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ON THE NUMBER OF HOLOMORPHIC MAPPINGS BETWEEN RIEMANN SURFACES OF FINITE ANALYTIC TYPE

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Abstract The set of non-constant holomorphic mappings between two given compact Riemann surfaces of genus greater than 1 is always finite. This classical statement was made by de Franchis. Furthermore, bounds on the cardinality of the set depending only on the genera of the surfaces have been obtained by a number of mathematicians. The analysis is carried over in this paper to the case of Riemann surfaces of finite analytic type (i.e. compact Riemann surfaces minus a finite set of points) so that the finiteness result, together with a crude but explicit bound depending only on the topological data, may be extended for the number of holomorphic mappings between such surfaces.

Keywords: holomorphic mapping; Riemann surface of finite analytic type; Euler–Poincaré characteristic

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1. Introduction

A classical theorem of de Franchis [**1**] implies that the set of non-constant holomorphic mappings between two given compact Riemann surfaces of genus greater than 1 is finite. Having seen that the set is finite, we naturally want to obtain a bound on its cardinality. It is known that there exist bounds that depend only on the genera of the surfaces. A number of papers treating related questions exist (see [**8**] for some historical accounts).

What if we worked with non-compact surfaces obtained by removing a finite set of points from compact Riemann surfaces? We intend to give an answer to this slightly generalized question. The case of Riemann surfaces of low signature will also be considered.

There is probably little chance of surprise regarding the problem, even if one works with Riemann surfaces of finite analytic type, since any holomorphic mapping between two such hyperbolic Riemann surfaces (i.e. each surface must have negative Euler–Poincaré characteristic) extends to a holomorphic mapping between their uniquely defined compactifications (see § 2). In fact, we have chosen to restrict to the compact case for clarity in Ito and Yamamoto [**8**]. Thus, it is clear that in parts of this paper we will be re-proving results that are well known to the experts. We hope that our exposition of these topics includes material not previously formulated and that the leisurely approach that we have taken will be of benefit to the reader.

We first fix the notation that will be used throughout the paper. Let M be a compact Riemann surface of genus $g_M \geqslant 0$ and let $\mathcal{P}^M = \{P_1^M, \dots, P_m^M\}$ be a finite set of m distinct points on it. We say that $\dot{M}$ is a Riemann surface of finite analytic type (g_M, m) if $\dot{M}$ is (biholomorphically equivalent to) $M \setminus \mathcal{P}^M$. The 'distinct points' above are called punctures of $\dot{M}$ and the case when the set of its punctures is empty is of course included in the case of finite analytic type; that is, $\dot{M} = M$ if $\mathcal{P}^M = \emptyset$. For a type (g_M, m), the quantity

$$\chi(\dot{M}) = 2 - 2g_M - m \tag{1.1}$$

will be referred to as the Euler–Poincaré characteristic of $\dot{M}$.

When $\chi(\dot{M}) < 0$ (the only exceptions are the sphere, the once-punctured sphere, the twice-punctured sphere and the torus), the universal covering space of $\dot{M}$ can be regarded as the Poincaré disc, with covering transformations given by isometries of the Poincaré (hyperbolic) metric. Let Γ be a torsion-free Fuchsian group that uniformizes smoothly a Riemann surface $\dot{M}$ with one or more punctures. Then there is an obvious one-to-one correspondence between the punctures of $\dot{M} = U/\Gamma$ and the conjugacy classes of maximal parabolic subgroups of Γ and there is a natural way of topologizing the union of $\dot{M}$ and the set of its punctures $\{P_1^M, \dots, P_m^M\}$, and of making this union into a compact Riemann surface with all punctures filled in (see [**9**, pp. 198–203] for details).

Let $\mathrm{Hol}(\dot{M}, \dot{N})$ denote the set of non-constant holomorphic mappings from a source Riemann surface $\dot{M}$ of type (g_M, m) to a target Riemann surface $\dot{N}$ of type (g_N, n). Assume that the Euler–Poincaré characteristic $\chi(\dot{N})$ of the target surface $\dot{N}$ is negative. In this case we obtain an explicit upper bound on $\#\,\mathrm{Hol}(\dot{M}, \dot{N})$ depending only on the topological data. (Here and in what follows, $\#$ denotes the cardinality of a set.)

Theorem 1.1. *Let $\dot{M}$ and $\dot{N}$ be Riemann surfaces of finite analytic types (g_M, m) and (g_N, n), respectively, with $\chi(\dot{N}) < 0$. Then there exist bounds on the cardinality of $\mathrm{Hol}(\dot{M}, \dot{N})$ which depend only on the signatures of the surfaces. When $g_M \geqslant g_N$ and $m \geqslant n$, we have the following.*

(i) *If $g_N \geqslant 2$, then*

$$\#\,\mathrm{Hol}(\dot{M}, \dot{N}) \leqslant \left[2\sqrt{g_N \left[\frac{\chi(\dot{M})}{\chi(\dot{N})}\right]} + 1\right]^{4g_N g_M}. \tag{1.2}$$

(ii) *If $g_N = 1$ and (thus) $n \geqslant 1$, then*

$$\#\,\mathrm{Hol}(\dot{M}, \dot{N}) \leqslant m\left[2\sqrt{\left[\frac{\chi(\dot{M})}{\chi(\dot{N})}\right]} + 1\right]^{4g_M}. \tag{1.3}$$

(iii) *If* $g_N = 0$ *and (thus)* $n \geqslant 3$, *then*

$$\#\,\mathrm{Hol}(\dot{M}, \dot{N}) \leqslant (m-2)\left(2\left[\frac{\chi(\dot{M})}{\chi(\dot{N})}\right] + 1\right)^{m}. \tag{1.4}$$

Here, $[x]$ *is the greatest integer less than or equal to* x.

Remark 1.2.

(i) Verification of the finiteness of non-constant holomorphic mappings $f\colon \dot{M} \to \dot{N}$ is not difficult when $\chi(\dot{M}) \geqslant 0$ and $\chi(\dot{N}) < 0$. Indeed, a holomorphic mapping between such surfaces is (trivially observed to be) constant and hence $\mathrm{Hol}(\dot{M}, \dot{N})$ is empty (see § 5).

(ii) We therefore present arguments bounding the number of non-constant holomorphic mappings under the condition on Euler–Poincaré characteristics which amounts to the existence of the Poincaré metric, i.e. $\chi(\dot{M}) < 0$ and $\chi(\dot{N}) < 0$ unless otherwise specified.

(iii) Also note that even when $\chi(\dot{M}) < 0$ and $\chi(\dot{N}) < 0$, a necessary condition for the existence of a non-constant holomorphic mapping from $\dot{M}$ to $\dot{N}$ is that $g_M \geqslant g_N$ and $m \geqslant n$ (see § 2).

Theorem 1.1 gives rough estimates on the number of holomorphic mappings between two (not necessarily compact) Riemann surfaces of finite analytic type. Our bounds are of the same nature as those obtained by Martens [**10**, **11**] and Tanabe [**13**] for the case of compact Riemann surfaces. In particular, the estimate of part (i) of Theorem 1.1 in the compact case, i.e. $m = n = 0$, is part of Tanabe's Theorem 2. We hope that we have properly attributed credit to their work.

