ON THE SEMIGROUP OF DIFFERENTIABLE MAPPINGS

SADAYUKI YAMAMURO

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To Bernhard Hermann Neumann on his 60th birthday

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The purpose of this paper is to improve a result in [2] on the automorphisms of the semigroup \( \mathcal{D} = \mathcal{D}(E) \) of all (Fréchet)-differentiable mappings of a real Banach space \( E \) into itself.

We denote the derivative of \( f \in \mathcal{D} \) at \( a \in E \) by \( f'(a) \). This means that \( f'(a) \in \mathcal{L} = \mathcal{L}(E) \) (the Banach algebra of all continuous linear mappings of \( E \) into itself with the usual upper bound norm) and

\[
\lim_{||x|| \to 0} ||r(f; a, x)|| = 0,
\]

where

\[
r(f; a, x) = f(a+x) - f(a) - f'(a)(x)
\]

for \( x \in E \).

It is well-known that, for \( fg \), which is defined by

\[
(fg)(x) = f(g(x))
\]

for every \( x \in E \), we have \( fg \in \mathcal{D} \) whenever \( f \in \mathcal{D} \) and \( g \in \mathcal{D} \), and

\[
(gf)'(a) = f'(g(a))g'(a).
\]

This product defines a semigroup structure in \( \mathcal{D} \). An automorphism \( \phi \) of \( \mathcal{D} \) is a bijection of \( \mathcal{D} \) such that

\[
\phi(fg) = \phi(f) \phi(g)
\]

for every \( f \in \mathcal{D} \) and \( g \in \mathcal{D} \).

An automorphism \( \phi \) is said to be inner if there exists a bijection \( h \in \mathcal{D} \) such that \( h^{-1} \in \mathcal{D} \) and

\[
\phi(f) = hfh^{-1}
\]

for every \( f \in \mathcal{D} \).

We denote the set of real numbers by \( \mathbb{R} \). For \( \alpha \in \mathbb{R} \), the mapping \( x \to \alpha x \) of \( E \) into itself is obviously continuous and linear. We denote this mapping by \( \alpha \). Since \( \alpha \in \mathcal{D} \), for an automorphism \( \phi \) of \( \mathcal{D} \), we can consider \( \{\phi(x) | \alpha \in \mathbb{R}\} \) which is a one-parameter group of diffeomorphisms (i.e. bijective and bi-differentiable mappings).

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For \( a = \phi(0)(0) \) and the translation \( t_a : x \to x + a \), the mapping \( \phi_0 : \mathcal{D} \to \mathcal{D} \) defined by

\[
\phi_0(f) = t_a^{-1} \phi(f) t_a
\]

is an automorphism which satisfies \( \phi_0(0) = 0 \).

**Definition.** An automorphism \( \phi \) of \( \mathcal{D} \) is said to be uniform if, for any positive \( \varepsilon \in \mathbb{R} \) and every \( \{x_n\} \subset \mathbb{R} \) such that \( x_n \to 0 \), there exists a positive \( \delta \in \mathbb{R} \) such that \( ||x|| < \delta \) implies

\[
\sup_{u \geq 1} ||x_n^{-1}\phi_0(x_n)(x) - x|| \leq \varepsilon ||x||.
\]

The main result of this paper is the following theorem.

**Theorem.** An automorphism of \( \mathcal{D} \) is inner if and only if it is uniform.

If \( \phi(\alpha) \in \mathcal{L} \) for every \( \alpha \in \mathbb{R} \), \( \{\phi(\alpha)\} \) is a one-parameter group of topological linear isomorphisms of \( E \) into itself. The continuity with respect to the parameter (see (2) below) leads to the conclusion that \( \phi(\alpha) = \alpha \) for every \( \alpha \in \mathbb{R} \), from which the uniformity immediately follows and, therefore, \( \phi \) is inner. This is the result obtained in [2].

If we take the sum \( f + g \) as well as product \( fg \) into consideration, the set \( \mathcal{D} \) is a near-ring. If \( \phi \) is a near-ring automorphism, then it is easy to see that \( \phi(\alpha) = \alpha \) for every \( \alpha \in \mathbb{R} \), which implies that \( \phi \) is uniform. This implies that the near-rings \( \mathcal{D}(E_1) \) and \( \mathcal{D}(E_2) \) are isomorphic if and only if the Banach spaces \( E_1 \) and \( E_2 \) are diffeomorphic. On the other hand, from our theorem it follows that the semigroups \( \mathcal{D}(E_1) \) and \( \mathcal{D}(E_2) \) are isomorphic by a uniform isomorphism if and only if \( E_1 \) and \( E_2 \) are diffeomorphic.

