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# THE JOLY TOPOLOGY AND THE MOSCO-BEER TOPOLOGY REVISITED

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Some extensions to the non reflexive case of continuity results for the Legendre-Fenchel transform are presented following an approach due to J.-L. Joly. We compare the topology introduced by J.-L. Joly and the Mosco-Beer topology introduced by G. Beer. In particular, in the case of the space of closed proper convex functions defined on the dual of a normed vector space they coincide.

## 1. INTRODUCTION AND NOTATIONS

The continuity of the Legendre-Fenchel transform represents a crucial property for convergence questions dealing with optimisation problems in duality. Wijsman [**16, 17**] pointed out this fact for his "infimal convergence" in finite dimensional spaces. Mosco [**11, 12**] extended this result to reflexive Banach spaces for the convergence he introduced. The corresponding fact in general Banach spaces for the stronger bounded Hausdorff topology has been studied by Walkup and Wets in [**15**], Attouch and Wets [**2**], Beer [**6**], Penot [**13**]. However this topology is so strong that an increasing sequence of finite dimensional subspaces $(X_n)$ whose union is dense in a separable Banach space cannot converge to $X$.

It is the purpose of the present paper to consider this question for a topology close to the Mosco topology as introduced by Beer in [**3**] (we call it the Mosco-Beer topology henceforth). The reason why we look for a modification of this topology lies in the fact it has been shown by Beer and Borwein [**7**] that outside the class of reflexive Banach spaces the Mosco-Beer topology does not have the pleasant properties one would expect: it is not Hausdorff and the Legendre-Fenchel transform is not continuous. Our strategy consists in examining anew a proposal made by Joly [**9, 10**] which, by brute force, makes bicontinuous the Legendre-Fenchel transform and in proving that this proposal has decent properties and coincides with the Mosco-Beer topology for convex functions on reflexive Banach spaces. We also show that the Joly topology is well suited for optimisation problems (Propositions 1.1 and 2.4)

Given a topological space $P$, a subset $T$ of $P$, a point 0 of $P$ in the closure of $T$ and given a family of subsets $(C_t)_{t\in T}$ of a topological space $(X,\tau)$ parametrised by $T$,

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the *Kuratowski-Painlevé limit inferior (respectively superior)* of the family $(C_t)_{t\in T}$ as $t$ goes to 0 is the set denoted by $\liminf_{t\to 0} C_t$ (respectively $\limsup_{t\to 0} C_t$) of $x \in X$ such that each neighbourhood of $x$ meets $C_t$ eventually (respectively frequently). Equivalently, $x \in \limsup_{t\to 0} C_t$ if and only if there exist a net $(t_i)_{i\in I}$ converging to 0 in $T$ and a net $(x_i)_{i\in I}$ converging to $x$ in $X$ such that $x_i \in C_{t_i}$ for each $i$, and $x \in \liminf_{t\to 0} C_t$ if and only if for each net $(t_i)_{i\in I}$ converging to 0 in $T$ there exists a subnet $(t_j)_{j\in J}$ and a net $(x_j)_{j\in J}$ converging to $x$ in $X$ such that $x_j \in C_{t_j}$ for each $j \in J$. In the case when $X$ is metrisable and the sets $C_t$ are closed, one has $x \in \liminf_{t\to 0} C_t$ (respectively $x \in \limsup_{t\to 0} C_t$) if and only if $\limsup_{t\to 0} d(x, C_t) = 0$ (respectively $\liminf_{t\to 0} d(x, C_t) = 0$). We say that $C = \lim_{t\to 0} C_t$ if $\limsup_{t\to 0} C_t \subset C \subset \liminf_{t\to 0} C_t$.

Given a function $f$ defined on a set $X$ with values in $\mathbb{R} \cup \{+\infty\}$, we denote by $E(f) \subset X \times \mathbb{R}$ its *epigraph*, $E(f) = \{(x,r) \in X \times \mathbb{R} : f(x) \leqslant r\}$ and by $S(f,r) = \{x \in X : f(x) \leqslant r\}$ its *sublevel sets*. The *indicator function* of a subset $A \subset X$ is the function $i_A$ defined by $i_A(x) = 0$ if $x \in A$ and $i_A(x) = +\infty$ if $x \notin A$.