We note that the results of Ito and Yamamoto [**8**] and Tanabe [**14**] will yield a significantly lower bound and that the bound we derive could be lowered accordingly. However, we do not intend to include these discussions (of the sharpness of the estimates obtained) since the topics fall outside the scope of the paper.*

A plan of the proof of the theorem is as follows. Part (i) of Theorem 1.1 is established in § 4, while parts (ii) and (iii) are proven in § 5. In § 5 we shall also mention the excluded cases in which the target surface belongs to the short list of 'exceptional' types (including several examples). In §§ 2 and 3 we have collected all the preliminary details, which will play a direct role in the proof of the theorem.

2. Estimates for degrees of mappings

We now wish to 'generalize' the well-known Hurwitz combinatorial formula relating various topological indices connected with a holomorphic mapping between compact Riemann surfaces. For this purpose, we assume throughout the section (as previously

* I. A. Mednykh (personal communication, 2009) has stated that the sharp bound for the number of holomorphic mappings from S_3 to S_2 has been obtained, where S_3 and S_2 are compact Riemann surfaces of genera 3 and 2, respectively, as a consequence of his work [**12**]. This development seems to be based essentially on the concept of hyperellipticity.

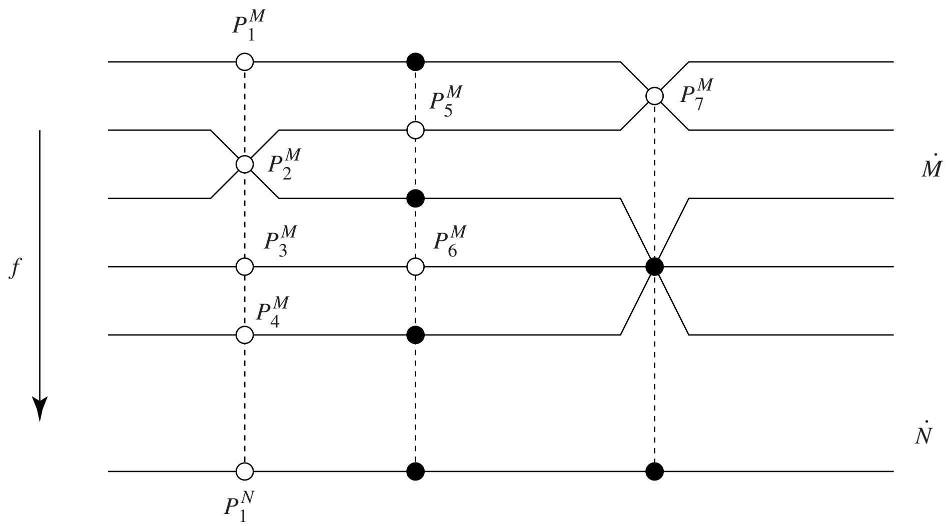

Figure 1. Removable punctures. P_5^M, P_6^M and P_7^M are not the inverse images of the punctures of $\dot{N}$ under $\bar{f}$.

explained) that the source and target Riemann surfaces both have negative Euler–Poincaré characteristics.

Since a non-constant holomorphic mapping f from $\dot{M}$ to $\dot{N}$ extends holomorphically to their compact closures (see below), we simply define the *degree*, d_f, of f as that of the holomorphic extension $\bar{f}$. Also, we define the *total branching number*, B_f, of f by

$$B_f = \sum_{P \in \dot{M}} b_f(P),$$

where $b_f(P)$ is the *branch number* of f at P.

The degree and the total branching number are well defined, because every non-constant holomorphic mapping between compact Riemann surfaces is a finite-sheeted covering map that is branched possibly at isolated points.

One pitfall to avoid is that of thinking that d_f is the number (counted with multiplicity) of inverse images of *any* point of $\dot{N}$. Indeed, more explicitly, the punctures of $\dot{N}$ may not contain the image of the punctures of $\dot{M}$ under the extended mapping $\bar{f}$, although the inverse image of the punctures of $\dot{N}$ are contained in the punctures of $\dot{M}$ (see Figure 1). The exceptional punctures of $\dot{M}$ are considered to be such only due to lack of information. Punctures with this character will be referred to as *removable punctures* with respect to f.

Proposition 2.1. *Let $\dot{M}$ and $\dot{N}$ be Riemann surfaces of finite analytic types (g_M, m) and (g_N, n), respectively, with $\chi(\dot{M}) < 0$ and $\chi(\dot{N}) < 0$. Then, for a possible holomorphic mapping $f \in \mathrm{Hol}(\dot{M}, \dot{N})$,*

$$\chi(\dot{M}) + \#\mathcal{P}_f^M = d_f \chi(\dot{N}) - \biggl(B_f + \sum_{P \in \mathcal{P}_f^M} b_{\bar{f}}(P)\biggr), \tag{2.1}$$

where $\mathcal{P}_f^M = \mathcal{P}^M \setminus \bar{f}^{-1}(\mathcal{P}^N)$ is the set of removable punctures of $\dot{M}$ with respect to f.

A consequence of the above blanket assumption is, as remarked before, that the relevant universal covering space can be regarded as the Poincaré upper half-plane U. Before proceeding further, we verify the following.

Assertion 2.2. *Every non-constant holomorphic mapping from $\dot{M}$ to $\dot{N}$ extends as a holomorphic mapping of M onto N. In particular, if either $g_M < g_N$ or $m < n$, the set* $\mathrm{Hol}(\dot{M}, \dot{N})$ *is empty.*

Remark 2.3. The above assertion (and probably Proposition 2.1 as well) are very well known. We have included them here for the convenience of the reader.

Proof of Assertion 2.2. We shall show that a holomorphic mapping $f \in \mathrm{Hol}(\dot{M}, \dot{N})$ extends as a holomorphic mapping $\bar{f} \in \mathrm{Hol}(M, N)$.

We lift f to a holomorphic mapping $\tilde{f}$ from U to itself which is uniquely specified by prescribing the action at a single point consistent with the f action. Let Γ (respectively, G) be a torsion-free Fuchsian group uniformizing $\dot{M}$ (respectively, $\dot{N}$). We can then define a homomorphism $\theta\colon \Gamma \to G$ that satisfies

$$\tilde{f} \circ \gamma = \theta(\gamma) \circ \tilde{f} \quad \text{for every } \gamma \in \Gamma. \tag{2.2}$$

Let us choose any puncture P_i^M of $\dot{M}$. There exists a parabolic transformation $T \in \Gamma$ that determines the puncture P_i^M. We assume that T is normalized, that is, that the transformation is given by

$$T\colon z \mapsto z + 1. \tag{2.3}$$

Clearly, the rather specialized assumption above is not essential, but should be looked upon as the result of a normalization.

We claim that $\theta(T)$ must be either parabolic or the identity. Indeed, if $\theta(T)$ is hyperbolic, it involves no loss of generality to assume that $\theta(T)$ is of the form

$$\theta(T)\colon z \mapsto kz, \quad \text{positive } k \neq 1 \tag{2.4}$$

(by conjugation). An easy application of the Schwartz–Pick distance-decreasing principle demonstrates that

$$d(z, T(z)) \geqslant d(f(z), f(T(z))) = d(f(z), \theta(T)(f(z))), \tag{2.5}$$

where d is the Poincaré (hyperbolic) distance on U; that is, if $d(z, T(z))$ is small, then $d(f(z), \theta(T)(f(z)))$ is correspondingly small. But direct calculations yield

$$\cosh d(z, T(z)) = 1 + \frac{1}{2(\operatorname{Im} z)^2} \tag{2.6}$$

and

$$\cosh d(f(z), \theta(T)(f(z))) = 1 + \frac{(k-1)^2 |f(z)|^2}{2k(\operatorname{Im} f(z))^2} \geqslant 1 + \frac{(k-1)^2}{2k}, \tag{2.7}$$

where cosh is the hyperbolic cosine function and Im denotes the imaginary part of the complex number. This is an obvious contradiction.