We believe that the answer to the following problem is affirmative.

**Problem.** Is every automorphism of \( \mathcal{D} \) uniform?

Therefore, in the proof of sufficiency, we shall avoid using the uniformity wherever possible, which sometimes makes the proof unnecessarily long.

**Proof of the necessity**

We assume that \( \phi \) is an inner automorphism of the semigroup \( \mathcal{D} \). Therefore, there exists a diffeomorphism \( h : E \to E \) such that \((*)\) is true. Then, since \( \phi(1) = 1 \), we have \((h_0^{-1})(0) = h_0'(0)^{-1}\) and \( h_0(0) = 0 \) where \( h_0 = t_a^{-1}h \) with \( a = h(0) \).

Let \( \varepsilon \) be an arbitrary positive number. There exists \( \varepsilon_1 > 0 \) such that

\[
||h_0'(0)||\varepsilon_1 + (||h_0'(0)^{-1}|| + \varepsilon_1)\varepsilon_1 < \varepsilon.
\]

Putting \( r_1(x) = r(h_0^{-1}; 0, x) \) and \( r_2(x) = r(h_0^{-1}; 0, x) \), we can take \( \delta_1 > 0 \)
such that $0 < |x| < \delta_1$ implies $|r_i(x)| < \varepsilon_i|x|$ $(i = 1, 2)$. Since $h$ is continuous, there exists $\delta > 0$ such that

$$0 < \delta < \delta_1 \text{ and } |h_0^{-1}(x)| < \delta_1 \text{ if } |x| < \delta.$$  

Then, for $x \in \mathcal{A}$ such that $0 < |x| < 1$, if $|x| < \delta$,

$$||x^{-1}r_1(xh_0^{-1}(x))|| \leq ||h_0^{-1}(x)|| (||xh_0^{-1}(x)||)^{-1}||r_1(xh_0^{-1}(x))||$$

$$< (||h_0'(0)||^{-1} ||x|| + ||r_2(x)||) \varepsilon_1.$$

Therefore, since

$$x^{-1}\phi_0(x)(x) - x = h_0'(0)r_2(x) + x^{-1}r_1(xh_0^{-1}(x)),$$

we have, if $|x| < \delta$,

$$||x^{-1}\phi_0(x)(x) - x|| \leq ||h_0'(0)|| ||r_2(x)|| + ||x^{-1}r_1(xh_0^{-1}(x))||$$

$$< ||h_0'(0)|| \varepsilon_1 ||x|| + (||h_0'(0)||^{-1} ||x|| + ||r_2(x)||) \varepsilon_1$$

$$\leq (||h_0'(0)|| \varepsilon_1 + (||h_0'(0)||^{-1} + \varepsilon_1) \varepsilon_1) ||x||$$

$$< \varepsilon ||x||.$$

**Proof of the sufficiency**

Let $\phi$ be an automorphism of $\mathcal{A}$. The following fact has been proved by K. D. Magill, Jr. [1].

*There exists a bijection $h : E \to E$ which satisfies $(\ast)$.*

All we know about this $h$ at this stage is that it is a bijection (i.e., one-to-one and onto). We are going to prove that $h \in \mathcal{A}$ and $h^{-1} \in \mathcal{A}$.

Since $\phi^{-1}(f) = h^{-1}fh$ and $\phi^{-1}$ is also an automorphism, any statement about $h$ can be replaced by the same statement about $h^{-1}$. We shall use this fact freely.

Moreover, we can assume that $h(0) = 0$, because, if $h(0) = a \neq 0$, we have only to consider the bijection $h_0 = t_a^{-1}h$, which corresponds to the automorphism $\phi_0$.

For the sake of convenience, we denote the set of all sequences $\{\varepsilon_n\} \subset \mathcal{A}$ such that $\lim_{n \to \infty} \varepsilon_n = 0$ by $(c_0)$.