We denote by $\mathcal{C}(X)$ the set of closed convex subsets of a normed vector space $(X, \|\,.\,\|)$ whose closed unit ball is denoted by $B$ and dual space by $X^*$. The closed ball with centre $x$ and radius $r$ is denoted by $B(x,r)$. We say that two normed vector spaces $X$, $Y$ are in *duality* (respectively in *metric duality*) if they are coupled by a bilinear function $\langle .,. \rangle$ in such a way that the mappings $x \longmapsto \langle x, . \rangle$ and $y \longmapsto \langle ., y \rangle$ are embeddings (respectively isometric embeddings) of $X$ into $Y^*$ and of $Y$ into $X^*$. We denote by $\mathcal{C}_Y(X)$ the set of $\sigma(X,Y)$-closed nonempty convex subsets of $X$ and we adopt an analogous definition for $\mathcal{C}_X(Y)$. In the sequel $\mathrm{Aff}_Y(X)$ stands for the set of mappings $x \longmapsto \langle x, y \rangle - r$, where $(y,r) \in Y \times \mathbb{R}$ and $\Gamma_Y(X)$ ($\Gamma(X)$ if there is no ambiguity on $Y$, $\Gamma_0(X)$ if $Y = X^*$) denotes the set of proper functions on $X$ whose epigraphs belong to $\mathcal{C}_{Y\times\mathbb{R}}(X \times \mathbb{R})$, with an analogous definition for $\mathrm{Aff}_X(Y)$ and $\Gamma_X(Y)$. The sets $\Gamma_Y(X)$ and $\Gamma_X(Y)$ are connected by a one to one mapping $\mathcal{L}$, namely the Legendre-Fenchel transform, defined by $\mathcal{L}(f) = f^*$ where

$$f^*(y) = \sup\{\langle x, y \rangle - f(x) : x \in X\}.$$

A classical way to define topologies on a subset $\mathcal{S}$ of the power set $2^X$ of a set $X$ is to use the sets

$$A^- = \{C \in \mathcal{S} : A \cap C \neq \emptyset\}$$

and

$$A^+ = \{C \in \mathcal{S} : C \subset A\}$$

where $A$ runs over some family $\mathcal{A} \subset 2^X$. The topologies $\mathcal{A}^-$ and $\mathcal{A}^+$ are the weakest topologies on $\mathcal{S}$ containing respectively the families $\{A^- : A \in \mathcal{A}\}$ and $\{A^+ : A \in \mathcal{A}\}$.

In the sequel we shall mainly deal with $\mathcal{S} = \mathcal{C}_Y(X)$ where $X$, $Y$ are normal vector spaces in metric duality, in particular when $Y = X^*$ (in which case we write $\mathcal{C}(X)$) and when $X = Y^*$ (in such a case we write $\mathcal{C}^*(Y^*)$ for $\mathcal{C}_Y(X)$). We identify a function $f$ with its epigraph $E(f)$ so that, for $W \subset X \times \mathbb{R}$, $f \in W^-$ means $E(f) \cap W \neq \emptyset$.

We also recall that the *topology of bounded hemiconvergence* or the *bounded-Hausdorff topology* $\tau_{BH}$ is the weakest topology on $\mathcal{C}_Y(X)$ which makes continuous the functions $e_r(A,.)$ and $e_r(.,A)$ for $r \in \mathbb{R}_+$, $A \in \mathcal{C}_Y(X)$, where

$$e_r(A,C) = \sup\{d(x,C) : x \in A \cap rB\},$$

and

$$d(x,C) = \inf\{d(x,y) : y \in C\},$$

(see [**1, 6, 11, 13, 14**]). Let us introduce the topologies we intend to study in this paper. We define them either on $\mathcal{C}(X)$ or on $\Gamma_0(X)$. Considering the mappings $f \longmapsto E(f)$ from $\Gamma_0(X)$ into $\mathcal{C}(X \times \mathbb{R})$ and $C \longmapsto i_C$ from $\mathcal{C}(X)$ into $\Gamma_0(X)$ as embeddings shows that these two possibilities immediately induce a topology on the other space.

DEFINITION 1.1: Let $X$ be a normed vector space (n.v.s.).

