \mathcal{O}_X(sf_0) \otimes \pi^* L_1 \otimes I_Y \to 0\]

it follows the exact sequence

\[0 \to \mathcal{O}_C(r) \otimes L_2 \to \pi_* E(-dC_0) \to \mathcal{O}_C(s) \otimes L_1 \otimes \mathcal{O}_C(-Z_1) \to 0,\]

where $Z_1$ is an effective divisor on $C$ with the support $\pi(Y)$. With the notation $Z = \pi^{-1}(Z_1)$, by applying $\pi^*$ ($\pi$ is a flat morphism) we obtain the following commutative diagram with exact rows

\[
\begin{array}{ccc}
0 & \to & \mathcal{O}_X(rf_0) \otimes \pi^* L_2 \to E(-dC_0) \to \mathcal{O}_X(sf_0) \otimes \pi^* L_1 \otimes I_Y \to 0 \\
\downarrow \psi & & \downarrow \psi \\
0 & \to & \mathcal{O}_X(rf_0) \otimes \pi^* L_2 \to \pi^* \pi_* E(-dC_0) \to \mathcal{O}_X(sf_0) \otimes \pi^* L_1 \otimes I_Z \to 0
\end{array}
\]
From the injectivity of $\psi$ we obtain the injectivity of $\varphi$. Because of
\[ O_X(sf_0) \otimes \pi^*L_1 \otimes I_Y \cong \text{Coker } \psi \cong \text{Coker } \varphi \]
we conclude.

Recall that a set $M$ of vector bundles on a $\mathbb{C}-$scheme $X$ is called \textit{bounded} if there exists an algebraic $\mathbb{C}$-scheme $T$ and a vector bundle $V$ on $T \times X$ such that every $E \in M$ is isomorphic with $V_t = V|_{t \times X}$ for some closed point $t \in T$ (see [K]).

For the next result see [B]:

\textbf{THEOREM 2.} The set $M(c_1, c_2, d, r)$ is bounded.

\section*{2. Uniform bundles}

In what follows, we keep the notations from Section 1.

\textbf{DEFINITION 3.} A 2-vector bundle $E$ is called an \textit{uniform bundle} if the splitting type is preserved on all fibres of $X$.

Theorem 1 allows us to give a criterion for uniformness.

\textbf{LEMMA 4.} Let $f$ be a fibre of $X$ and let us suppose that $I_Y \cap f \cong O_f(-n)$. Then $E|_f \cong O_f(d + n) \oplus O_f(d' - n)$.

\textit{Proof.} We suppose that $E|_f \cong O_f(a) \oplus O_f(a')$, where $a \geq a'$. Then we have a surjective morphism
\[ E|_f \rightarrow O_f(d') \otimes I_Y \otimes O_f \]
in virtue of Theorem 1. On the other hand, the restriction of the sequence
\[ 0 \rightarrow I_Y \rightarrow O_X \rightarrow O_Y \rightarrow 0 \]
to $f$ gives a surjective morphism
\[ I_Y \otimes O_f \rightarrow I_Y \cap f \cong O_f(-n). \]
So, we obtain another surjective morphism
\[ O_f(a) \oplus O_f(a') \rightarrow O_f(d' - n). \]
By using the inequalities $a \geq a'$, $d \geq d' \geq d' - n$ and the equality $a + a' = d + d' = \alpha$ it follows that $a' = d' - n$ and $a = d + n$. 
Corollary 5. \( E \) is an uniform bundle if and only if \( l(c_1, c_2, d, r) = 0 \).

By means of Corollary 5 the uniform bundles are given by extensions of line bundles. It is naturally to ask if the converse is true. Unfortunately, this question has a negative answer, as proved by the following

Proposition 6. On the rational ruled surface \( \mathbb{F}_e \) with \( e \geq 1 \) there exist non-uniform bundles given by extensions of line bundles.

For the proof we need some preparations.

Let \( E \) be a 2-vector bundle given by an extension

\[
0 \to F \to E \to G \to 0,
\]

where \( F = \mathcal{O}_X(aC_0 + r'f_0) \otimes \pi^*L_2', G = \mathcal{O}_X(a'C_0 + s f_0) \otimes \pi^*L_1' (L_1', L_2' \in \text{Pic}_0(C)) \) are line bundles on \( X \). By means of Theorem 1, \( E \) sits also in a canonical extension (1). If \( a \geq a' \) then \( E \) is obviously uniform. Then, we shall suppose that \( a < a' \).

