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TORSION IN K_0 OF UNIT-REGULAR RINGS*

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We construct examples of unit-regular rings R for which $K_0(R)$ has torsion, thus answering a longstanding open question in the negative. In fact, arbitrary countable torsion abelian groups are embedded in K_0 of simple unit-regular algebras over arbitrary countable fields. In contrast, in all these examples $K_0(R)$ is strictly unperforated.

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1. Introduction

A longstanding open problem has been whether K_0 of a unit-regular ring R must be torsionfree. Equivalently, if A and B are finitely generated projective right R-modules and the direct sum of m copies of A is isomorphic to the direct sum of m copies of B, for some positive integer m, is $A \cong B$? (Cf. [**8**, p. 200] and [**4**, Open Problem 27, p. 347].) Positive answers are known for various classes of regular rings, including regular rings whose primitive factors are artinian [**4**, Proposition 6.11], regular rings satisfying general comparability [**4**, Theorem 8.16], right $\aleph_0$-continuous regular rings [**11**, Corollary 2.2; **6**, Corollary 2.6; **1**, Theorem 2.13], and N*-complete regular rings [**5**, Theorem 2.6]. On the other hand, among non-unit-regular rings some negative examples are known; e.g., every finite cyclic group is isomorphic to K_0 of some regular ring [**4**, Example 15.1]. These examples are directly infinite, as are the examples with stable rank 2 constructed in [**12**, Example 4].

Here we demonstrate that torsion can occur in K_0 of unit-regular rings; in fact, given an arbitrary countable torsion abelian group G, we construct simple unit-regular rings R for which $K_0(R)$ contains a copy of G. Interestingly, torsion is the only source of perforation here, for in these examples $K_0(R)$ is *strictly unperforated*, meaning that whenever $x \in K_0(R)$ and $n \in \mathbb{N}$ with $nx > 0$, then $x > 0$. We proceed by building a tower of several constructions, starting with directly infinite examples of the sort mentioned above. Such an example may be cut down to a countable dimensional algebra over a field, and a construction of Tyukavkin [**14**] then allows us to represent the latter algebra as a factor of a subalgebra of a direct product of matrix algebras. This produces examples which are at least directly finite. Finally, we use a direct limit construction to

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embed these directly finite examples into simple unit-regular algebras which inherit the nontrivial torsion in K_0.

We generally follow the notation and conventions of [4]. In particular, nA denotes the direct sum of n copies of a module A. In Section 4, it will be convenient to work within the monoid of isomorphism classes of finitely generated projective right modules over a ring R. We denote this monoid by $\mathrm{FP}_{\cong}(R)$ and write its elements in the form $\langle A\rangle$. Thus $\langle A\rangle$ denotes the isomorphism class of a finitely generated projective module A, as opposed to its stable isomorphism class $[A]\in K_0(R)$. The operation in $\mathrm{FP}_{\cong}(R)$ is induced from direct sum: $\langle A\rangle+\langle B\rangle=\langle A\oplus B\rangle$. Any ring homomorphism $\phi: R\to S$ induces a monoid homomorphism $\mathrm{FP}_{\cong}(\phi): \mathrm{FP}_{\cong}(R)\to \mathrm{FP}_{\cong}(S)$ sending isomorphism classes $\langle A\rangle$ to isomorphism classes $\langle A\otimes_R S\rangle$, where S is viewed as an R-S-bimodule via ϕ. Thus we obtain a functor $\mathrm{FP}_{\cong}(-)$ from rings to monoids. We shall need two basic properties of this functor. First, observe that $\mathrm{FP}_{\cong}(-)$ preserves direct limits. Second, when R and S are Morita-equivalent rings, the categories of finitely generated projective right modules over R and S are equivalent, and so $\mathrm{FP}_{\cong}(S)\cong \mathrm{FP}_{\cong}(R)$. In particular, if $S=M_n(R)$ for some n, there is an isomorphism $\mathrm{FP}_{\cong}(S)\to \mathrm{FP}_{\cong}(R)$ given by $\langle A\rangle\mapsto\langle A\otimes_S P\rangle$, where P is the left-hand column of S. Note that this isomorphism sends $\langle S\rangle$ to $n\langle R\rangle$. On the other hand, the diagonal map $\Delta: R\to S$ usually does *not* induce an isomorphism of $\mathrm{FP}_{\cong}(R)$ onto $\mathrm{FP}_{\cong}(S)$: the composition of $\mathrm{FP}_{\cong}(\Delta)$ with the isomorphism just discussed is the endomorphism of $\mathrm{FP}_{\cong}(R)$ given by multiplication by n.