(a) The Mosco-Beer topology $\tau_M$ is the topology defined on $\mathcal{C}(X)$ by $\tau_M^- \vee \tau_M^+$ with $\tau_M^- = \mathcal{O}^-, \tau_M^+ = \mathcal{W}^+$ where $\mathcal{O}$ (respectively $\mathcal{W}$) denotes the family of open subsets of $X$ (respectively of complements $X \backslash K$, $K$ weakly compact).

(b) The dual Mosco-Beer topology $\tau_M^*$ is defined on the set $\mathcal{C}^*(X^*) := \mathcal{C}_X(X^*)$ of $w^*$-closed convex subsets of $X^*$ by $\mathcal{O}^- \vee \mathcal{W}^{*+}$ where $\mathcal{W}^*$ stands for the family of complements of $w^*$-compact subsets of $X^*$.

Let us mention the following lemma whose proof mimics the one of Lemma 5.4 in [**8**].

LEMMA 1.1. *Let $X$, $Y$ be n.v.s. in duality. Assume that bounded subsets of $X$ are relatively $\sigma(X,Y)$-compact. Let $\mathcal{W}_C$ be the family of complements of $\sigma(X,Y)$-compact convex subsets of $X$. Then the topologies $\mathcal{W}_C^+$ and $\mathcal{W}^+$ coincide on $\mathcal{C}_Y(X)$.*

DEFINITION 1.2: Let $X$, $Y$ be *normed vector spaces in duality*. The Joly topology $\tau_J$ is the weakest topology on $\Gamma_Y(X)$ which makes continuous the mappings $E : f \longmapsto E(f)$ and $E^* : f \longmapsto E(f^*)$ from $\Gamma_Y(X)$ into $(\mathcal{C}(X \times \mathbb{R}), \mathcal{O}^-)$ and $(\mathcal{C}(Y \times \mathbb{R}), \mathcal{O}^-)$ respectively.

Equivalently, the Joly topology is the weakest topology on $\Gamma_Y(X)$ for which $f \longmapsto E(f)$ and $f \longmapsto E \circ \mathcal{L}(f)$ are lower semicontinuous multifunctions. Let us give a still more concrete description of this topology, first in terms of convergence, then in terms of open sets.

LEMMA 1.2. *A net $(f_i)_{i \in I}$ of $\Gamma_Y(X)$ converges to $f \in \Gamma_Y(X)$ if and only if for any $x \in dom\ f$, $y \in dom\ f^*$, there exists a subnet $(f_j)_{j \in J}$, a net $(x_j)_{j \in J}$ with limit*

*$x$, a net $(y_j)_{j\in J}$ with limit $y$ such that*

$$\liminf_{j\in J} f_j(x_j) \leqslant f(x),$$
$$\liminf_{j\in J} f_j^*(y_j) \leqslant f^*(y).$$

Using the fact that the conjugate function of the indicator function of a cone $C$ is the indicator function of the polar $C^\circ$ of $C$ we get the following

COROLLARY 1.1. *A parametrised family $(C_t)_{t\in T}$ of closed convex cones in $\mathcal{C}_Y(X)$ converges to $C \in \mathcal{C}_Y(X)$ for $\tau_J$ if and only if*

$$C \subset \liminf_{t\to 0} C_t,$$
$$C^\circ \subset \liminf_{t\to 0} C_t^\circ.$$

Let us give a still more concrete description of this topology. Let $\Omega$ be an open subset of $X \times \mathbb{R}$, let $f \in E^{-1}(\Omega^-)$ and let $(x,s) \in E(f) \cap \Omega$. Let $U$ be an open neighbourhood of $x$ and let $s_1 < s < s_2$ be such that $U \times (s_1, s_2) \subset \Omega$. Identifying $f$ with $E(f)$ one has

$$f \in (U \times (-\infty, s_2))^-,$$

and since $E(h) \cap U \times (-\infty, s_2) \neq \emptyset$ implies that $E(h) \cap \Omega \neq \emptyset$ one gets

$$f \in E^{-1}(\Omega^-) \supset E^{-1}\Big((U \times (-\infty, s_2))^-\Big).$$

It ensues that the Joly topology is generated by the sets:

$$E^{-1}(U \times (-\infty, s)) = \{f \in \Gamma_Y(X) : E(f) \cap U \times (-\infty, s) \neq \emptyset\},$$
$$(E \circ \mathcal{L})^{-1}(V \times (-\infty, s)) = \{f \in \Gamma_Y(X) : E(f^*) \cap V \times (-\infty, s) \neq \emptyset\},$$

where $s \in \mathbb{R}$ and $U$ (respectively $V$) is an open subset of $X$ (respectively $Y$). Let us set

$$[U, s] = E^{-1}(U \times (-\infty, s))$$

and $$]V, s[ = (E \circ \mathcal{L})^{-1}(V \times (-\infty, s)),$$

so that $$[U, s] = \{f \in \Gamma_Y(X) : \exists\, x \in U, f(x) < s\},$$
$$]V, s[ = \{f \in \Gamma_Y(X) : \exists\, y \in V, \exists\, t < s, E(f) \subset D_{y,t}\},$$

where $$D_{y,t} = \{(x, r) \in X \times \mathbb{R} : \langle x, y\rangle \leqslant r + t\}.$$

Using the embedding $i : C \longmapsto i_C$, we obtain

$$i^{-1}([U,s]) = \begin{cases} \{C \in \mathcal{C}_Y(X) : \exists\, x \in U,\ i_C(x) < s\} = U^- & \text{for } s \in \mathbb{P} = (0,+\infty), \\ \emptyset & \text{otherwise,} \end{cases}$$

and

$$i^{-1}(]V,s[) = \{C \in \mathcal{C}_Y(X) : \exists\, y \in V,\ \sigma_C(y) < s\},$$

where $\sigma_C$ denotes the *support function* of $C$, $\sigma_C = (i_C)^*$. Thus $\tau_J = \tau_J^- \vee \tau_J^+$ where $\tau_J^- = \mathcal{O}^-$ is the lower-Vietoris topology and $\tau_J^+$ is generated by the sets

$$\{C \in \mathcal{C}_Y(X) : \inf_V \sigma_C < s\}$$

where $V$ is an open set of $Y$ and $s \in \mathbb{R}$.

The following property follows easily from the definition of $\tau_J$ and shows the relevance of the Joly topology for optimisation problems.

**Proposition 1.1.** *Let $X$, $Y$ be normed vector spaces in duality. The Joly topology $\tau_J$ is the weakest topology on $\Gamma_Y(X)$ such that for each open subset $U$ (respectively $V$) of $X$ (respectively $Y$), the functions $f \longmapsto \inf_U f$ and $f \longmapsto \inf_V f^*$ are upper semicontinuous.*

Proof: The weakest topology on $\Gamma_Y(X)$ such that for each open subset $U$ (respectively $V$) of $X$ (respectively $Y$), the functions $f \longmapsto \inf_U f$ and $f \longmapsto \inf_V f^*$ are upper semicontinuous is generated by the sets

$$\{f \in \Gamma_Y(X) : \inf_U f < s\} = [U,s],$$
$$\{f \in \Gamma_Y(X) : \inf_V \mathcal{L}(f) < s\} = ]V,s[,$$

hence the result. □

Observe that, from the very definition of the Joly topology, the Legendre-Fenchel transform $\mathcal{L}$ is an homeomorphism from $(\Gamma_Y(X), \tau_J)$ onto $(\Gamma_X(Y), \tau_J)$. Therefore it will be useful only if one disposes of workable characterisations of this topology.

## 2. Identification of the Joly Topology

The definition of the Joly topology is sensible as shown by the following result.

**Proposition 2.1.** *(Compare with* [9, 10] *and* [4, Theorem 3.4]*). The Joly topology $\tau_J$ induces on the space $\mathrm{Aff}_Y(X)$ of $Y$-affine functions on $X$ the topology transported by the isomorphism $a : (y,r) \longmapsto \langle \cdot, y\rangle - r$ of $Y \times \mathbb{R}$ onto $\mathrm{Aff}_Y(X)$.*

Proof: Let $(y_0, r_0) \in Y \times \mathbb{R}$ and let $U$ be an open subset of $X \times \mathbb{R}$ such that $a(y_0,r_0) \in U^-$. Let $(x_0, s_0) \in U$ be such that $\langle x_0, y_0\rangle - r_0 \leqslant s_0$ and let $\varepsilon > 0$

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be such that $B(x_0, \varepsilon) \times [s_0 - \varepsilon, s_0 + \varepsilon] \subset U$. Then for $r \in [r_0 - \varepsilon/2, r_0 + \varepsilon/2]$ and $y \in B\Big(y_0, 2^{-1}(\|x_0\| + 1)^{-1}\varepsilon\Big)$ we have $\langle x_0, y\rangle - r \leqslant s_0 + \varepsilon$, hence $a(y,r) \in U^-$.