(1) $\inf_{n \geq 1} ||\varepsilon_n^{-1}h(\varepsilon_n a)|| > 0$ for every $a \in E$ and any $\{\varepsilon_n\} \in (c_0)$.

Assume that there exist $a \in E$ and $\{\varepsilon_n\} \in (c_0)$ such that

$$\lim_{n \to \infty} ||\varepsilon_n^{-1}h(\varepsilon_n a)|| = 0.$$

For any $\{\delta_n\} \in (c_0)$, taking one of its subsequences if necessary, we can assume that $\delta_n \varepsilon_n^{-1} \to 0$. Then,

$$\delta_n^{-1}h(\delta_n a) = \delta_n^{-1}h(\delta_n \varepsilon_n^{-1} \varepsilon_n a) = \delta_n^{-1}\phi(\delta_n \varepsilon_n^{-1})h(\varepsilon_n a).$$
On the other hand, the uniformity implies that there exists \( \delta > 0 \) such that 
\[
|\|x\|| < \delta \Rightarrow \sup_{n \geq 1} |\|\phi(x)\|| \leq |\|x\||.
\]
Since \( \lim_{n \to \infty} h(e_n a) = 0 \), we get 
\[
|\|\delta^{-1} h(\delta_n a)\|| = |\|\delta^{-1} e_n \phi(\delta_n e_n^{-1}) h(e_n a)\|| 
\leq |\|e_n^{-1} h(e_n a)\||,
\]
which implies 
\[
\lim_{n \to \infty} \delta^{-1} h(\delta_n a) = 0.
\]
Therefore, 
\[
\lim_{\varepsilon \to 0} \varepsilon^{-1} h(\varepsilon x) = 0,
\]
which means that \( h \) is Gateaux-differentiable at 0, because, for any \( x \), if we take \( \chi \in \mathcal{L} \) such that \( \chi(a) = x \), 
\[
\varepsilon^{-1} h(\varepsilon x) = \varepsilon^{-1} h(\varepsilon \chi(a)) = \varepsilon^{-1} \phi(\chi) h(\varepsilon a),
\]
from which it follows that \( \lim_{\varepsilon \to 0} \varepsilon^{-1} h(\varepsilon a) = 0 \). Moreover, \( h \) is Gateaux-differentiable at every point, because, for \( t_x : x \to x+z \), we have \( t_x \in \mathcal{D} \) and 
\[
\varepsilon^{-1}[h(x+\varepsilon z) - h(x)] = \varepsilon^{-1}[\phi(t_x) h(\varepsilon z) - \phi(t_x) h(0)],
\]
from which it follows that 
\[
\lim_{\varepsilon \to 0} \varepsilon^{-1}[h(x+\varepsilon z) - h(x)] = \phi(t_x)'(0) \lim_{\varepsilon \to 0} \varepsilon^{-1} h(\varepsilon z).
\]
If we denote the Gateaux-derivative of \( h \) at \( x \) by \( h^*(x) \), then we have \( h^*(x) = 0 \) for every \( x \in E \). The mean value theorem then implies that \( h = 0 \), which is a contradiction.

*For the conjugate space \( E \), the value of \( \bar{a} \in E \) at \( x \in E \) is denoted by \( \langle x, \bar{a} \rangle \).*

(2) *For any \( \bar{a} \in E \), \( \langle h(x), \bar{a} \rangle \) is continuous with respect to \( x \).*

To prove the continuity at \( a \in E \), we use the method used by K. D. Magill, Jr. [1]. We take positive \( \varepsilon \in \mathcal{B} \) and non-zero \( b \in E \) and consider the mapping \( g \in \mathcal{D} \) such that
\[
g(x) = \beta(\langle x-h(a), \bar{a} \rangle) b + h(a),
\]
where \( \beta : \mathcal{B} \to \mathcal{B} \) is a differentiable function such that 
\[
\beta(\xi) = 0 \text{ if } |\xi| \leq \varepsilon; = 1 \text{ if } \xi = 0.
\]
We take \( f \in \mathcal{D} \) such that \( \phi(f) = g \). Then, \( f(a) \neq a \), because, if \( f(a) = a \), we have 
\[
h(a) = hf(a) = \phi(f)h(a) = gh(a) = b + h(a),
\]
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which is a contradiction. Since \( f \) is continuous, there exists \( \delta > 0 \) such that \( ||x-a|| < \delta \) implies \( f(x) \neq a \). Therefore, if \( ||x-a|| < \delta \), we have \( gh(x) = hf(x) \neq h(a) \), which means that \( \beta(\langle h(x) - h(a), \tilde{a} \rangle) \neq 0 \). By the definition of \( \beta \), we have \( \langle h(x) - h(a), \tilde{a} \rangle < \epsilon \).