There are now two possibilities to consider.

Case 1. $\theta(T)$ is a parabolic transformation belonging to a puncture P_j^N. We may assume that $\theta(T) = T$ by conjugation. Thus, $\tilde{f}$ induces a holomorphic mapping w from Δ^* to itself so that the following diagram commutes

$$\begin{array}{ccc} U & \xrightarrow{\tilde{f}} & U \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \Delta^* & \xrightarrow[w]{} & \Delta^* \end{array} \tag{2.8}$$

where $p(z) = \mathrm{e}^{2\pi \mathrm{i} z}$, $z \in U$, and Δ^* is the punctured disc $\{0 < |z| < 1\}$. Again, from the Schwartz–Pick Lemma we obtain (using the same notation as above)

$$1 + \frac{1}{2(\operatorname{Im} z)^2} = \cosh d(z, T(z)) \geqslant \cosh d(\tilde{f}(z), T(\tilde{f}(z))) = 1 + \frac{1}{2(\operatorname{Im} f(z))^2}. \tag{2.9}$$

Hence, w is extendable with $w(0) = 0$, and consequently f has a holomorphic extension from $\dot{M} \cup \{P_i^M\}$ to $\dot{N} \cup \{P_j^N\}$.

Case 2 ($\boldsymbol{\theta(T)}$ is the identity). Here it is more convenient to assume that $\tilde{f}$ is a mapping from U to Δ, the unit disc. Obviously, $\tilde{f}$ induces a holomorphic mapping (also denoted by w) from Δ^* to Δ so that the diagram

$$\begin{array}{ccc} U & & \\ {\scriptstyle p}\downarrow & \searrow^{\tilde{f}} & \\ \Delta^* & \xrightarrow[w]{} & \Delta \end{array} \tag{2.10}$$

commutes. With the aid of the Riemann Removable Singularity Theorem, w can be extended to Δ, since $|w|$ is bounded. (w thus turns out to be a mapping from Δ to Δ.) Hence f has a removable puncture at P_i^M.

We have shown that a holomorphic mapping $f \in \operatorname{Hol}(\dot{M}, \dot{N})$ extends as a holomorphic mapping $\bar{f} \in \operatorname{Hol}(M, N)$.

The last part of the assertion follows readily when we apply the standard Riemann–Hurwitz formula to the extended mapping, which will conclude the proof. (See also the digression in § 5, for example.) □

Proof of Proposition 2.1. We now extend the Riemann–Hurwitz formula from compact surfaces to surfaces with punctures, under the above condition on Euler characteristics. Noting that f could possibly have removable punctures, we first observe that

$$\#\bar{f}^{-1}(\mathcal{P}^N) = m - \#\mathcal{P}_f^M \tag{2.11}$$

and that

$$\sum_{P \in M \setminus \bar{f}^{-1}(\mathcal{P}^N)} b_{\bar{f}}(P) = B_f + \sum_{P \in \mathcal{P}_f^M} b_{\bar{f}}(P). \tag{2.12}$$

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From the relevant definitions, it is immediate that

$$d_{\bar{f}} = \sum_{P \in \bar{f}^{-1}(P_j^N)} b_{\bar{f}}(P) + \#\bar{f}^{-1}(P_j^N), \quad 1 \leqslant j \leqslant n.$$

Summing the above expressions over j and using the fact that the degree of f is defined by that of $\bar{f}$, we see that

$$\begin{aligned} nd_f &= \sum_{P \in \bar{f}^{-1}(\mathcal{P}^N)} b_{\bar{f}}(P) + \#\bar{f}^{-1}(\mathcal{P}^N) \\ &= \Big(B_{\bar{f}} - \sum_{P \in M \setminus \bar{f}^{-1}(\mathcal{P}^N)} b_{\bar{f}}(P)\Big) + \#\bar{f}^{-1}(\mathcal{P}^N). \end{aligned} \tag{2.13}$$

In what follows, we will use the standard Riemann–Hurwitz relation with respect to the extended mapping in the form

$$B_{\bar{f}} = d_f \chi(N) - \chi(M).$$

We return to (2.13) and continue with our calculation. Then we conclude that (using (2.11), (2.12) and the Riemann–Hurwitz relation)

$$nd_f = (d_f \chi(N) - \chi(M)) - \Big(B_f + \sum_{P \in \mathcal{P}_f^M} b_{\bar{f}}(P)\Big) + (m - \#\mathcal{P}_f^M)$$

or

$$(\chi(M) - m) + \#\mathcal{P}_f^M = d_f(\chi(N) - n) - \Big(B_f + \sum_{P \in \mathcal{P}_f^M} b_{\bar{f}}(P)\Big),$$

from which (2.1) follows. □

We stress that Proposition 2.1 can also give upper estimates on the degree of a holomorphic mapping for some low signatures. Consideration of the standard special example for the case of $(g_N, n) = (0, 3)$ will convince the reader of the usefulness of this proposition. We note the following explicit corollary for later use.

Corollary 2.4. *Under the hypothesis of Proposition* 2.1,

$$d_f \leqslant \left[\frac{\chi(\dot{M})}{\chi(\dot{N})}\right], \tag{2.14}$$

where, as usual, $[x]$ *denotes the greatest integer in* x*.*

Remark 2.5. Since it is immediate that, by the very definition of degree,

$$d_f \leqslant \left[\frac{\chi(M)}{\chi(N)}\right], \tag{2.15}$$

we could use the estimate

$$d_f \leqslant \min\left\{\left[\frac{\chi(\dot{M})}{\chi(\dot{N})}\right], \left[\frac{\chi(M)}{\chi(N)}\right]\right\} \tag{2.16}$$

instead of (2.14), when an inequality $[\chi(M)/\chi(N)] < [\chi(\dot{M})/\chi(\dot{N})]$ may hold.

3. Martens's Rigidity Theorem

The exposition here is due to Martens. The proof is along the lines of [**10**,**11**]. (Of course, there are many good references for the background material. The reader can consult [**2**–**5**] for more details on the concepts discussed in this section.)

We start with a compact Riemann surface X of positive genus $g = g_X \geqslant 1$, and a canonical homology basis

$$\{\gamma_1, \gamma_2, \dots, \gamma_{2g}\} = \{\gamma_1^X, \gamma_2^X, \dots, \gamma_{2g_X}^X\}$$

on X. (Whenever there can be no confusion, the surface X will be dropped from the symbols.) Recall that the statement '$\{\gamma_1, \dots, \gamma_{2g}\}$ is a canonical homology basis' means that the intersection matrix for this basis is

$$J_g = \begin{pmatrix} 0_g & I_g \\ -I_g & 0_g \end{pmatrix}, \tag{3.1}$$

where 0_g is the $g \times g$ zero matrix and I_g is the $g \times g$ identity matrix.