Now let $V$ be an open subset of $Y \times \mathbb{R}$ such that $[a(y_0, r_0)]^* \in V^-$. Let $(z_0, t_0) \in V$ be such that $[a(y_0, r_0)]^*(z_0) \leqslant t_0$. Since

$$[a(y_0, r_0)]^*(z_0) = i_{\{y_0\}}(z_0) + r_0,$$

we have $z_0 = y_0$ and $r_0 \leqslant t_0$ and for $(y, r)$ close enough to $(y_0, r_0)$ we have

$$(y, t_0 + r - r_0) = (y_0, t_0) + (y - y_0, r - r_0) \in V$$

and

$$(y, r + t_0 - r_0) = (y, [a(y,r)]^*(y) + t_0 - r_0) \in E([a(y,r)]^*),$$

so that $[a(y,r)]^* \in V^-$. Thus we have shown that $a(.,.)$ is continuous from $Y \times \mathbb{R}$ onto $(\mathrm{Aff}_Y(X), \tau_J)$.

Now let $(y_0, r_0) \in Y \times \mathbb{R}$ and let $\varepsilon > 0$ be given. Setting $V = \operatorname{int} B(y_0, \varepsilon) \times (r_0 - \varepsilon, r_0 + \varepsilon)$ what precedes shows that for any $(y,r) \in Y \times \mathbb{R}$ with $[a(y,r)]^* \in V^-$ we have $y \in \operatorname{int} B(y_0, \varepsilon)$ and $r < r_0 + \varepsilon$. If moreover for

$$U = \operatorname{int} B\Big(0, (2\|y_0\| + 2\varepsilon)^{-1}\varepsilon\Big) \times (-r_0 - \varepsilon/2, -r_0 + \varepsilon/2)$$

we have $a(y,r) \in U^-$ then we can find $(x_0, t_0) \in U$ with $\langle y, x_0\rangle - r \leqslant t_0$ and hence

$$r \geqslant -|\langle x_0, y\rangle| - t_0 \geqslant -(\|y_0\| + \varepsilon)(2\|y_0\| + 2\varepsilon)^{-1}\varepsilon + r_0 - \varepsilon/2 > r_0 - \varepsilon.$$

Thus $(y,r) \in V$ whenever $(y,r)$ is such that $a(y,r) \in U^-$ and $[a(y,r)]^* \in V^-$. As $a(y_0, r_0) \in U^-$ and $[a(y_0, r_0)]^* \in V^-$ we conclude that the mapping $a^{-1}$ is continuous from $(\mathrm{Aff}_Y(X), \tau_J)$ onto $Y \times \mathbb{R}$. ∎

An important property of the Joly topology is the following. It is in sharp contrast with what occurs in general for the Mosco-Beer topology.

**Proposition 2.2.** *For any pair $(X, Y)$ of n.v.s. in duality the topological space $(\Gamma_Y(X), \tau_J)$ is Hausdorff.*

**Proof:** Let $f$, $g$ be distincts elements of $\Gamma_0(X)$. Interchanging $f$ and $g$ if necessary, we can find $\bar{x} \in X$, $\bar{r} \in \mathbb{R}$ such that $f(\bar{x}) < \bar{r} < g(\bar{x})$. Since $g \in \Gamma_Y(X)$, we can find $(\bar{y}, \bar{t}) \in Y \times \mathbb{R}$ such that

$$g(.) \geqslant \langle ., \bar{y}\rangle - \bar{t}$$

and

$$\bar{r} < \langle \bar{x}, \bar{y}\rangle - \bar{t}.$$

Since $(x, y, t) \longmapsto \langle x, y\rangle - t$ is continuous we can find $\varepsilon > 0$ such that

$$\bar{r} < \langle x, y\rangle - t,$$

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for each $(x,y) \in B(\overline{x},\varepsilon) \times B(\overline{y},\varepsilon)$, $t \in [\overline{t}-\varepsilon, \overline{t}+\varepsilon]$. Let

$$W = \operatorname{int} B(\overline{x},\varepsilon) \times (-\infty, \overline{r}),$$
$$Z = \operatorname{int} B(\overline{y},\varepsilon) \times \left(\overline{t}-\varepsilon, \overline{t}+\varepsilon\right).$$