Lemma 7. With the above notations we have \( d \leq a' \).

Proof. Indeed, by the restriction of the sequence (2) to a general fibre \( f \) we obtain a surjective morphism

\[
\mathcal{O}_f(d) \oplus \mathcal{O}_f(d') \to \mathcal{O}_f(a').
\]

If \( d > a' \), then it follows that \( d' = a' \) which contradicts the inequalities \( a < a', d \geq d' \). (\( a + a' = d + d' \)).

Lemma 8. If \( d = a' \) then \( E \) is uniform.

Proof. Let \( f \) be a fibre of \( X \) such that the splitting type of \( E|_f \) is different from the generic splitting type of \( E \). According to Lemma 4

\[
E|_f \cong \mathcal{O}_f(d + n) \oplus \mathcal{O}_f(d' - n),
\]

where \( n > 0 \).

By the restriction of (2) to \( f \) we obtain a surjective morphism

\[
\mathcal{O}_f(d + n) \oplus \mathcal{O}_f(d' - n) \to \mathcal{O}_f(d).
\]

Because of \( d + n > d \) it follows \( d' - n = d \), contradiction.
Lemma 9. In the above hypotheses, if \( d = a' \), then \( E \cong F \oplus G \).

Proof. Let us observe that we can suppose, without loss of generality, that \( a = 0 \) and \( r' = 0 \) (by twisting the sequences (1) and (2) with \( \mathcal{O}_X(-aC_0 - r'f_0) \)). Then, it follows that \( d = a' = \alpha > 0 \), \( s' = \beta \) and \( d' = 0 \). Therefore, the sequences (1) and (2) become:

\[
\begin{array}{ccccccccc}
0 & \rightarrow & \mathcal{O}_X(\alpha C_0 + \beta f_0) & \otimes & \pi^* L_1' & \rightarrow & E & \rightarrow & \mathcal{O}_X(s f_0) & \otimes & \pi^* L_1 & \otimes & I_Y & \rightarrow & 0 \\
& & \chi & \downarrow \varphi & & & & \psi & \downarrow \varphi & & & & \downarrow \varphi & & & \\
(1') & 0 & \rightarrow & \mathcal{O}_X(\alpha C_0 + r f_0) & \otimes & \pi^* L_2 & \rightarrow & E & \rightarrow & \mathcal{O}_X(s f_0) & \otimes & \pi^* L_1 & \otimes & I_Y & \rightarrow & 0 \\
& & & & & & & & & & & & & & \\
& & & & & \pi^* L_2' & & & & & & & & & & \\
\end{array}
\]

The computation of \( c_2(E) \) in (1') gives \( \deg(Y) = -\alpha s \). Moreover, by means of Lemma 8, \( \deg(Y) = 0 \), so \( s = 0 \) (we supposed \( \alpha > 0 \)).

The homomorphism \( \chi = \varphi \psi \) is non-zero, otherwise \( \mathcal{O}_X(\alpha C_0 + \beta f_0) \subset \pi^*(L_2') \) (which would contradict the condition \( \alpha > 0 \)), so \( L_2 = L_1' \) and \( \chi \) is the multiplication by a \( \lambda \in \mathbb{C}^* \), and the assertion follows.

In this moment, we are able to give the counter-example announced in Proposition 6.

Proof of Proposition 6. Let \( G \) be \( \mathcal{O}_X(2C_0) \) and let \( F \) be \( \mathcal{O}_X \). Then:

\[
\dim H^1(G^{-1}) = e + 1 \neq 0.
\]

For \( E \) given by an extension \( \xi \in \text{Ext}^1(G, \mathcal{O}_X) \), keeping the notations from Section 1, we have \( d \leq 2 \) (Lemma 7), \( d' \geq d' \), \( d + d' = 2 \) and \( r + s = 0 \).

There are only two possibilities:

(a) \( d = 2 \), \( d' = 0 \), which implies \( E \cong \mathcal{O}_X \oplus \mathcal{O}_X(2C_0) \) (Lemma 9).

(b) \( d = d' = 1 \) and, in this case, in the canonical extension (1) of \( E \), we have
\[ \text{deg}(Y) = dd'e - ds - d'r = e \geq 1. \]

By applying Corollary 5, all vector bundles given by non-zero extensions from \( \text{Ext}^1(G, \mathcal{O}_X) \) are non-uniform.