(3) \( \sup_{n \geq 1} ||\varepsilon_n^{-1}h(e_n a)|| < \infty \) for any \( a \in E \) and any \( \{e_n\} \in (c_0) \).

As a special case, \( \lim_{n \to \infty} h(e_n a) = 0 \).

Let us suppose that there exist \( a \in E \) and \( \{e_n\} \in (c_0) \) such that

\[
\lim_{n \to \infty} ||\varepsilon_n^{-1}h^{-1}(e_n a)|| = \infty.
\]

Then, for some \( \tilde{a} \in \tilde{E} \), we have

\[
\lim_{n \to \infty} \langle \varepsilon_n^{-1}h^{-1}(e_n a), \tilde{a} \rangle = \infty.
\]

For these \( a \in E \) and \( \tilde{a} \in \tilde{E} \), we consider the mapping \( a \otimes \tilde{a} \in \mathcal{L} \) that is defined by

\[
a \otimes \tilde{a}(x) = \langle x, \tilde{a} \rangle a.
\]

Then,

\[
\phi(a \otimes \tilde{a})(0)(a) = \lim_{n \to \infty} \varepsilon_n^{-1}\phi(a \otimes \tilde{a})(e_n a)
\]

\[
= \lim_{n \to \infty} \varepsilon_n^{-1}h[\langle h^{-1}(e_n a), \tilde{a} \rangle a]
\]

\[
= \lim_{n \to \infty} (\varepsilon_n^{-1}\langle h^{-1}(e_n a), \tilde{a} \rangle) (\langle h^{-1}(e_n a), \tilde{a} \rangle)^{-1}
\]

\[
\times h[\langle h^{-1}(e_n a), \tilde{a} \rangle a],
\]

from which it follows that

\[
\lim_{n \to \infty} (\langle h^{-1}(e_n a), \tilde{a} \rangle)^{-1}h[\langle h^{-1}(e_n a), \tilde{a} \rangle a] = 0,
\]

which contradicts the facts proved in (1) and (2).

(4) For any \( a \in E \) and any \( \{e_n\} \in (c_0) \), there exists a subsequence \( \{e_{n_k}\} \) such that

\[
\{\varepsilon_{n_k}^{-1}h(e_{n_k} a)\}
\]

is convergent.

Since \( a \) can be supposed to be non-zero, we can take \( \tilde{a} \in \tilde{E} \) such that \( \langle a, \tilde{a} \rangle \neq 0 \) and \( (\phi(a \otimes \tilde{a})(0)(a) \neq 0 \). For this \( a \otimes \tilde{a} \), we take \( \{\delta_n\} \in (c_0) \) such that

\[
\langle h^{-1}(\delta_n a), \tilde{a} \rangle = \varepsilon_n,
\]

which is possible because of (2). Since the sequence of real numbers

\[
\{\delta_n^{-1}\langle h^{-1}(\delta_n a), \tilde{a} \rangle\}
\]

is bounded, it contains a convergent subsequence

\[
\{\delta_{n_k}^{-1}\langle h^{-1}(\delta_{n_k} a), \tilde{a} \rangle\}.
\]
Then,
\[
0 \neq \phi(a \otimes \bar{a})'(0)(a) = \lim_{k \to \infty} \delta_{n_k}^{-1} \phi(a \otimes \bar{a})\left(\delta_{n_k} a\right)
\]
\[
= \lim_{k \to \infty} \delta_{n_k}^{-1} \langle h^{-1}(\delta_{n_k} a), \bar{a} \rangle \epsilon_{n_k}^{-1} h(\epsilon_{n_k} a),
\]
which implies that
\[
\lim_{k \to \infty} \delta_{n_k}^{-1} \langle h(\delta_{n_k} a), \bar{a} \rangle \neq 0.
\]
Therefore, we have the limit
\[
\lim_{k \to \infty} \epsilon_{n_k}^{-1} h(\epsilon_{n_k} a) = \left(\lim_{k \to \infty} \delta_{n_k}^{-1} \langle h^{-1}(\delta_{n_k} a), \bar{a} \rangle\right)^{-1} \phi(a \otimes \bar{a})'(0)(a).
\]
(5) The limit \( \lim_{\epsilon \to 0} \epsilon^{-1} h(\epsilon a) \) exists.