Since $(\overline{y},\overline{t}) \in E(g^*)$ and $(\overline{x}, f(\overline{x})) \in E(f)$ we have $g^* \in Z^-$ and $f \in W^-$. Now $W^-$ and $(E \circ \mathcal{L})^{-1}(Z^-)$ are disjoint since if $h \in W^- \cap (E \circ \mathcal{L})^{-1}(Z^-)$ we can find $x \in \operatorname{int} B(\overline{x},\varepsilon)$, $y \in \operatorname{int} B(\overline{y},\varepsilon)$, $r < \overline{r}$, $t \in \left(\overline{t}-\varepsilon, \overline{t}+\varepsilon\right)$ such that

$$h(x) \leqslant r,$$
$$t \geqslant h^*(y) \geqslant \langle x,y \rangle - r,$$

a contradiction with our choice of $\varepsilon$ which ensures that

$$r < \overline{r} < \langle x,y \rangle - t\,.$$

□

**Corollary 2.1.** *Let $X$ be a Banach space with dual space $Y$. If the Joly topology $\tau_J$ and the Mosco-Beer topology $\tau_M$ coincide on $\Gamma_Y(X)$ then $X$ is reflexive.*

Proof: This follows from the preceding proposition and from [7, Theorem 4.2] asserting that $\tau_M$ is not Hausdorff when $X$ is not reflexive. □

We will see later that the converse is true. An interesting property of the Joly topology is given in Proposition 2.4 which completes Proposition 1.1. In order to prove it, we shall need the following lemma. We present a proof for the sake of completeness.

**Lemma 2.1.** [5, Lemma 2.1] *Let $f \in \Gamma_Y(X)$ and let $M$ be a convex $\sigma(X \times \mathbb{R}, Y \times \mathbb{R})$-compact subset of $X \times \mathbb{R}$ such that $E(f) \cap M = \emptyset$. Then there exists $(y,t) \in E(f^*)$ and $s > t$ such that $\langle x,y \rangle - r > s$ for each $(x,r) \in M$. Moreover,*

$$(M^c)^+ = \{g \in \Gamma_Y(X) : E(g) \cap M = \emptyset\}$$

*is open in $\tau_J$.*

Proof: Observe that since $E(f) - M$ is convex, $\sigma(X \times \mathbb{R}, Y \times \mathbb{R})$-closed and stable by addition of elements of $\{0\} \times \mathbb{R}_+$, it is the epigraph of some $h \in \Gamma_Y(X)$ such that $(0,0) \notin E(h)$. Thus there exist $y \in Y$ and $\varepsilon > 0$ such that $h(.) > \langle .,y \rangle + 2\varepsilon$. Setting $s = \inf\{\langle y,x \rangle - r : (x,r) \in M\} - \varepsilon$ and $t = s - \varepsilon$, we get the first assertion of the lemma since for any $(x,r) \in M$ and any $(u,p) \in E(f)$ we have $p - r \geqslant h(u-x) \geqslant \langle u-x, y \rangle + 2\varepsilon$, hence

$$t = \inf\{\langle x,y \rangle - r - 2\varepsilon : (x,r) \in M\} \geqslant \sup\{\langle u,y \rangle - f(u) : u \in X\} = f^*(y).$$

Now let $V^*$ be a neighbourhood of $y$ such that $|\langle x, z-y\rangle| < \varepsilon$ for any $z \in V^*, (x,r) \in M$. Then $]V^*, s[$ is a neighbourhood of $f$ since $E(f) \subset D_{y,t} := \{(x,r) \in X \times \mathbb{R}, \langle x,y\rangle \leqslant r+t\}$ is equivalent to $(y,t) \in E(f^*)$ and for any $g \in ]V^*, s[$ we can find $z \in V^*, q \in \mathbb{R}$ with $q < s$, $E(g) \subset D_{z,q} \subset D_{z,s}$ and $D_{z,s} \cap M = \emptyset$ so that $]V^*, s[ \subset (M^c)^+$. ∎

Taking into account the way the Mosco-Beer topology is generated we deduce from the preceding lemma and from Lemma 1.1 the following result.