§3. Non-emptiness of moduli spaces

For a rank-2 vector bundle \( E \), we shall denote by \( d_E \) and \( r_E \) the invariants of \( E \), when confusions may appear.

**Theorem 10.** \( M(c_1, c_2, d, r) \) is non-empty if and only if \( l := l(c_1, c_2, d, r) \geq 0 \) and one of the following conditions holds:

(I) \( 2d > \alpha \) or,

(II) \( 2d = \alpha, \beta - 2r \leq g + l \).

**Proof.** We observe that if \( M \neq \emptyset \) then, by means of Theorem 1, the elements of \( M \) lie among 2-vector bundles given by extensions of type (1). Therefore, we conclude that \( M \neq \emptyset \) if and only if in the extensions of type (1) there are 2-vector bundles with \( d_E = d \) and \( r_E = r \).

It is clear that all the vector bundles given by an extension of type (1) have \( d_E = d \) so we shall look for bundles with \( r_E = r \).

We fix \( L_1, L_2 \in \text{Pic}_0(C) \) and \( Y \subset X \) a locally complete intersection (or the empty set) and we denote

\[ N_1 = \mathcal{O}_X(dC_0 + sf_0) \otimes \pi^*L_1 \]
\[ N_2 = \mathcal{O}_X(dC_0 + rf_0) \otimes \pi^*L_2 \]

and \( l = \text{deg}(Y) \).

Consider the spectral sequence of terms

\[ E_2^{p,q} = H^p(X, \mathcal{E}xt^q(I_Y \otimes N_1, N_2)) \]

which converges to

\[ \text{Ext}^{p+q}(I_Y \otimes N_1, N_2). \]

We have

\[ \mathcal{E}xt^0(I_Y \otimes N_1, N_2) \cong N_2 \otimes N_1^{-1} \quad \text{and} \quad \mathcal{E}xt^1(I_Y \otimes N_1, N_2) \cong \mathcal{O}_Y. \]

But \( H^2(X, N_2 \otimes N_1^{-1}) = 0 \) so the exact sequence of lower terms becomes

\[ 0 \rightarrow H^1(X, N_2 \otimes N_1^{-1}) \rightarrow \text{Ext}^1(I_Y \otimes N_1, N_2) \rightarrow H^0(Y, \mathcal{O}_Y) \rightarrow 0. \]
Now, by a result due to Serre (see [O-S-S], Chap.I, 5, [Se]), any element in the group \( \text{Ext}^1(Y \otimes N_1, N_2) \) which has an invertible image in \( H^0(Y, \mathcal{O}_Y) \) defines an extension of the desired form with \( E \) a 2-vector bundle.

We write the sequence (1) under the equivalent form

\[
0 \to \mathcal{O}_X \to E(-dC_0) \otimes \pi^* L' \to \mathcal{O}_X((d' - d)C_0 + (s - r)f_0) \otimes \pi^*(\tilde{L}) \otimes I_Y \to 0
\]

where \( \tilde{L} = L_1 \otimes L_2^{-1} \), \( L'' = \mathcal{O}_C(-r) \otimes L_2^{-1} \) and \( \deg(L'') = -r \).

From the definition, it follows \( r \leq r_E \) for every bundle \( E \) given by an extension (1). We distinguish three cases:

(I) \( d > d' \). In this case we shall prove that \( M \) is non-empty if and only if \( l \geq 0 \). To do this we prove that all vector bundles from extension (1) have \( r_E = r \).

We verify that for all \( L' \in \text{Pic}(C) \) with \( \deg(L') < 0 \) we have

\[
H^0(E(-dC_0) \otimes \pi^*(L'' \otimes L')) = 0,
\]

which is true because \( H^0(L') = 0 \) and

\[
H^0(\mathcal{O}_X((d' - d)C_0 + (s - r)f_0) \otimes \pi^*(L_1 \otimes L_2^{-1} \otimes L') \otimes I_Y) = 0.
\]

(II) \( a^o \). \( d = d' \), \( r \geq s \). Then \( M \) is non-empty if and only if \( l \geq 0 \). The proof runs like in the first case with the remark \( \deg(\mathcal{O}_C(s-r) \otimes L_1 \otimes L_2^{-1} \otimes L') < 0 \).

(II) \( b^o \). \( d = d' \), \( r < s \). Then \( M \) is non-empty if and only if \( l \geq 0 \) and \( \beta - 2r \leq g + l \).