2. Torsion over directly infinite regular rings

Examples are already known of directly infinite regular rings for which K_0 is a finite cyclic group of arbitrary order [4, Example 15.1]. We build on these examples to obtain larger torsion subgroups, and then we trim the rings to countable examples for use in the following section. If one only wishes to obtain examples of simple unit-regular rings R for which $K_0(R)$ contains a finite cyclic subgroup, the following proposition is not needed; it may be replaced by [4, Example 15.1] in the proof of Corollary 2.2, and the latter proof simplifies accordingly.

Proposition 2.1. *Given any field F and any torsion abelian group G, there exists a regular F-algebra T such that G embeds in $K_0(T)$.*

Proof. After replacing G by its divisible hull, we may assume that $G\cong\bigoplus_\alpha G_\alpha$ where each $G_\alpha=\mathbb{Z}(p_\alpha^\infty)$ for some prime p_α. If each G_α embeds in K_0 of a regular F-algebra T_α, then G embeds in $K_0(\prod_\alpha T_\alpha)$. Thus we need only consider the case that $G=\mathbb{Z}(p^\infty)$ for some prime p.

Set $K=F(x)$ for some indeterminate x; then K is a field which has finite dimensional extension fields of all possible dimensions. For $n\in\mathbb{N}$, choose a field $L_n\supset K$ such that $\dim_K(L_n)=p^n$. The construction in [4, Examples 6.13, 15.1] (with K and L_n taking the roles of F^* and F) yields a regular K-algebra R_n such that $K_0(R_n)$ is cyclic of order p^n, generated by a class $[e_nR_n]$ where e_n is an idempotent in R_n such that $p^n(e_nR_n)\oplus R_n\cong R_n$. It follows from this isomorphism that $p^n(e_nR_n)$ is cyclic.

Now set $T=R/I$ where $R=\prod_n R_n$ and $I=\bigoplus_n R_n$. We may build some right ideals of R by taking direct products of sequences of right ideals from the rings R_n. For $n\geqq k$, there is a principal right ideal in R_n isomorphic to $p^{n-k}(e_nR_n)$, and so R has a principal right ideal

$$B_k\cong\left(\prod_{n=1}^{k-1}0\right)\times\left(\prod_{n=k}^{\infty}p^{n-k}(e_nR_n)\right).$$

In particular, $B_1\cong\prod_n p^{n-1}(e_nR_n)$ and $pB_1\cong\prod_n p^n(e_nR_n)$. Since $p^n(e_nR_n)\oplus R_n\cong R_n$ for all n, we obtain that $pB_1\oplus R\cong R$. Thus $p(B_1/B_1I)\oplus T\cong T$, and so $p[B_1/B_1I]=0$ in $K_0(T)$. If $[B_1/B_1I]=0$, then $(B_1/B_1I)\oplus mT\cong mT$ for some $m\in\mathbb{N}$. The matrices implementing this isomorphism would lift to matrices over R whose components (projecting onto matrices over the R_n) would implement isomorphisms $p^{n-1}(e_nR_n)\oplus mR_n\cong mR_n$ for all but finitely many n. But such isomorphisms do not exist, because $p^{n-1}[e_nR_n]$ is a nonzero element of $K_0(R_n)$ for every n. Therefore $[B_1/B_1I]\neq 0$, and so this element of $K_0(T)$ has order p.

For all k, we have

$$B_k\cong\left(\prod_{n=0}^{k-1}0\right)\times(e_kR_k)\times\left(\prod_{n=k+1}^{\infty}pp^{n-k-1}(e_nR_n)\right)$$

$$\cong\left(\left(\prod_{n=0}^{k-1}0\right)\times(e_kR_k)\times\left(\prod_{n=k+1}^{\infty}0\right)\right)\oplus pB_{k+1},$$

whence $B_k/B_kI\cong p(B_{k+1}/B_{k+1}I)$ and so $[B_k/B_kI]=p[B_{k+1}/B_{k+1}I]$. Therefore $\mathbb{Z}(p^\infty)$ is isomorphic to the subgroup of $K_0(T)$ generated by the elements $[B_k/B_kI]$. □