**COROLLARY 2.2.**

(a) *The Joly topology $\tau_J$ on $\Gamma_Y(X)$ is stronger than the topology $\mathcal{O}^- \vee \mathcal{W}_C^+$ where $\mathcal{W}_C^+$ is the family of complements of $\sigma(X \times \mathbb{R}, Y \times \mathbb{R})$-compact convex subsets of $X \times \mathbb{R}$.*

(b) *The Joly topology $\tau_J$ is stronger on $\Gamma_Y(X)$ than the Mosco-Beer topology $\tau_M$ with $Y = X^*$, $X$ reflexive and is stronger on $\Gamma_X(X^*)$ than the dual Mosco-Beer topology $\tau_M^*$.*

Let us mention the following comparison result.

**PROPOSITION 2.3.** *The Joly topology $\tau_J$ on $\Gamma_Y(X)$ is weaker than the topology of bounded hemiconvergence (or bounded Hausdorff topology or Attouch-Wets topology $\tau_{BH}$).*

PROOF: Let $(f_t)_{t\in T}$ be a parametrised family in $\Gamma_Y(X)$ which converges to $f$ for $\tau_{BH}$ in the sense that $(E(f_t))_{t\in T}$ converges to $E(f)$ in $(\mathcal{C}_{Y\times\mathbb{R}}(X \times \mathbb{R}), \tau_{BH})$. Then we have

$$E(f) \subset \liminf_{t\to 0} E(f_t).$$

Since the Legendre-Fenchel transform is continuous with respect to $\tau_{BH}$ **[1, 6, 13]** we also have $(f_t^*)_{t\in T}$ converges to $f^*$ for $\tau_{BH}$ and $E(f^*) \subset \liminf_{t\to 0} E(f_t^*)$. ∎

**PROPOSITION 2.4.** *For any $\sigma(X,Y)$-compact convex subset $K$ of $X$, the function $f \longmapsto \inf_K f$ is lower semicontinuous on $\Gamma_Y(X)$ endowed with $\tau_J$. In the case $Y = X^*$, $X$ reflexive or $X = Y^*$ this result holds true for any $\sigma(X,Y)$ compact subset $K$ of $X$.*

PROOF: This assertion follows from Lemma 1.1 and Lemma 2.1 by observing that the set of $f \in \Gamma_Y(X)$ with $\inf_K f > s$ is the set $(M^c)^+$ with $M = K \times \{s\}$. ∎

The following theorem extends a result of Beer and Pai **[8**, Theorem 3.1] to the nonreflexive case.

**THEOREM 2.1.** *For any n.v.s. $Y$ with dual space $X = Y^*$ the Joly topology $\tau_J$ and the dual Mosco-Beer topology $\tau_M^*$ coincide on $\Gamma_Y(X)$.*

PROOF: By Corollary 2.2, it suffices to prove that $\tau_M^*$ is stronger than $\tau_J$. This amounts to showing that for any open subset $V$ of $Y$, $r \in \mathbb{R}$ the set

$$\Omega = \mathcal{L}^{-1}\Big(\big(V \times (-\infty, r)\big)^{-}\Big)$$

is open in $\tau_M^*$. Let $g \in \Omega$ and let $f = g^*$. Thus there exists $y_0 \in V$ with $f(y_0) < r$. Choose $\varepsilon \in (0,1)$, $s \in \mathbb{R}$ with $B(y_0, \varepsilon) \subset V$ and

$$(1) \qquad f(y_0) < s < f(y_0) + \varepsilon < r.$$

Since $g = f^*$, our choice of $s$ ensures

(i) $g(x) > \langle x, y_0 \rangle - s$ for each $x \in X$,
(ii) there exists $x_0 \in X$ with $g(x_0) < \langle x_0, y_0 \rangle - s + \varepsilon$.