We have only to show that, if the limits
\[
\lim_{n \to \infty} \epsilon_{n}^{-1} h(\epsilon_{n} a) = a_1 \text{ and } \lim_{n \to \infty} \delta_{n}^{-1} h(\delta_{n} a) = a_2
\]
exist for \( \{\epsilon_{n}\} \in (c_0) \) and \( \{\delta_{n}\} \in (c_0) \), then we have \( a_1 = a_2 \).

We can assume, taking a subsequence of \( \delta_{n} \) if necessary, that
\[
\lim_{n \to \infty} \delta_{n} = 0.
\]
Then,
\[
\delta_{n}^{-1} h(\delta_{n} a) = \delta_{n}^{-1} h(\delta_{n} \epsilon_{n}^{-1} \epsilon_{n} a) = \delta_{n}^{-1} \phi(\delta_{n} \epsilon_{n}^{-1}) h(\epsilon_{n} a)
\]
\[
= \epsilon_{n}^{-1} [\delta_{n}^{-1} \epsilon_{n} \phi(\delta_{n} \epsilon_{n}^{-1}) h(\epsilon_{n} a) - h(\epsilon_{n} a)] + \epsilon_{n}^{-1} h(\epsilon_{n} a).
\]
The uniformity then implies that
\[
\lim_{n \to \infty} ||\delta_{n}^{-1} h(\delta_{n} a) - \epsilon_{n}^{-1} h(\epsilon_{n} a)|| = 0.
\]
We denote this limit by \( h^*(0)(a) \).

(6) \( h \) is differentiable at every point in all directions.

Let \( a \) be an arbitrary point and consider the mapping \( t_a : x \mapsto x + a \).

Then, \( t_a \in \mathcal{D} \) and
\[
e^{-1}[h(a + \epsilon x) - h(a)] = e^{-1}[\phi(t_a) h(\epsilon x) - \phi(t_a) h(0)]
\]
\[
= e^{-1}[\phi(t_a)'(0) h(\epsilon x) + r(\phi(t_a) ; 0, h(\epsilon x))].
\]
Therefore,
\[
\lim_{\epsilon \to 0} e^{-1}[h(a + \epsilon x) - h(a)] = \phi(t_a)'(0) h^*(0)(x).
\]
We denote this limit by \( h^*(a)(x) \). Obviously,
\[
h^*(a)(ax) = a h^*(a)(x).
\]
(7) For any $a \otimes \tilde{a}$, $h(a \otimes \tilde{a}) \in \mathcal{D}$ and

$$(h(a \otimes \tilde{a}))'(x)(y) = \langle y, \tilde{a} \rangle h^*(\langle x, \tilde{a} \rangle a)(a).$$

Since

$$\varepsilon^{-1}[h(a \otimes \tilde{a})(x+\varepsilon y) - h(a \otimes \tilde{a})(x)]$$

$$= \varepsilon^{-1}[h(\langle x, \tilde{a} \rangle a + \varepsilon \langle y, \tilde{a} \rangle a) - h(\langle x, \tilde{a} \rangle a)],$$

it follows from (6) that the limit as $\varepsilon \to 0$ exists and it is

$$\langle y, \tilde{a} \rangle h^*(\langle x, \tilde{a} \rangle a)(a),$$

which is obviously continuous and linear with respect to $y$. Moreover,

$$\lim_{\|\varepsilon\| \to 0} \left| \begin{array}{c} |y|^{-1} |h(a \otimes \tilde{a})(x+y) - h(a \otimes \tilde{a})(x) - (h(a \otimes \tilde{a}))^* (x)(y)| \\
\end{array} \right|$$

$$\leq ||\tilde{a}|| \lim_{\|\varepsilon\| \to 0} \left| \begin{array}{c} \langle y, \tilde{a} \rangle^{-1}[h(\langle x, \tilde{a} \rangle a + \langle y, \tilde{a} \rangle a) - h(\langle x, \tilde{a} \rangle a)] \\
-h^*(\langle x, \tilde{a} \rangle a)(a) \\
= 0,
\right|$$

which means that $h(a \otimes \tilde{a}) \in \mathcal{D}$.