Corollary 2.2. *Given any countable field F and any countable torsion abelian group G, there exists a countable regular F-algebra T such that G embeds in $K_0(T)$.*

Proof. By Proposition 2.1, there exists a regular F-algebra U such that G is isomorphic to a subgroup H of $K_0(U)$. List the elements of H as $x_1,x_2,\ldots$. Then H can be presented with the x_n as generators and countably many relations, corresponding to countably many equations σ_i each of which says that some $\mathbb{Z}$-linear combination of finitely many of the x_n vanishes. Each σ_i can be rewritten in the form

$$\sum_{n=1}^{\infty}a_{in}x_n=\sum_{n=1}^{\infty}b_{in}x_n$$

where the a_{in} and b_{in} are nonnegative integers, all but finitely many of which vanish.

Write each $x_n=[e_n(t_nU)]-[f_n(t_nU)]$ for some $t_n\in\mathbb{N}$ and some idempotent matrices $e_n, f_n\in M_{t_n}(U)$. Each equation σ_i corresponds to an isomorphism

$$\left(\bigoplus_{n=1}^{\infty} a_{in}(e_n(t_nU))\right)\oplus\left(\bigoplus_{n=1}^{\infty} b_{in}(f_n(t_nU))\right)\oplus c_iU$$
$$\cong\left(\bigoplus_{n=1}^{\infty} b_{in}(e_n(t_nU))\right)\oplus\left(\bigoplus_{n=1}^{\infty} a_{in}(f_n(t_nU))\right)\oplus c_iU$$

for some $c_i\in\mathbb{N}$. Such an isomorphism can be implemented by a pair of matrices $u_i, v_i\in M_{s_i}(U)$ where $s_i=c_i+\sum_n(a_{in}+b_{in})t_n$.

Let T_0 be the F-subalgebra of U generated by the entries of all the matrices e_n, f_n, u_i, v_i. Since F is countable, so is T_0. Let G_0 be the subgroup of $K_0(T_0)$ generated by the elements $y_n=[e_n(t_nT_0)]-[f_n(t_nT_0)]$. Since all the u_i and v_i are matrices over T_0, the y_n satisfy the same relations as the x_n. Thus K_0 of the inclusion map $T_0\to U$ maps G_0 isomorphically onto H.

Finally, choose countable F-subalgebras $T_0\subseteq T_1\subseteq T_2\subseteq\cdots\subseteq U$ such that each element of T_j has a quasi-inverse in T_{j+1}. Then $T=\bigcup_j T_j$ is a countable regular F-subalgebra of U. Since the inclusion map $T_0\to U$ factors through the inclusion map $\eta: T_0\to T$, we conclude that $K_0(\eta)$ is injective on G_0. Therefore $K_0(\eta)(G_0)\cong G_0\cong G$. □

3. Transfer to residually finite dimensional algebras

Our next step is to build residually finite dimensional regular algebras S for which $K_0(S)$ contains an arbitrary countable torsion subgroup. This is achieved by applying a construction of Tyukavkin [14] as developed in [10, Section 2] to the algebras produced in Corollary 2.2. Recall that an algebra A over a field F is *residually finite dimensional* provided the intersection of the cofinite dimensional ideals of A is zero. We shall say that A is *countably residually finite dimensional* if some intersection of countably many cofinite dimensional ideals of A is zero. Note that this occurs if and only if A can be embedded in a direct product of countably many full matrix algebras over F.

Throughout this section, let F be a field, $B(F)$ the algebra of all row- and column-finite $\omega\times\omega$ matrices over F, and T a regular F-subalgebra of $B(F)$. Later we shall assume that F is countable and T is one of the examples produced in Corollary 2.2; any such algebra T—in fact, any countable dimensional F-algebra—can be embedded in $B(F)$ by [10, Proposition 2.1]. We use Tyukavkin's construction to build a subalgebra of the algebra $U=\prod_{n=1}^{\infty} M_n(F)$ with a factor algebra isomorphic to T. Namely, let S be the set of those sequences $x=(x_n)\in U$ for which there is an element $\phi(x)\in T$ satisfying the following property: for all $k\in\mathbb{N}$, there exists an index $m_k\geq k$ such that for $n\geq m_k$, all entries of the first k rows and columns of x_n agree with the corresponding entries of $\phi(x)$. Then S is an F-subalgebra of U and ϕ is a well-defined surjective F-algebra homomorphism of S onto T. Moreover, S is regular, and the ideal $I = \ker(\phi)$ satisfies $IUI \subseteq I$. Let us identify T with S/I via ϕ; then we may view ϕ as the quotient map.