Let

$$A = B(x_0, r_0) \text{ with } r_0 = \max(3, \|x_0\|)$$

and let

$$K = (A \times \mathbb{R}) \cap Gr(a_0) \text{ where } a_0 = \langle ., y_0 \rangle - s.$$

Then $K$ is a $\sigma(X,Y)$-compact convex subset of $X \times \mathbb{R}$ and $E(g) \subset K^c$ by (i). By (ii) there is an open neighbourhood $U$ of $x_0$ contained in $x_0 + B$ such that for each $x \in U$ we have

$$g(x_0) - \varepsilon < \langle x, y_0 \rangle - s.$$

Let us show that the $\tau_M^*$-neighbourhood

$$N = (K^c)^+ \cap (U \times (-\infty, g(x_0) + \varepsilon))^-$$

of $g$ is contained in $\Omega$. Let $h \in N$ and let $x_1 \in U$ with $h(x_1) < g(x_0) + \varepsilon$. Since the $\sigma(X,Y)$-compact convex set $K$ is disjoint from $E(h)$, using Lemma 2.1 we can find $(y_1, r_1) \in E(h^*)$ such that $K$ lies below the graph of $\langle ., y_1 \rangle - r_1$. This means that

$$(2) \qquad \langle ., y_0 \rangle - s + i_A < \langle ., y_1 \rangle - r_1 + i_A.$$

Taking conjugates and observing that

$$(\langle ., y_0 \rangle - s + i_A)^* = (i_A)^*(. - y_0) + s,$$
$$(i_A)^* = \left(i_{r_0 B_X}\right)^* + \langle x_0, . \rangle = r_0 \|.\| + \langle x_0, . \rangle,$$

we get for each $y \in X$

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(iii) $s + r_0 \|y - y_0\| + \langle x_0, y - y_0 \rangle \geqslant r_1 + r_0 \|y - y_1\| + \langle x_0, y - y_1 \rangle$.

Now, by our choice of $(y_1, r_1) \in E(h^*)$, we have

$$\langle x_1, y_1 \rangle - r_1 \leqslant h(x_1) < g(x_0) + \varepsilon,$$

and by definition of $U$
$$s + g(x_0) - \varepsilon < \langle x_1, y_0 \rangle,$$

which entails

$$s - r_1 < \langle x_1, y_0 - y_1 \rangle + 2\varepsilon, \tag{3}$$

so that, using (iii) with $y = y_0$ we get

$$r_0 \|y_1 - y_0\| \leqslant s - r_1 + \langle x_0, y_1 - y_0 \rangle.$$

Hence from (3)
$$r_0 \|y_1 - y_0\| \leqslant \langle x_1 - x_0, y_0 - y_1 \rangle + 2\varepsilon.$$

Since $\|x_1 - x_0\| \leqslant 1$ and $3 \leqslant r_0$ it ensues that $2\|y_0 - y_1\| \leqslant 2\varepsilon$. Therefore $y_1 \in V$ and, as $0 \in A$, we derive from (2) that

$$-s < -r_1$$

thus from (1)
$$r_1 < s < r$$

hence $(y_1, r_1) \in (V \times (-\infty, r)) \cap E(h^*)$, and $h \in \Omega$. ∎

**Corollary 2.3.** *Let $X$ be a Banach space with dual space $Y$. Then the Joly topology $\tau_J$ and the Mosco-Beer topology $\tau_M$ coincide if and only if $X$ is reflexive.*

Proof: This follows from Theorem 2.1 and Corollary 2.2. ∎

**Corollary 2.4.** *For any n.v.s. $X$ the Legendre-Fenchel transform $\mathcal{L}$ is continuous from $(\Gamma_X(X^*), \tau_M^*)$ into $(\Gamma_{X^*}(X), \widetilde{\tau}_M)$ with $\widetilde{\tau}_M = \mathcal{O}^+ \vee \mathcal{W}_C^+$.*

Proof: This follows from the fact that $\tau_J$ and $\tau_M^*$ coincide on $\Gamma_X(X^*)$ (Theorem 2.1), that $\mathcal{L}$ is continuous from $(\Gamma_X(X^*), \tau_J)$ into $(\Gamma_{X^*}(X), \tau_J)$ and that $\widetilde{\tau}_M$ is weaker than $\tau_J$ (Corollary 2.2, (a)). ∎

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