Let us briefly review the basic facts (to fix notation). There exists a unique basis

$$\{\omega_1, \dots, \omega_g\} = \{\omega_1^X, \dots, \omega_{g_X}^X\}$$

for the vector space $A(X)$ of holomorphic abelian differentials on X with the property

$$\int_{\gamma_j} \omega_i = \delta_{ij} (= \text{Kronecker delta}), \quad i, j = 1, 2, \dots, g. \tag{3.2}$$

With these choices of bases, the matrix

$$Z = Z_X = (\pi_{ij}), \quad \pi_{ij} = \int_{\gamma_{g+j}} \omega_i, \quad i, j = 1, 2, \dots, g, \tag{3.3}$$

is symmetric with positive definite imaginary part, and we call the *period matrix of* X the following $g \times 2g$ complex matrix:

$$\Pi = \Pi_X = (I_g, Z). \tag{3.4}$$

Let us denote by $L = L(\Pi)$ the lattice (over $\mathbb{Z}$) generated by the columns of the $g \times 2g$ matrix Π. These columns form $2g$ vectors in $\mathbb{C}^g$ which are linearly independent over the reals. We define the *Jacobian variety of* X as the compact complex torus of complex dimension g by

$$J(X) = \mathbb{C}^g / L. \tag{3.5}$$

Given any two points P_1 and P_2 on X, we can associate with them the vector

$$\left(\int_{P_1}^{P_2} \omega_1, \dots, \int_{P_1}^{P_2} \omega_g \right)^{\mathrm{T}} \in \mathbb{C}^g, \tag{3.6}$$

where the integrations are along some path joining P_1 to P_2. If we consider the vector (3.6) modulo L, then the choice of path becomes immaterial. We can thus map X holomorphically into $J(X)$ in an obvious fashion; define

$$\kappa\colon X \to J(X)$$

by choosing a base point $P_0 \in X$ and setting

$$\kappa(P) = \left(\int_{P_0}^{P} \omega_1, \dots, \int_{P_0}^{P} \omega_g\right)^{\mathrm{T}} \pmod{L}.$$

The mapping κ is known as the *Abel–Jacobi embedding* of X. This mapping depends on the choice of the base point P_0. Hence, it is denoted by κ_{P_0} when its dependence on the base point is to be emphasized.

The following technical proposition will be needed later.

Proposition 3.1. *Let X be a compact Riemann surface of genus $g \geqslant 2$. Then $\kappa_{P_0}(X) = \kappa_{P_0'}(X)$ if and only if $P_0 = P_0'$.*

Proof. Let $P_0, P_0' \in X$, $P_0 \neq P_0'$. If $\kappa_{P_0}(X) = \kappa_{P_0'}(X)$, then, for any $P \in X$, we can find $P' \in X$ such that

$$\int_{P_0}^{P} \omega = \int_{P_0'}^{P'} \omega \quad \text{for every } \omega \in A(X),$$

and thus $P'P_0/P_0'P$ is a principal divisor by Abel's Theorem. (Here we write divisors multiplicatively rather than additively.) This contradiction establishes the proposition. See also the analysis below. □

Remark 3.2. We let $\operatorname{div}(X)$ denote the *group of divisors* on X. For $D \in \operatorname{div}(X)$, we set

$$L(D) = \{f;\ f \text{ is a meromorphic function with } (f) \geqslant D\}$$

and

$$\Omega(D) = \{\omega;\ \omega \text{ is an abelian differential with } (\omega) \geqslant D\}$$

by using obvious notation. Let $P_1 \in X$ be arbitrary. The Riemann–Roch Theorem implies that

$$\dim \Omega(P_1) = \dim L(1/P_1) - \deg P_1 + g - 1 = g - 1 > 0$$

since $\mathbb{C} = L(1/P_1)$, because a surface of positive genus does not admit a meromorphic function with a single pole. Let us choose a non-trivial abelian differential $\omega \in \Omega(P_1)$. We take $P_2 \in X$ to be a point such that ω does not vanish at P_2 (ω has $2g - 2$ zeros, counting multiplicity). Since P_2 was chosen so that $\omega \in \Omega(P_1) \setminus \Omega(P_1P_2)$, it follows that $\dim \Omega(P_1P_2) \leqslant g - 2$. Thus, again by Riemann–Roch, we conclude that

$$\dim L(1/P_1P_2) = 1.$$

(The statement $\dim L(1/P_1P_2) = 1$ is an immediate consequence of the inequality $\dim L(1/P_1P_2) \leqslant 1$, which we showed in the discussion, in view of the obvious inequality $\dim L(1/P_1P_2) \geqslant 1$.) We have already seen that a divisor greater than or equal to $1/P_1P_2$ cannot be principal.

Let M and N be any two compact Riemann surfaces with $g_M \geqslant g_N \geqslant 1$. (Note that if $g_M < g_N$, then there does not exist a non-constant holomorphic mapping from M to N.) We shall assume that every such Riemann surface M (respectively, N) has on it a fixed canonical homology basis $\{\gamma_1^M, \dots, \gamma_{2g_M}^M\}$ (respectively, $\{\gamma_1^N, \dots, \gamma_{2g_N}^N\}$).

For $f \in \mathrm{Hol}(M, N)$, let us denote by H_f the $2g_N \times 2g_M$ matrix with integer entries m_{ij} representing the induced homology map

$$f_*\colon H_1(M;\mathbb{Z}) \to H_1(N;\mathbb{Z})$$

with respect to the canonical homology bases; that is,

$$f_*\gamma_j^M = \sum_{i=1}^{2g_N} m_{ij}\gamma_i^N, \quad j = 1, \dots, 2g_M. \tag{3.7}$$

We will see that our approach to the finiteness problem is based on Martens's idea that a holomorphic mapping between compact Riemann surfaces is determined by the way it operates on the homology of the surface, and that this brings the problem down to counting homology actions.

Finally, let z be a local coordinate on M and let ζ be a local coordinate on N and suppose that, in terms of these local coordinates, we have

$$\zeta = f(z).$$

If ω is a holomorphic abelian differential on N, then locally

$$\omega = h(\zeta)\,\mathrm{d}\zeta.$$

The pullback of ω via f, $f^*\omega$, is the holomorphic abelian differential defined in terms of the local coordinate z by

$$f^*\omega = h(f(z))f'(z)\,\mathrm{d}z.$$

Thus f naturally induces a complex linear map

$$f^*\colon A(N) \to A(M).$$

Associated to f^* is a $g_N \times g_M$ matrix A_f with complex entries c_{ij}; that is,

$$f^*\omega_i^N = \sum_{j=1}^{g_M} c_{ij}\omega_j^M, \quad i = 1, \dots, g_N, \tag{3.8}$$

where $\{\omega_1^M, \dots, \omega_{g_M}^M\}$ and $\{\omega_1^N, \dots, \omega_{g_N}^N\}$ are the respective dual bases with respect to the canonical homology bases.