Each of the projection maps $S \to M_n(F)$ induces a functor mod-$S \to$ mod-$M_n(F)$. Composing with the length function, we obtain a function d_n from the finitely generated right S-modules to $\mathbb{Z}^+$ which is invariant under isomorphism and additive on direct sums, and has the additional property that $d_n(S) = n$. Note that $d_n(xS) = \operatorname{rank}(x_n)$ for all $x \in S$.

Lemma 3.1. *Let A and B be finitely generated projective right S-modules such that $AI = A$ and $BI = B$.*

(a) *If $A \otimes_S U \cong B \otimes_S U$, then $A \cong B$.*

(b) *If t is a positive integer that divides $d_n(A)$ for all n, then A has a submodule C such that $tC \cong A$.*

(c) *If $c_1, c_2, \ldots$ is a bounded sequence of nonnegative integers, there exists a finitely generated projective right S-module C such that $CI = C$ and $d_n(C) = c_n$ for all n.*

Proof. We may assume that $A = e(mS)$ and $B = f(mS)$ for some $m \in \mathbb{N}$ and some idempotent matrices $e, f \in M_m(S)$. Since $AI = A$, we find that $e \in M_m(I)$, and similarly $f \in M_m(I)$.

(a) We are given $e(mU) \cong A \otimes_S U \cong B \otimes_S U \cong f(mU)$, and so there exist $u \in eM_m(U)f$ and $v \in fM_m(U)e$ such that $uv = e$ and $vu = f$. Since $IUI \subseteq I \subseteq S$, we see that $u, v \in M_m(S)$, and consequently $A \cong B$.

(b) After identifying $M_m(U)$ with $\prod_n M_{mn}(F)$, we have $e = (e_1, e_2, \ldots)$ for some idempotent matrices $e_n \in M_{mn}(F)$ such that $\operatorname{rank}(e_n) = d_n(A)$ is divisible by t. Hence, each e_n is a sum of t pairwise orthogonal equivalent idempotents. Thus, at least in $M_m(U)$, we have $e = f_1 + \cdots + f_t$ for some pairwise orthogonal equivalent idempotents f_i. Observe that $f_i = ef_ie \in M_m(I)M_m(U)M_m(I) \subseteq M_m(S)$. Now $A = f_1(mS) \oplus \cdots \oplus f_t(mS)$, and each $f_i(mU) \cong f_1(mU)$. By part (a), each $f_i(mS) \cong f_1(mS)$, and thus $C = f_1(mS)$ is the desired submodule of A.

(c) Since the sequence $(c_1, c_2, \ldots)$ is a finite sum of 0, 1 sequences, it suffices to consider the case that all $c_n \in \{0, 1\}$. Now define matrices $x_n \in M_n(F)$ as follows:

$$x_n = \begin{cases} 0 & \text{if } c_n = 0 \\ e_{nn} & \text{if } c_n = 1, \end{cases}$$

where e_{nn} denotes the usual matrix unit. This gives us a sequence $x = (x_n) \in U$, and we observe that $x \in I$. Therefore xS is a finitely generated projective right S-module such that $(xS)I = xS$ and $d_n(xS) = \operatorname{rank}(x_n) = c_n$ for all n. □

Proposition 3.2. *The restriction of $K_0(\phi)$ to the torsion subgroup of $K_0(S)$ provides an isomorphism onto the torsion subgroup of $K_0(T)$.*

Proof. First consider a torsion element $x \in \ker K_0(\phi)$. By [4, Proposition 15.15],

$x=[A]-[B]$ for some finitely generated projective right S-modules A and B such that $AI=A$ and $BI=B$. Further, $tx=0$ for some $t\in\mathbb{N}$, and so $tA\oplus kS\cong tB\oplus kS$ for some $k\in\mathbb{N}$. It follows that $d_n(A)=d_n(B)$ for all n, and hence $A\otimes_S U\cong B\otimes_S U$. Then $A\cong B$ by Lemma 3.1, and so $x=0$. Therefore $K_0(\phi)$ is injective on the torsion subgroup of $K_0(S)$.