(8) For any $a \otimes \tilde{a}$, $(a \otimes \tilde{a})h \in \mathcal{D}$ and

$$( (a \otimes \tilde{a})h )'(x)(y) = \langle h^*(x)(y), \tilde{a} \rangle a.$$ 

By (7), we have

$$(a \otimes \tilde{a})h = \phi^{-1}(h(a \otimes \tilde{a})) \in \mathcal{D}.$$ 

The formula for $((a \otimes \tilde{a})h)'(x)(y)$ is obvious.

(9) $h^*(a) \in \mathcal{L}$ for every $a \in E$.

The linearity follows immediately from (8). To prove the continuity, let us take an arbitrary non-zero $b \otimes \tilde{b}$. Then

$$|\langle h^*(a)(x), \tilde{b} \rangle| = ||b||^{-1} ||(b \otimes \tilde{b})h)'(a)(x)||$$

$$\leq ||b||^{-1} ||(b \otimes \tilde{b})h)'(a)|| ||x||,$$

which means the set

$$\{h^*(a)(x) ||x|| \leq 1\}$$

is weakly bounded. Therefore, $h^*(a)$ is continuous.

We define $r_1(x)$ and $r_2(x)$ by

$$h(x) - h^*(0)(x) = r_1(x) \text{ and } h^{-1}(x) - (h^{-1})^*(0)(x) = r_2(x).$$

(10) For any sequence $\{x_n\}$ such that $\lim_{n \to \infty} x_n = 0$, the sequence $\{|x_n|^{-1} r_i(x_n)\}$ converges weakly to 0 for $i = 1, 2$. Therefore, the sequence $\{|x_n|^{-1} h(x_n)\}$ is bounded, which implies that $\lim_{n \to \infty} h(x_n) = 0$. 

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From (8) it follows that \((a \otimes \bar{a})r_1(x)\) is the remainder of \((a \otimes \bar{a})h\) at 0. Therefore,
\[
\lim_{n \to \infty} ||x_n||^{-1} (a \otimes \bar{a})r_1(x_n) = 0,
\]
which implies that
\[
\lim_{n \to \infty} \langle ||x_n||^{-1} r_1(x_n), \bar{a} \rangle = 0
\]
for every \(\bar{a} \in E\).

(11) \(\lim_{|x| \to 0} |x|^{-1} r_i(x) = 0\) \((i = 1, 2)\). Therefore, \(h \in \mathcal{D}\) and \(h^{-1} \in \mathcal{D}\). Assume that there exists a sequence \({x_n} \subset E\) such that
\[
\lim_{n \to \infty} x_n = 0 \text{ and } ||x_n||^{-1} ||r_1(x_n)|| \geq \gamma > 0 \quad (n = 1, 2, \cdots)
\]
for some positive \(\gamma \in \mathbb{R}\). By (5), we can take \({e_n} \in (c_0)\) such that
\[
||e_n^{-1} r_1(e_n x_n)|| \leq ||x_n||^2 \quad (n = 1, 2, \cdots).
\]
Then, for large \(n\), we have
\[
||e_n^{-1} \phi(e_n)h(x_n) - h(x_n)|| = ||e_n^{-1} h(e_n x_n) - h(x_n)||
\]
\[
= ||e_n^{-1} r_1(e_n x_n) - r_1(x_n)|| \geq ||r_1(x_n)|| - ||e_n^{-1} r_1(e_n x_n)||
\]
\[
\geq (\gamma - ||x_n||) ||x_n|| \geq (\gamma - ||x_n||) (\inf_{n \geq 1} ||x_n|| ||h(x_n)||^{-1}) ||h(x_n)||.
\]
Since, by (10), \(\inf_{n \geq 1} ||x_n|| ||h(x_n)||^{-1} > 0\) which implies that \(\lim_{n \to \infty} h(x_n) = 0\), this contradicts the uniformity.

References


Australian National University
and
State University of New York at Buffalo