Let us see first that the natural isomorphism

\[
M(2d, \beta, \gamma, d, r) \to M(0, \beta, l, 0, r)
\]

allows us to suppose \( d = d' = 0 \).

In this case, the sequence (3) becomes

\[
0 \to \mathcal{O}_X \to E \otimes \mathcal{O}_X(-rf_0) \otimes \pi^* L_2^{-1} \to \mathcal{O}_X((s - r)f_0) \otimes \pi^*(L_1 \otimes L_2^{-1}) \otimes I_Y \to 0.
\]

The definition of the second invariant implies that \( r_E = r \) if and only if \( \pi^*E \otimes \mathcal{O}_C(-rp_0) \otimes L_2^{-1} \) is normalised. \( E' \) belong to an extension

\[
0 \to \mathcal{O}_C \to E' \to L \to 0.
\]

where \( L = \mathcal{O}_C((s-r)p_0) \otimes L_1 \otimes L_2^{-1} \otimes \mathcal{O}_C(-Z_1) \) with \( Z_1 \) an effective divisor on \( C \) with support \( \pi(Y) \) and \( \text{card}(Y) \leq \deg(Z_1) \leq l = \deg(Y) \).

According to a result of Nagata ([N] or [Ha] Ex.V.2.5), if \( E' \) is normalised, then
\[-\deg(E') = r - s + \deg(Z_1) \geq -g\]

which proves “only if” part of (II)b°.

For “if” part we choose \( Y \) reduced, obtained by intersection between \( C_0 \) and \( l \) distinct fibres of \( X \). In this case, we have the following short exact sequence

\[0 \to I_Z \to I_Y \to I_{Y \subset Z} \to 0\]

where \( Z_1 = \pi(Y) = p_1 + \cdots + p_l \), \( Y \subset Z = \pi^{-1}(Z_1) = f_1 + \cdots + f_l \) with \( f_i \) distinct fibres, \( \mathcal{O}_Z = \mathcal{O}_{f_1} \oplus \cdots \oplus \mathcal{O}_{f_l} \), \( I_{Y \subset Z} = \mathcal{O}_{f_1}(-1) \oplus \cdots \oplus \mathcal{O}_{f_l}(-1) \).

So, the sequence (5) becomes

\[0 \to I_Z \to I_Y \to \mathcal{O}_{f_1}(-1) \oplus \cdots \oplus \mathcal{O}_{f_l}(-1) \to 0.\]

Tensoring by \( K_X \otimes N_2^{-1} \otimes N_1 \) and taking the long cohomology sequence we obtain an injective map:

\[H^1(K_X \otimes N_2^{-1} \otimes N_1 \otimes I_Z) \to H^1(K_X \otimes N_2^{-1} \otimes N_1 \otimes I_Y).\]

By dualizing, it follows that the natural map

\[\text{Ext}^1(I_Y \otimes N_1, N_2) \xrightarrow{\varphi} \text{Ext}^1(I_Z \otimes N_1, N_2) \cong \text{Ext}^1(L, \mathcal{O}_C)\]

is surjective, which shows that all bundles in (4) are coming from (1) by applying \( \pi_* \).

According to [Ha] (Ex. V.2.5), there is a non-empty open set \( V \subset \text{Ext}^1(L, \mathcal{O}_C) \) (don’t forget the condition \( s - r \leq g + l \)) such that all \( \xi \in V \) define normalised vector bundles on \( C \).

Now, in \( \text{Ext}^1(I_Y \otimes N_1, N_2) \) the set of vector bundles is a non-empty open set \( U \). It is clear that \( \varphi^{-1}(V) \cap U \neq \emptyset \) (being open sets in Zariski topology), so we conclude.

§4. Moduli of stable bundles

There is an interesting relation between the moduli spaces \( M(c_1, c_2, d, r) \) and the Qin’s sets \( E_\zeta(c_1, c_2) \) (see [Q1], [Q2] for precised definitions).

As in the proof of Theorem 10, case (I) we conclude that if \( \zeta \) is a normalized class representing a non-empty wall of type \( (c_1, c_2) \) such that \( l_\zeta(c_1, c_2) > 0 \), then, for \( (2d - \alpha, 2r - \beta) = \zeta : E_\zeta(c_1, c_2) \) and \( M(c_1, c_2, d, r) \) are coincident modulo a factor of \( \text{Pic}_0(C) \) (Qin workes with first Chern class \( c_1 \) as an element in \( \text{Pic}(X) \)).