Now consider a torsion element $y\in K_0(T)$. Then $y=[A']-[B']$ for some finitely generated projective right T-modules A' and B', and $tA'\oplus kT\cong tB'\oplus kT$ for some $t,k\in\mathbb{N}$. Since $y=[A'\oplus kT]-[B'\oplus kT]$ and $t(A'\oplus kT)\cong t(B'\oplus kT)$, there is no loss of generality in assuming that $tA'\cong tB'$.

Choose finitely generated projective right S-modules A and B such that $A/AI\cong A'$ and $B/BI\cong B'$; then $(tA)/(tA)I\cong(tB)/(tB)I$. By [4, Proposition 2.19], there exist decompositions $tA=A_1\oplus A_2$ and $tB=B_1\oplus B_2$ such that $A_1\cong B_1$ while $A_2I=A_2$ and $B_2I=B_2$. Then $tA\oplus B_2\cong tB\oplus A_2$, from which we see that $d_n(B_2)\equiv d_n(A_2)\pmod t$ for all n. Choose integers $c_n\in\{0,1,\dots,t-1\}$ such that $d_n(A_2)+c_n$ is divisible by t for all n. By Lemma 3.1, there is a finitely generated projective right S-module C such that $CI=C$ and $d_n(C)=c_n$ for all n. Hence, $d_n(A_2\oplus C)$ is divisible by t for all n. Since $d_n(B_2)\equiv d_n(A_2)$ $(\mathrm{mod}\, t)$ for all n, we also have $d_n(B_2\oplus C)$ divisible by t for all n.

By Lemma 3.1, $A_2\oplus C\cong tD$ and $B_2\oplus C\cong tE$ for some finitely generated projective right S-modules D and E; moreover, $DI=D$ and $EI=E$. Hence, the element $z=[A\oplus E]-[B\oplus D]\in K_0(S)$ satisfies

$$K_0(\phi)(z)=[A/AI]-[B/BI]=[A']-[B']=y.$$

On the other hand, $t(A\oplus E)\cong tA\oplus B_2\oplus C\cong tB\oplus A_2\oplus C\cong t(B\oplus D)$, and so $tz=0$. Therefore $K_0(\phi)$ maps the torsion subgroup of $K_0(S)$ onto the torsion subgroup of $K_0(T)$. □

4. Transfer to unit-regular algebras

Our final construction step provides a means of embedding a countably residually finite dimensional regular algebra into a simple unit-regular algebra while preserving the torsion in K_0. This construction is an analog of the C^*-algebra construction investigated in [7].

Throughout this section, let S be a countably residually finite dimensional regular algebra over a field F. Later, we shall let S be one of the algebras produced by the Tyukavkin construction. Since finite dimensional algebras embed in matrix algebras, there exists a countable sequence of F-algebra homomorphisms $\delta_n\colon S\to M_{t(n)}(F)$ such that $\bigcap_{n=1}^{\infty}\ker(\delta_n)=0$. Replace the sequence $\delta_1,\delta_2,\dots$ by a sequence in which each map is repeated infinitely often, say $\delta_1,\delta_1,\delta_2,\delta_1,\delta_2,\delta_3,\delta_1,\delta_2,\delta_3,\delta_4,\dots$. Hence, after renumbering we may assume that $\bigcap_{n=k}^{\infty}\ker(\delta_n)=0$ for all k.

Identify F with the subalgebra $F\cdot 1\subseteq S$; consequently, each $M_{t(n)}(F)$ is identified with a subalgebra of $M_{t(n)}(S)$, and so we may view δ_n as a homomorphism $S\to M_{t(n)}(S)$. We shall also use δ_n to denote the induced homomorphism $M_k(S)\to M_{kt(n)}(F)\to M_{kt(n)}(S)$ for any k.

Set $v(1)=1$ and $v(n+1)=v(n)(1+t(n))$ for $n=1,2,\ldots$. Moreover, for $n\in\mathbb{N}$, set $R_n=M_{v(n)}(S)$, let $\phi_n\colon R_n\to R_{n+1}$ be the block diagonal map given by the rule

$$\phi_n(a)=\begin{pmatrix} a & 0 \\ 0 & \delta_n(a)\end{pmatrix},$$

and let $\phi_{nk}=\phi_{n-1}\phi_{n-2}\ldots\phi_k\colon R_k\to R_n$ for $k<n$. Finally, define R to be the direct limit of the sequence