We move on to the actual description of Martens's Rigidity Theorem. Let $\mathit{\Pi}_M$ and $\mathit{\Pi}_N$ denote the respective period matrices with respect to the canonical homology bases and the dual bases for their holomorphic abelian differentials. We must establish the *Hurwitz relation*

$$A_f \mathit{\Pi}_M = \mathit{\Pi}_N H_f, \tag{3.9}$$

which is another way of saying that there is a commutative diagram

$$\begin{array}{ccc} \mathbb{R}^{2g_M} & \xrightarrow{\mathit{\Pi}_M} & \mathbb{C}^{g_M} \\ {\scriptstyle H_f}\downarrow & & \downarrow{\scriptstyle A_f} \\ \mathbb{R}^{2g_N} & \xrightarrow[\mathit{\Pi}_N]{} & \mathbb{C}^{g_N} \end{array} \tag{3.10}$$

But it is easy to verify that

$$\int_{f_*\gamma_i^M} \omega_j^N = \int_{\gamma_i^M} f^*\omega_j^N, \quad 1 \leqslant i \leqslant 2g_M,\ 1 \leqslant j \leqslant g_N, \tag{3.11}$$

which is equivalent to (3.10). Hence H_f (and thus f) induces a holomorphic mapping F from $J(M)$ to $J(N)$. The following diagram commutes:

$$\begin{array}{ccccccc} \mathbb{R}^{2g_M} & \xrightarrow{\mathit{\Pi}_M} & \mathbb{C}^{g_M} & \longrightarrow & J(M) & \xleftarrow{\kappa_{P_0}} & M \\ {\scriptstyle H_f}\downarrow & & {\scriptstyle A_f}\downarrow & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle f} \\ \mathbb{R}^{2g_N} & \xrightarrow[\mathit{\Pi}_N]{} & \mathbb{C}^{g_N} & \longrightarrow & J(N) & \xleftarrow[\kappa_{f(P_0)}]{} & N \end{array} \tag{3.12}$$

where unmarked horizontal arrows denote canonical projections.

It is now easy to deduce the following.

Proposition 3.3 (Martens's Rigidity Theorem). *Let M and N be compact Riemann surfaces of genera g_M and g_N, respectively, with $g_M \geqslant g_N \geqslant 1$. Suppose that $H_f = H_g$ for $f, g \in \mathrm{Hol}(M, N)$. Then*

(i) *if $g_N = 1$, then there exists a unique translation τ of N such that $f = \tau \circ g$,*

(ii) *if $g_N \geqslant 2$, then $f = g$.*

Proof. Say $P_0 \in M$ is the fixed base point of the Abel–Jacobi embedding $\kappa_{P_0}\colon M \to J(M)$. As before, F and G are holomorphic mappings from $J(M)$ to $J(N)$ induced by f and g, respectively. Then the following diagrams commute:

$$\begin{array}{ccccc} \mathbb{R}^{2g_M} & \longrightarrow & J(M) & \xleftarrow{\kappa_{P_0}} & M \\ {\scriptstyle H_f}\downarrow & & {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbb{R}^{2g_N} & \longrightarrow & J(N) & \xleftarrow[\kappa_{f(P_0)}]{} & N \end{array} \qquad \begin{array}{ccccc} \mathbb{R}^{2g_M} & \longrightarrow & J(M) & \xleftarrow{\kappa_{P_0}} & M \\ {\scriptstyle H_g}\downarrow & & {\scriptstyle G}\downarrow & & \downarrow{\scriptstyle g} \\ \mathbb{R}^{2g_N} & \longrightarrow & J(N) & \xleftarrow[\kappa_{g(P_0)}]{} & N \end{array}$$

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where unmarked horizontal arrows again denote natural projections (which *do not* depend on the base points). Therefore, we have $F = G$. In particular, we see that $\kappa_{f(P_0)}(N) = \kappa_{g(P_0)}(N)$. Thus, if $g_N \geqslant 2$, we conclude that $f(P_0) = g(P_0)$ by Proposition 3.1. This completes the proof of part (ii).

We are left with the case where $g_N = 1$, which can be dealt with as follows. Every torus has $\mathbb{C}$ as its universal covering space. We may thus write N as $\mathbb{C}$ modulo a fixed-point free subgroup of $\operatorname{Aut}\mathbb{C}$. It follows immediately that $\mathrm{d}\zeta$, since it is invariant even under $\operatorname{Aut}\mathbb{C}$, can be regarded as the only holomorphic differential on the torus, up to multiplication by constants (the holomorphic fixed-point free self-mappings on $\mathbb{C}$ are of the form $z \mapsto z + b$, with $b \in \mathbb{C}$). Let τ be a translation of N which takes $g(P_0)$ to $f(P_0)$. Since $\tau^*\mathrm{d}\zeta = \mathrm{d}\zeta$, it is quite easy to conclude that $f = \tau \circ g$. □

4. Finiteness theorem for the case of $g_N \geqslant 2$

As mentioned before, we will complete the proof of Theorem 1.1 (i) in this section; the development given here is based very significantly on the work of Tanabe [**13**]. We also remark that Proposition 4.2 is established under the assumption that $g_N \geqslant 1$.

We formalize some notions needed for the remainder of this paper. In the preceding section, we saw that if $g_N \geqslant 2$, then a holomorphic mapping from M to N can be recovered from the induced homology map. Thus, there is a natural map

$$\mu_0\colon \operatorname{Hol}(\dot{M}, \dot{N}) \to M(2g_N, 2g_M; \mathbb{Z}) \tag{4.1}$$

given by $f \mapsto H_{\bar{f}}$, which is injective when $g_N \geqslant 2$, where $M(2g_N, 2g_M; \mathbb{Z})$ denotes the set of $2g_N \times 2g_M$ matrices with integer entries (recall that $\bar{f}$ denotes the holomorphic extension of f).

We wish to study the images under μ_0 of holomorphic mappings from $\dot{M}$ to $\dot{N}$. What are the conditions on a $2g_N \times 2g_M$ matrix H so that H represents some induced homology map? First of all, H must be an integral matrix. Next, as we have already observed, it is also necessary that H satisfies the Hurwitz relation between $\varPi_M$ and $\varPi_N$; that is, H satisfies the equation $A\varPi_M = \varPi_N H$ for some $g_N \times g_M$ complex matrix A (which depends on H). The remarks we have made lead us to the following definition.

Definition 4.1. Let M and N be compact Riemann surfaces of genera g_M and g_N, respectively, with $g_M \geqslant g_N \geqslant 1$. The set of $2g_N \times 2g_M$ matrices H with rational entries which satisfy

$$A\varPi_M = \varPi_N H \quad \text{for some } g_N \times g_M \text{ matrix } A \text{ with complex entries} \tag{4.2}$$

is called the Hurwitz $\mathbb{Q}$-vector space of $\varPi_M$ and $\varPi_N$ and is denoted by $\mathcal{S}(\varPi_M, \varPi_N)$.

The matrices of the form under consideration form a ‘discrete’ subset of $\mathcal{S}(\varPi_M, \varPi_N)$. Now we start a more detailed investigation of the images under μ_0 of holomorphic mappings from $\dot{M}$ to $\dot{N}$. We define the *Castelnuovo–Severi inner product* of $H_1, H_2 \in \mathcal{S}(\varPi_M, \varPi_N)$ by

$$\langle H_1, H_2\rangle = \operatorname{tr}(J_{g_M} H_1^{\mathrm{T}} J_{g_N}^{-1} H_2). \tag{4.3}$$

Here J_{g_M} and J_{g_N} are the intersection matrices with respect to the relevant canonical homology bases. (We are using the notation and conventions introduced in the discussion near the beginning of § 3.) Positive definiteness of the bilinear form is the only issue, but the proof is omitted; see the proof of the lemma in [**11**] (cf. [**7**]). Using obvious notational conventions, we also define the norm of $H \in \mathcal{S}(\Pi_M, \Pi_N)$ by

$$\|H\| = \langle H, H\rangle^{1/2}. \tag{4.4}$$

Proposition 4.2. *Let $\dot{M}$ and $\dot{N}$ be Riemann surfaces of finite analytic types (g_M, m) and (g_N, n), respectively, with $\chi(\dot{M}) < 0$ and $\chi(\dot{N}) < 0$, and let $g_M \geqslant g_N \geqslant 1$. Then, for a possible holomorphic mapping $f \in \mathrm{Hol}(\dot{M}, \dot{N})$,*

$$\|H_{\bar{f}}\|^2 = 2g_N d_f. \tag{4.5}$$

Proof. For $f \in \mathrm{Hol}(M, N)$, we have $\|H_f\|^2 = 2g_N d_f$ because

$$H_f J_{g_M} H_f^{\mathrm{T}} = d_f J_{g_N} \tag{4.6}$$

(Martens [**11**] mentioned the above condition, and it seems he credited Hopf's paper [**6**] with the treatment of the last equality.) Then the verification of (4.5) is straightforward.