This is a consequence of the following facts:
(a) \( l_\zeta(c_1, c_2) = l(c_1, c_2, d, r) \)
(b) condition \( \zeta^2 < 0 \) implies \( 2d > \alpha \)
(c) in the case \( 2d > \alpha \) the bundles \( L_1, L_2 \) and the set \( Y \) from the sequence (1) are uniquely determined by \( E \).
(d) if \( l(c_1, c_2, d, r) > 0 \) then in the sequence (1) the bundles are given only by non-trivial extensions.

In fact it is not hard to see that \( M(c_1, c_2, d, r) = \emptyset \) if \( E_\zeta(c_1, c_2) = \emptyset \) so, by means of Theorem 10, \( E_\zeta(c_1, c_2) \neq \emptyset \) if \( l_\zeta(c_1, c_2) > 0 \). But we have even more:

**Corollary 11.** Let \( X \) be a ruled surface different from \( \mathbb{P}^1 \times \mathbb{P}^1 \) and let \( \mathcal{C} \) be a chamber of type \( (c_1, c_2) \) different from \( \mathcal{C}_f \). Then the moduli space \( M_\mathcal{C}(c_1, c_2) \neq \emptyset \).

**Proof.** From Theorem 1.3.3 in [Q2] it follows that
\[
M_\mathcal{C}(c_1, c_2) = (M_{\mathcal{C}_1}(c_1, c_2) - \bigsqcup_{\zeta} E_{(-\zeta)}(c_1, c_2)) \bigsqcup E_\zeta(c_1, c_2),
\]
where \( \mathcal{C}_1 \) is the chamber lying above \( \mathcal{C} \) and sharing with \( \mathcal{C} \) a non-empty common wall \( W \) and \( \zeta \) runs over all normalised classes representing \( W \). By the above considerations, it follows that \( E_\zeta(c_1, c_2) \neq \emptyset \) if \( l(c_1, c_2, d, r) > 0 \). It remains the case \( l(c_1, c_2, d, r) = 0 \) and it will be sufficient to prove that
\[
h^1(X, N_2 \otimes N_1^{-1}) := \dim H^1(X, N_2 \otimes N_1^{-1}) > 0
\]
(see the proof of Theorem 10).

We have
\[
N_2 \otimes N_1^{-1} = \mathcal{O}_X((d - d')C_0 + (r - s)f_0) \otimes \pi^*(L_2 \otimes L_1^{-1}),
\]
where \( d - d' = 2d - \alpha = u \) and \( r - s = 2r - \beta = v \). But \( \zeta = uC_0 + vf_0 \) is a normalized class and this implies that \( u > 0 \) and \( v < 0 \) (see [Q1]).

Because \( H^2(X, N_2 \otimes N_1^{-1}) = 0 \), the Riemann-Roch Theorem gives the equality:
\[
\chi = h^0(X, N_2 \otimes N_1^{-1}) - h^1(X, N_2 \otimes N_1^{-1}) = 1 - g + (1/2)((u+1)(2v-ue)+u(2-2g)).
\]

But \( \zeta^2 < 0 \) gives \( u(2v-ue) < 0 \); it follows \( 2v - ue < 0 \).

If \( g \geq 1 \), then obviously \( \chi < 0 \). If \( g = 0 \), then \( e \geq 0 \) and
\[
\chi = 1 + v + (u/2)(2(v+1) - e(u+1)).
\]
If \( e \geq 1 \), then \( \chi < 0 \). For \( e = 0 \) we get \( X = \mathbb{P}^1 \times \mathbb{P}^1 \), which we excluded. Thus, in all cases \( \chi < 0 \); it follows \( h^1(X, N_2 \otimes N_1^{-1}) > 0 \) and the proof is over.

**Remark.** Let us suppose that \( X = \mathbb{P}^1 \times \mathbb{P}^1 \) and that \( C \) is a chamber of type \((c_1, c_2)\) lying below a non-empty wall defined by a normalized class \( \zeta = uC_0 + vf_0 \) with \( v \leq -2 \). Then the same conclusion as in the above corollary holds.

Indeed, in this case we have \( \chi = (1 + v)(1 + u) \). Since \( v < -1 \), then again \( \chi < 0 \).

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