$$R_1\xrightarrow{\phi_1}R_2\xrightarrow{\phi_2}\cdots,$$

and let $\eta_1\colon S=R_1\to R$ be the natural embedding. Observe that

$$\phi_{k+1}\phi_k(a)=\begin{pmatrix} \begin{pmatrix} a & 0 \\ 0 & \delta_k(a)\end{pmatrix} & 0 \\ 0 & \delta_{k+1}\begin{pmatrix} a & 0 \\ 0 & \delta_k(a)\end{pmatrix}\end{pmatrix}=\begin{pmatrix} a & 0 & 0 & 0 \\ 0 & \delta_k(a) & 0 & 0 \\ 0 & 0 & \delta_{k+1}(a) & 0 \\ 0 & 0 & 0 & \delta_{k+1}\delta_k(a)\end{pmatrix}$$

for $a\in R_k$, and similarly for $\phi_{k+2}\phi_{k+1}\phi_k(a)$ etc. In particular, for any $n>k$ the matrix $\phi_{nk}(a)$ can be written as a block diagonal matrix in which $a,\delta_k(a),\delta_{k+1}(a),\ldots,\delta_{n-1}(a)$ all appear as blocks.

Now set $V=\mathrm{FP}_{\cong}(S)$ and $u=\langle S\rangle\in V$. Note that for any $v\in V$, there exist $v'\in V$ and $m\in\mathbb{N}$ such that $v+v'=mu$. As noted in Section 1, the category equivalences mod-$R_n\to$ mod-S induce monoid isomorphisms $\mu_n\colon \mathrm{FP}_{\cong}(R_n)\to V$ such that $\mu_n(\langle R_n\rangle)=v(n)u$. In particular, if p_n denotes the matrix unit $e_{11}\in R_n$, then $\mu_n(\langle p_nR_n\rangle)=u$. Set f_n equal to the composition

$$V\xrightarrow{\mu_n^{-1}}\mathrm{FP}_{\cong}(R_n)\xrightarrow{\mathrm{FP}_{\cong}(\phi_n)}\mathrm{FP}_{\cong}(R_{n+1})\xrightarrow{\mu_{n+1}}V,$$

and set $f_{nk}=f_{n-1}f_{n-2}\ldots f_k$ for $n>k$. Note that $\phi_n(p_n)$ is a diagonal matrix in which 1 appears $1+t(n)$ times and all other entries are 0. Hence,

$$f_n(u)=\mu_{n+1}(\langle\phi_n(p_n)R_{n+1}\rangle)=\mu_{n+1}((1+t(n))\langle p_{n+1}R_{n+1}\rangle)=(1+t(n))u.$$

Since the functor $\mathrm{FP}_{\cong}(-)$ preserves direct limits, $\mathrm{FP}_{\cong}(R)$ is isomorphic to the monoid

$$W=\varinjlim(V\xrightarrow{f_1}V\xrightarrow{f_2}\cdots).$$

Finally, set s_n equal to the composition

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$$V \xrightarrow{\mu_n^{-1}} \mathrm{FP}_{\cong}(R_n) \xrightarrow{\mathrm{FP}_{\cong}(\delta_n)} \mathrm{FP}_{\cong}(M_{v(n)t(n)}(F)) \xrightarrow{\text{length}} \mathbb{Z}^+,$$

and observe that $s_n(v(n)u)=v(n)t(n)$, whence $s_n(u)=t(n)$. Since the isomorphism μ_n^{-1} can be given by the rule $\langle A\rangle \mapsto \langle A \otimes_S Q\rangle$ where Q is the top row of R_n, we see that

$$s_n(\langle e(rS)\rangle)=\operatorname{rank}\delta_n(e)$$

for all idempotent matrices $e\in M_r(S)$.

Lemma 4.1. *Let $v\in V$ and $n>k$ in $\mathbb{N}$.*
(a) *If $s_i(v)=0$ for all $i>k$, then $v=0$.*
(b) $f_{nk}(v)=v+(\sum_{i=k}^{n-1} s_i(v)\prod_{j=i+1}^{n-1}(1+t(j)))u$.

Proof. (a) This follows from the fact that $\bigcap_{i=k+1}^{\infty}\ker(\delta_i)=0$.