To check (4.6), let $\{\alpha_1^M, \dots, \alpha_{2g_M}^M\}$ (respectively, $\{\alpha_1^N, \dots, \alpha_{2g_N}^N\}$) be the basis of the real-valued harmonic differentials on M (respectively, N) that is dual to the given canonical homology bases. Since $f^*\alpha_i^N \wedge f^*\alpha_j^N = f^*(\alpha_i^N \wedge \alpha_j^N)$, we immediately see from related definitions that

$$\iint_M f^*\alpha_i^N \wedge f^*\alpha_j^N = d_f \iint_N \alpha_i^N \wedge \alpha_j^N, \quad i, j = 1, 2, \dots, 2g_N. \tag{4.7}$$

Now we write any harmonic differential α^N on N as

$$\alpha^N = \sum_{k=1}^{2g_N} a_k \alpha_k^N \tag{4.8}$$

with $a = (a_1, \dots, a_{2g_N})^{\mathrm{T}} \in \mathbb{R}^{2g_N}$. Then we can rewrite (4.7) as

$$(H_f^{\mathrm{T}} a)^{\mathrm{T}} J_{g_M} (H_f^{\mathrm{T}} a) = d_f a^{\mathrm{T}} J_{g_N} a \quad \text{for all } a \in \mathbb{R}^{2g_N}, \tag{4.9}$$

because the matrix representing f^* with respect to $\{\alpha_1^M, \dots, \alpha_{2g_M}^M\}$ and $\{\alpha_1^N, \dots, \alpha_{2g_N}^N\}$ is H_f^{T}. This last statement is equivalent to (4.6). □

As a direct consequence of Proposition 4.2 and Corollary 2.4, we obtain the following.

Corollary 4.3. *Under the hypothesis of Proposition 4.2,*

$$\|H_{\bar{f}}\|^2 \leqslant 2g_N \left[\frac{\chi(\dot{M})}{\chi(\dot{N})}\right], \tag{4.10}$$

where $[x]$ is the greatest integer $\leqslant x$.

Remark 4.4. This corollary is a (minor) extension of [**13**, Lemma 2]. The inequality of Tanabe's Lemma follows easily if we make use of (4.10); indeed,

$$2g_N\left[\frac{\chi(\dot{M})}{\chi(\dot{N})}\right] \leqslant \frac{g_N}{2(g_N-1)}4(g_M-1) \leqslant 4(g_M-1),$$

when $g_M \geqslant g_N \geqslant 2$ and $m = n = 0$.

On the basis of the above development, we can strengthen the result that a holomorphic mapping is determined by the way it operates on the homology of the surface.

Let us denote by $M(2g_N, 2g_M; \mathbb{Z}/\mathbb{Z}l)$ the set of $2g_N \times 2g_M$ matrices with coefficients in the integers mod l. Then, when $g_N \geqslant 2$, the map

$$\mu_l\colon \operatorname{Hol}(\dot{M}, \dot{N}) \to M(2g_N, 2g_M; \mathbb{Z}/\mathbb{Z}l) \tag{4.11}$$

given by $f \mapsto [H_{\bar{f}}]_l$ is already injective for sufficiently large l (we denote the equivalence class of matrices corresponding to H by $[H]_l$). This fact, which will be verified in the next lemma, plays a crucial role in the proof of Theorem 1.1 (i).

Lemma 4.5 (see also Tanabe [13]). *Let $g_M \geqslant g_N \geqslant 2$ and $m \geqslant n$. Then there is a natural map*

$$\mu_l\colon \operatorname{Hol}(\dot{M}, \dot{N}) \to M(2g_N, 2g_M; \mathbb{Z}/\mathbb{Z}l).$$

For $l > 2\sqrt{g_N[\chi(\dot{M})/\chi(\dot{N})]}$, the map μ_l is precisely one-to-one.

Proof. The map μ_l has already been described in the remarks preceding the statement of the lemma. We shall therefore assume that

$$l > 2\sqrt{g_N[\chi(\dot{M})/\chi(\dot{N})]}.$$

Suppose now that $\mu_l(f) = \mu_l(g)$ for f, $g \in \operatorname{Hol}(\dot{M}, \dot{N})$ and let $D = H_{\bar{f}} - H_{\bar{g}}$. If we write

$$D = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix} \tag{4.12}$$

in $g_N \times g_M$ blocks, then

$$J_{g_M} D^{\mathrm{T}} J_{g_N}^{-1} D = \begin{pmatrix} -D_2^{\mathrm{T}} D_3 + D_4^{\mathrm{T}} D_1 & * \\ * & D_1^{\mathrm{T}} D_4 - D_3^{\mathrm{T}} D_2 \end{pmatrix}. \tag{4.13}$$

Hence, in particular, $2l^2$ divides $\|D\|^2$ since $D \equiv 0 \pmod{l}$. Recall, however (see Corollary 4.3), that

$$\|D\|^2 \leqslant 2(\|H_{\bar{f}}\|^2 + \|H_{\bar{g}}\|^2) \leqslant 8g_N\left[\frac{\chi(\dot{M})}{\chi(\dot{N})}\right]; \tag{4.14}$$

so that it necessarily is the case that

$$D = 0 \quad \text{or} \quad H_{\bar{f}} = H_{\bar{g}}.$$

We have shown (Proposition 3.3 (ii)) that holomorphic mappings from M to N are in a one-to-one correspondence with the induced homology maps under the assumption that $g_N \geqslant 2$. Hence, we conclude that $f = g$. □

It is then easily seen that, for $l > 2\sqrt{g_N[\chi(\dot{M})/\chi(\dot{N})]}$,

$$\#\,\mathrm{Hol}(\dot{M},\dot{N}) \leqslant \#M(2g_N, 2g_M; \mathbb{Z}/\mathbb{Z}l) = l^{4g_N g_M}.$$

Choosing the integer l as small as possible, part (i) of the theorem follows.

5. Finiteness theorem for the case where $g_N \leqslant 1$

Next we extend the considerations of the previous section to the remaining cases. We adopt the notation and conventions introduced so far.

First, we examine once-punctured tori. Note that almost all the arguments above work also for the cases $g_N = 1$; indeed we proved Proposition 4.2 under the assumption that $g_N \geqslant 1$. This time, however, a holomorphic mapping from $\dot{M}$ to $\dot{N}$ is not recovered from the induced homology map.

Lemma 5.1. *Let $g_M \geqslant g_N = 1$ and $m \geqslant n \geqslant 1$. For $l > 2\sqrt{[\chi(\dot{M})/\chi(\dot{N})]}$, the map*

$$\mu_l\colon\ \mathrm{Hol}(\dot{M},\dot{N}) \to M(2g_N, 2g_M; \mathbb{Z}/\mathbb{Z}l)$$

is at most m-to-one.

Proof. Under the hypothesis, in the light of the proof of Lemma 4.5, $H_{\bar{f}}$ can be recovered from its projection in the set of $2g_N \times 2g_M$ $(= 2\times 2g_M)$ matrices with coefficients in the integers mod l.