(b) Observe first that $f_i(w)=w+s_i(w)u$ for all $w\in V$ and $i\in\mathbb{N}$. The given formula for $f_{nk}(v)$ follows by an obvious induction on n. □

Proposition 4.2. *The algebra R is simple and unit-regular, and $K_0(R)$ is strictly unperforated. Moreover, $K_0(\eta_1)$ restricts to an embedding of the torsion subgroup of $K_0(S)$ into $K_0(R)$.*

Proof. Obviously R is regular. To prove simplicity, it suffices to show that for any nonzero element $a\in R_k$, there exists $n>k$ such that $R_n\phi_{nk}(a)R_n=R_n$. Since $\bigcap_{n=k}^{\infty}\ker(\delta_n)=0$, there is an index $n>k$ such that $\delta_{n-1}(a)\neq 0$. Now $\delta_{n-1}(a)$ is a matrix with scalar entries, and so at least one entry is invertible. Further, $\delta_{n-1}(a)$ appears as a block in $\phi_{nk}(a)$. Thus $\phi_{nk}(a)$ has at least one invertible entry, and hence $R_n\phi_{nk}(a)R_n=R_n$ as desired. Therefore R is simple.

To prove that R is unit-regular, it suffices to show that the monoid W has cancellation. Hence, it is enough to show that for any $k\in\mathbb{N}$ and any $x,y,v\in V$ satisfying $x+v=y+v$, there exists $n>k$ such that $f_{nk}(x)=f_{nk}(y)$. There exist $v'\in V$ and $m\in\mathbb{N}$ such that $v+v'=mu$, whence $x+mu=y+mu$. Moreover, $s_i(x)=s_i(y)$ for all i. If $x=0$, then $s_i(y)=0$ for all i, and consequently $y=0$ by Lemma 4.1. Thus we may assume that $x\neq 0$.

Now there exists an integer $\ell\geqq k$ such that $s_\ell(x)>0$. Choose an integer $n\geqq \ell+2$ such that $2^{n-\ell-1}\geqq m$. In view of Lemma 4.1, $f_{nk}(x)=x+hu$ for some integer h such that

$$h\geqq s_\ell(x)\prod_{j=\ell+1}^{n-1}(1+t(j))\geqq 2^{n-\ell-1}\geqq m.$$

Since $s_i(y)=s_i(x)$ for all i, it also follows from Lemma 4.1 that $f_{nk}(y)=y+hu$. Thus

$$f_{nk}(x)=(x+mu)+(h-m)u=(y+mu)+(h-m)u=f_{nk}(y),$$

as desired. Therefore R is unit-regular.

Since R is unit-regular, $W \cong K_0(R)^+$. Thus to prove that $K_0(R)$ is strictly unperforated, it suffices to show that for any $k, m \in \mathbb{N}$ and any $x, y, v \in V$ such that $mx + v = my$ and $v \neq 0$, there exist $n > k$ and a nonzero element $v' \in V$ such that $f_{nk}(x) + v' = f_{nk}(y)$. There exist $x' \in V$ and $p \in \mathbb{N}$ such that $x + x' = pu$, and there exists an integer $\ell \geqq k$ such that $s_\ell(v) > 0$. Choose an integer $n \geqq \ell + 2$ such that $2^{n-\ell-1} > mp$. As above, it follows from Lemma 4.1 that $f_{nk}(v) = v + hu$ for some integer $h > mp$. Further, $f_{nk}(x) = x + au$ and $f_{nk}(y) = y + bu$ for some $a, b \in \mathbb{Z}^+$ such that

$$mb = \sum_{i=k}^{n-1} s_i(my) \prod_{j=i+1}^{n-1} (1 + t(j)) = \sum_{i=k}^{n-1} s_i(mx + v) \prod_{j=i+1}^{n-1} (1 + t(j)) = ma + h.$$

Hence, $b - a = h/m > p$, and so

$$f_{nk}(x) + x' + y + (b - a - p)u = x + x' + y + (b - p)u = y + bu = f_{nk}(y);$$

moreover, the element $x' + y + (b - a - p)u$ is nonzero because $b - a - p > 0$. Therefore $K_0(R)$ is strictly unperforated.

Finally, to prove that $K_0(\eta_1)$ is injective on the torsion subgroup of $K_0(S)$, it suffices to show that each $K_0(\phi_n)$ is injective on the torsion subgroup of $K_0(R_n)$. Hence, it is enough to show that whenever $m, n \in \mathbb{N}$ and $x, y, v, v' \in V$ with $mx + v = my + v$ and $f_n(x) + v' = f_n(y) + v'$, there exists $w \in V$ such that $x + w = y + w$. Applying s_n to the equation $mx + v = my + v$ and cancelling $s_n(v), m$, we obtain $s_n(x) = s_n(y)$. Setting $w = s_n(x)u + v'$, we conclude that

$$x + w = f_n(x) + v' = f_n(y) + v' = y + w,$$

as desired. Therefore $K_0(\eta_1)$ is injective on the torsion subgroup of $K_0(S)$. □

Ara has pointed out that the algebra R has a unique rank function N, which may be described as follows. For $a \in R_k$ and $n > k$, observe that

$$\phi_{nk}(a) = \begin{pmatrix} a & 0 \\ 0 & \psi_{nk}(a) \end{pmatrix}$$

where $\psi_{nk}(a) \in M_{v(n)-v(k)}(F) \subseteq M_{v(n)-v(k)}(S)$; if $a' \in R$ is the image of a, then

$$N(a') = \lim_{n \to \infty} \frac{\operatorname{rank} \psi_{nk}(a)}{v(n)}.$$

5. Summary

The results of Sections 2–4 immediately combine to produce the desired examples.

Namely, given any countable field F and any countable torsion abelian group G, Corollary 2.2 provides us with a countable regular F-algebra T such that G embeds in $K_0(T)$. By [**10**, Proposition 2.1], T can be embedded in $B(F)$, and so we can construct a countably residually finite dimensional regular F-algebra S as in Section 3. By Proposition 3.2, the torsion subgroups of $K_0(S)$ and $K_0(T)$ are isomorphic, and so G embeds in $K_0(S)$. Finally, we use S to construct an algebra R as in Section 4; the desired properties of R are given by Proposition 4.2. To summarize:

Theorem 5.1. *Given any countable field F and any countable torsion abelian group G, there exists a simple unit-regular F-algebra R such that G embeds in $K_0(R)$ and $K_0(R)$ is strictly unperforated.*

Theorem 5.1 provides a negative solution to [**4**, Open Problem 27], but it immediately suggests a substitute problem: Is K_0 of every simple unit-regular ring necessarily strictly unperforated? An appropriate general version of this question for non-simple unit-regular rings is the following: If A and B are finitely generated projective modules over a unit-regular ring and $(n+1)A \lesssim nB$ for some positive integer n, is $A \lesssim B$?

Another obvious question is whether countability is necessary here: If R is a simple unit-regular ring with uncountable center, is $K_0(R)$ necessarily torsionfree or even unperforated? As "moral support" for a positive answer to this last question, recall the role of countability in various unit-regularity problems. For instance, the first example of a regular, non-unit-regular ring with a rank function was an algebra over a countable field [**3**], whereas any regular ring with a rank function which is an algebra over an uncountable field must be unit-regular [**9**, Corollary 5.3]. Further, any simple regular ring R with uncountable center such that all matrix rings $M_n(R)$ are directly finite is unit-regular [**9**, Corollary 5.4], while the question whether all directly finite simple regular rings are unit-regular [**4**, Open Problem 3] remains open.

O'Meara has asked whether unperforatedness might provide a means to prove unit-regularity of directly finite simple regular rings. He showed that a directly finite simple regular ring R is unit-regular provided R satisfies the following property: whenever $x, y \in R$ and $n \in \mathbb{N}$ such that $n(xR) \lesssim n(yR)$, then $xR \lesssim yR$ [**13**, Corollary 3]. In fact, in view of a general cancellation result of Blackadar [**2**, Theorem 3.1.4], it would suffice to know that $n(xR) \cong n(yR)$ always implies $xR \cong yR$. However, our examples show that neither of the above properties hold for all simple unit-regular rings, and hence they do not hold for all directly finite simple regular rings.

We conclude by mentioning the matrix-isomorphism problem for the class of unit-regular rings (cf. [**4**, Open Problem 47]): If R and S are unit-regular rings such that $M_n(R) \cong M_n(S)$ for some positive integer n, is $R \cong S$? One might hope to obtain a counterexample from Theorem 5.1. By that theorem, there exists a simple unit regular ring R with finitely generated projective modules A_1 and A_2 such that $nA_1 \cong nA_2$ but $A_1 \ncong A_2$, and the rings $E_i = \mathrm{End}_R(A_i)$ are then simple unit-regular rings such that $M_n(E_1) \cong M_n(E_2)$. However, it is unclear whether or not E_1 and E_2 are isomorphic. We leave this question to the reader.

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