So let $f_0 \in \mathrm{Hol}(\dot{M},\dot{N})$ be arbitrary. It suffices to prove the inequality

$$\#\{f \in \mathrm{Hol}(\dot{M},\dot{N}); H_{\bar{f}} = H_{\bar{f}_0}\} \leqslant m. \tag{5.1}$$

If $H_{\bar{f}} = H_{\bar{f}_0}$, then part (i) of Proposition 3.3 gives that there exists a translation τ_f defined on the compact ambient surface N such that

$$\bar{f} = \tau_f \circ \bar{f}_0. \tag{5.2}$$

Since the point $\tau_f^{-1}(P_1^N)$ must be the image of a puncture of $\dot{M}$ under $\bar{f}_0$, we have (5.1). □

Remark 5.2. One might expect that the quantity on the right-hand side of (5.1) can be replaced by n. However, a counterexample is provided by taking $\dot{M}$ to be a Riemann surface of genus 2 with two punctures (at 'P_1^M' and 'P_2^M'), a two-sheeted holomorphic branched covering $M \to N$ ramified of order 2 at P_1^M and P_2^M over P_1^N and one distinguished point, say Q^N (*not* a puncture), of $\dot{N}$, respectively, and letting the translation defined on N to send Q^N to P_1^N (see Figure 2 and also Figure 3).

While we have shown in § 2 that a holomorphic mapping $f \in \mathrm{Hol}(\dot{M},\dot{N})$ extends as a holomorphic mapping $\bar{f} \in \mathrm{Hol}(M,N)$, it is observed that a holomorphic mapping from M to N may be regarded as an extension of some $f \in \mathrm{Hol}(\dot{M},\dot{N})$ if and only if its inverse image of the punctures of $\dot{N}$ is contained in the punctures of $\dot{M}$.

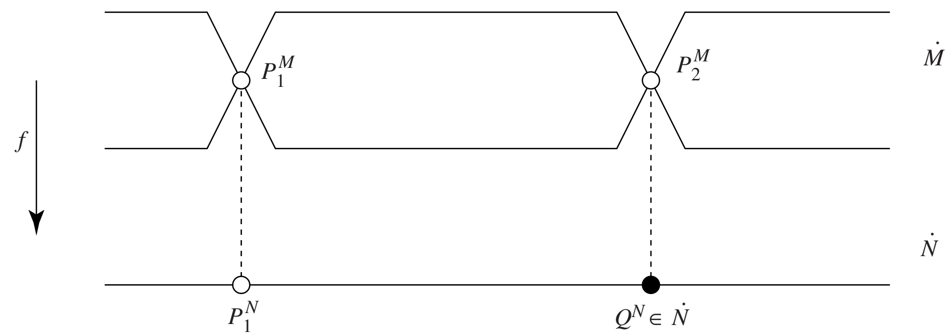

Figure 2. Possible restriction of a mapping between compact ambient surfaces to punctured surfaces. Since the inverse image of Q^N under $\bar{f}$ is a puncture P_2^M, one can, following $\bar{f}$ by a translation of N which takes Q^N to P_1^N, exhibit a holomorphic mapping from $\dot{M}$ to $\dot{N}$ as a restriction of the composition; however, such a translation is not an extension of a self-mapping on $\dot{N}$ itself.

We use the map of the lemma to complete the proof of Theorem 1.1 (ii);

$$\#\operatorname{Hol}(\dot{M},\dot{N}) \leqslant m \cdot \#M(2g_N, 2g_M; \mathbb{Z}/\mathbb{Z}l) = ml^{4g_M} \quad \text{for } l > 2\sqrt{[\chi(\dot{M})/\chi(\dot{N})]}.$$

So far we have considered Riemann surfaces with negative Euler–Poincaré characteristics. Can we find a bound on the cardinality of $\operatorname{Hol}(\dot{M},\dot{N})$ even when either $\chi(\dot{M}) \geqslant 0$ or $\chi(\dot{N}) \geqslant 0$? To change the pace slightly, we digress a little to investigate the infiniteness of $\operatorname{Hol}(\dot{M},\dot{N})$ for 'exceptional' cases.

Let f be a holomorphic mapping from $\dot{M}$ to $\dot{N}$ and let $\tilde{f}$ be its lift to the universal covering space. Then $\tilde{f}$ is a holomorphic mapping between simply connected Riemann surfaces. Recall that any simply connected Riemann surface is biholomorphically equivalent to exactly one of

(1) the sphere $\hat{\mathbb{C}}$,

(2) the plane $\mathbb{C}$ and

(3) the upper half-plane U.

We now apply a classical result that goes by the name of *Liouville's Theorem* to $\tilde{f}$ and conclude that if the universal covering Riemann surface of $\dot{M}$ is either $\hat{\mathbb{C}}$ or $\mathbb{C}$, and that of $\dot{N}$ is U, then $\tilde{f}$ must reduce to a constant; that is, *if $\chi(\dot{M}) \geqslant 0$ and $\chi(\dot{N}) < 0$, there does not exist a non-constant holomorphic mapping from $\dot{M}$ to $\dot{N}$*. Thus, we require that, in this digression, the target surface has non-negative Euler–Poincaré characteristic, i.e. $\chi(\dot{N}) \geqslant 0$.

Theorem 1.1 (ii) was established under the assumption that $g_M \geqslant g_N = 1$ and $m \geqslant n \geqslant 1$. We claim now that if $g_N = 1$ and $m > n = 0$, then

$$\#\operatorname{Hol}(\dot{M},\dot{N}) = +\infty.$$

We represent the target surface $\dot{N} = N$ as the complex plane factored by a lattice G. Clearly, the case when the source surface $\dot{M}$ is a once-punctured surface implies all the other cases; in fact, $\mathrm{Hol}(\dot{M}, \dot{N})$ in this case can be only at least as small as $\mathrm{Hol}(\dot{M}, \dot{N})$ with more punctures on M (obviously). Hence, it is enough to show $\mathrm{Hol}(\dot{M}, \dot{N})$ with $g_N = 1$ and $1 = m > n = 0$ to be an infinite set. Observe that (by the Weierstrass 'Gap' Theorem, for example), for each integer $k \geqslant 2g_M$, there exists a meromorphic function f_k on M which is regular except at P_1^M and which has a pole of order k at P_1^M. Following f_k by a natural projection $\mathbb{C} \to \mathbb{C}/G$, we can construct a non-constant holomorphic mapping (also denoted by f_k) from $\dot{M}$ to N. Since the sequence $\{f_k\}$ is distinct, we are done.

What happens if, in addition to our assumption that the target surface $\dot{N}$ is a compact torus, $\dot{M} = M$, i.e. the source surface is also compact without punctures? As an easy consequence of the Riemann–Hurwitz formula, we see that, for $g_M = 0$, $g_N = 1$ and $m = n = 0$, there does not exist a non-constant holomorphic mapping from M to N. Furthermore, we find that, for $g_M = g_N = 1$ and $m = n = 0$,

$$\#\,\mathrm{Hol}(\dot{M}, \dot{N}) = \begin{cases} +\infty & \text{if } M \text{ can be represented as an unbranched covering of } N, \\ 0 & \text{otherwise,} \end{cases}$$

since the Riemann–Hurwitz relation also shows that a holomorphic mapping between compact tori cannot be a branched covering. (In the former case, via standard topology, the set of equivalence classes of unbranched coverings of N is in a natural one-to-one correspondence with the set of the subgroups of the fundamental group of N.) We are left with the case $g_M \geqslant 2$. The cardinality of $\mathrm{Hol}(\dot{M}, \dot{N})$ is again either 0 or $+\infty$, since $\mathrm{Aut}\, N$ is an infinite group. The latter possibility occurs; for example, given a torus with two distinguished points there is a unique Riemann surface of genus 2 formed by taking two copies of the torus and ramifying the sheets above the two distinguished points (see Figure 3). In the same spirit, one can consider the case $g_M \geqslant 3$. (What is an explicit example of the former possibility for which the set of non-constant holomorphic mappings is demonstrably empty?)

While it is possible to consider the cases $g_N = 0$ and $0 \leqslant n \leqslant 2$ at this point, we shall delay the argument until after we have established part (iii) of the theorem. We will continue with this problem in the remark at the end of the section.

We now return to the proof of Theorem 1.1 (iii). To study the cases where $g_N = 0$ and $n \geqslant 3$, we represent $\dot{N}$ as a submanifold of $\hat{\mathbb{C}}$. We say the surface $\dot{N}$ is *normalized* if

$$P_1^N = \infty, \quad P_2^N = 0 \quad \text{and} \quad P_3^N = 1.$$

Henceforth, $\dot{N}$ will be assumed to be normalized unless the contrary is stated explicitly.

Again, the finiteness must be proved, but this time we have little analytic (or geometric) information at our disposal, and we are forced to use a more algebraic method.

Now the extension $\bar{f}$ can clearly be viewed as a meromorphic function on M, with the property that the points in the divisor $(\bar{f})$ are punctures of $\dot{M}$; so if $f \in \mathrm{Hol}(\dot{M}, \dot{N})$, then

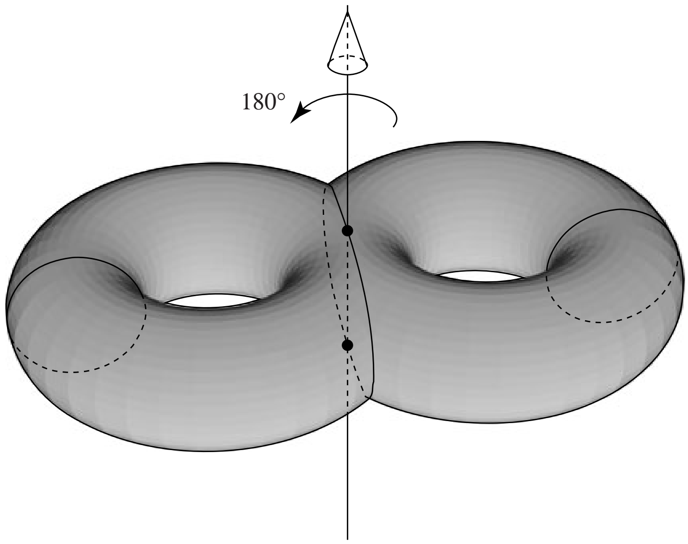

Figure 3. Involution on a surface of genus 2. A torus is associated to a Riemann surface of genus 2. The 'involution' is the rotation by π radians through the axis shown. We can quotient the surface of genus 2 by using the involution to give the original torus.

f determines a divisor $(\bar{f}) \in \mathrm{div}(M)$ by

$$(\bar{f}) = \sum_{P \in \mathcal{P}_M} v_P(\bar{f})P, \tag{5.3}$$

where $v_P(\bar{f})$ is the valuation of $\bar{f}$ at P and all valuations are of the form

$$v_P(\bar{f}) = \mathrm{ord}_P(\bar{f}),$$

where, in the last equation $\mathrm{ord}_P(\bar{f}) = \nu$ (or $-\nu$) if $\bar{f}$ has a zero (or pole) of order ν at P. Thus, we have established a map

$$\lambda\colon\ \mathrm{Hol}(\dot{M}, \dot{N}) \to \mathrm{div}(M) \tag{5.4}$$

from the set of holomorphic mappings to the group of divisors on M which is defined by $f \mapsto (\bar{f})$.

Lemma 5.3. *Let $g_M \geqslant g_N = 0$ and $m \geqslant n \geqslant 3$. Assume that $\dot{N}$ is normalized. Then there is a natural map*

$$\lambda\colon\ \mathrm{Hol}(\dot{M}, \dot{N}) \to \mathrm{div}(M).$$

The map λ is at most $(m-2)$-to-one.

Proof. We let $f, g \in \mathrm{Hol}(\dot{M}, \dot{N})$. Note that if $\lambda(f) = \lambda(g)$, then $\bar{f}$ and $\bar{g}$ have the same zeros and poles and the same multiplicative behaviour. Hence g is a constant multiple of f.

We therefore conclude that the number of inverse images of any point of $\mathrm{div}(M)$ under the map λ is at most $m-2$ (see the proof of Lemma 5.1 for an analogous argument). □

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It is now a simple matter to complete the proof of part (iii) of the theorem. Without loss of generality, we may of course assume that $\dot{N}$ is normalized. We merely note that

$$-\left[\frac{\chi(\dot{M})}{\chi(\dot{N})}\right] \leqslant v_{P_i^M}(\bar{f}) \leqslant \left[\frac{\chi(\dot{M})}{\chi(\dot{N})}\right] \quad \text{for all } P_i^M \in \mathcal{P}_M. \tag{5.5}$$

But this is an immediate consequence of Corollary 2.4 (and the remarks preceding the statement of the lemma). Since we have (5.5), it is clear that the image in $\operatorname{div}(M)$ of $\operatorname{Hol}(\dot{M}, \dot{N})$ under the map λ consists of at most

$$\left(2\left[\frac{\chi(\dot{M})}{\chi(\dot{N})}\right] + 1\right)^m$$

points. It, together with the fact that λ is at most $(m-2)$-to-one, yields the desired inequality (1.4).

Our main theorem is now fully established.

Remark 5.4. We end the paper with a few more trivial remarks about the cases $g_N = 0$ and $0 \leqslant n \leqslant 2$, for the sake of completeness. We may assume that $\dot{N}$ is either the sphere $\hat{\mathbb{C}}$, the plane $\mathbb{C}$ or the punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

By Riemann–Roch, the set of non-constant meromorphic functions is ample for every genus $g \geqslant 0$ Riemann surface. This statement implies that if $g_N = 0$ and $n = 0$, then

$$\#\operatorname{Hol}(\dot{M}, \dot{N}) = +\infty.$$

What if the target Riemann surface has one or two punctures? We claim that, for $g_N = 0$ and $1 \leqslant n \leqslant 2$,

$$\#\operatorname{Hol}(\dot{M}, \dot{N}) = \begin{cases} 0 & \text{if } m = 0, \\ +\infty & \text{if } m \geqslant 1. \end{cases}$$

The only aspect needing some comment is the equality for $m \geqslant 1$. This is almost established by a previous argument. Recall that we have observed, in the digression above, that every non-compact Riemann surface with punctures carries infinitely many non-constant holomorphic functions. Hence, we conclude that the statement is true for $n = 1$. It is also true for $n = 2$ in view of the fact that $\mathbb{C}^*$ has $\mathbb{C}$ as its holomorphic universal covering space (the covering map $\pi\colon \mathbb{C} \to \mathbb{C}^*$ is given by $\pi(z) = \exp(2\pi i z)$ for all $z \in \mathbb{C}$).

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