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Markov measures determine the zeta function

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Abstract. With the purpose of understanding when two subshifts of finite type are equivalent from the point of view of their spaces of Markov measures we propose the notion of Markov equivalence. We show that a Markov equivalence must respect the cycles (periodic orbits) of the subshifts. In particular, Markov equivalent subshifts of finite type have the same zeta function.

1. *Markov equivalence*

One of the important characteristics of a subshift of finite type is the linear space of Markov measures it supports. Global consideration of this space has been useful in both the study of codes between subshifts of finite type and classifications of individual Markov measures. (See, for example, [**3**], [**5**], [**7**] and [**8**].)

This note is a first step towards deciding and understanding when two subshifts of finite type are equivalent from the point of view of their spaces of Markov measures. We propose the notion of Markov equivalence of subshifts of finite type, which involves a sequence of hyperbolic homeomorphisms between their doubly transitive points. We show that the maps effecting a Markov equivalence must respect the cycles (periodic orbits) of the subshifts. This forces, in particular, Markov equivalent subshifts of finite type to have the same zeta function. The constraints on the periodic orbits assume a special form when we have a sequence of almost topological conjugacies (in the sense of [**1**]) establishing the Markov equivalence.

A $k \times k$ matrix A of zeros and ones defines a *subshift of finite type* Σ_A, which is the topological subspace of $\{1, \ldots, k\}^{\mathbf{Z}}$ consisting of those points $x = (x_n)$ with $A(x_n, x_{n+1}) = 1$ for all $n \in \mathbb{Z}$, and the left shift homeomorphism σ is understood to act on Σ_A. (We shall also take A to be irreducible.) A point $x \in \Sigma_A$ is *doubly transitive* if both $\{\sigma^n x\colon n \in \mathbb{N}\}$ and $\{\sigma^{-n} x\colon n \in \mathbb{N}\}$ are dense in Σ_A; we denote the set of doubly transitive points of Σ_A by Ω_A. Ω_A is a dense σ-invariant G_δ, and every ergodic Borel measure supported by Σ_A gives full measure to Ω_A (see [**13**]). On Ω_A, we have the subspace topology and the restriction of the shift, which is also denoted by σ.

Suppose Σ_A, Σ_B are subshifts of finite type. The maps $\varphi: \Omega_A \to \Omega_B$ we consider will always be shift-commuting. Let $\varphi: \Omega_A \to \Omega_B$ be a (shift-commuting) continuous surjection. It is not hard to see that the continuity of φ is equivalent to the requirement

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that it be *finitary* in the following sense. There exist functions $m, a: \Omega_A \to \mathbb{N}$, called the *memory and anticipation of* φ, such that $\varphi(x')_0 = \varphi(x)_0$ whenever $x, x' \in \Omega_A$ and $x'_n = x_n$ for $-m(x) \le n \le a(x)$. Thus, $(\varphi x)_0$ is determined by $x_{-m(x)}x_{-m(x)+1}\cdots x_{a(x)}$, and the maps m, a are continuous. For $x \in \Omega_A$, put

$$m^*(x) = \sup_{n\in\mathbb{N}} \{m(\sigma^n x) - n\},$$

$$a^*(x) = \sup_{n\in\mathbb{N}} \{a(\sigma^{-n} x) - n\}.$$

Possibly, $m^*(x) = \infty$ or $a^*(x) = \infty$. The map φ is called *hyperbolic* if $m^*(x), a^*(x) < \infty$ for all $x \in \Omega_A$. (This is equivalent to requiring φ to preserve stable and unstable manifolds - see [**12**].) A homeomorphism $\varphi: \Omega_A \to \Omega_B$ is called *hyperbolic* if both φ and φ^{-1} are.

If λ is the maximum eigenvalue of A, the *topological entropy* of Σ_A is

$$h(\Sigma_A) = \log\lambda = \sup\{h(\mu): \mu \text{ is } \sigma\text{-invariant Borel probability on } \Sigma_A\},$$

where $h(\mu)$ denotes the entropy of μ. Letting

$$\pi_A(n) = \operatorname{trace}(A^n) = \operatorname{card}\{x \in \Sigma_A: \sigma^n x = x\},$$

the *zeta function* of Σ_A is

$$\zeta_A(t) = \exp\left(\sum_{1\le n<\infty} \pi_A(n)t^n/n\right) = \det(I - tA)^{-1},$$

which is defined and analytic for $|t| < 1/\lambda$. (See [**1**], [**2**], [**9**], [**13**].)

When A is $k\times k$, an element of $\{1,\ldots,k\}$ is a *symbol* of Σ_A. Consider a string $i_1i_2\cdots i_n$ where $i_1, i_2, \ldots, i_n$ are symbols of Σ_A. If $1 = A(i_1, i_2) = A(i_2, i_3) = \cdots = A(i_{n-1}, i_n)$, then $i_1i_2\cdots i_n$ is called a *word* (or *block*) of length n; when there is a need to emphasize Σ_A, we use the terms Σ_A-word and Σ_A-block. We denote by A_n the 0-1 matrix whose rows and columns are indexed by Σ_A-words of length n, and having $A_n(i_1\cdots i_n, i'_1\cdots i'_n) = 1$ if and only if $i'_1 = i_2$, $i'_2 = i_3, \ldots, i'_{n-1} = i_n$. By definition, $A_1 = A$. The subshifts of finite type Σ_{A_n}, $n \ge 1$, are naturally topologically conjugate; Σ_{A_n} is called the *n-block system* of Σ_A. A stochastic matrix compatible with one of the matrices A_n, $n \ge 1$, defines an ergodic Markov measure p on Σ_A. Since $p(\Omega_A) = p(\Sigma_A) = 1$, we may think of p as a measure on Ω_A. The Markov measure p is said to have *memory* n if it may be defined by a stochastic matrix compatible with A_n, but not by one compatible with A_{n-1}. We denote by $\mathcal{M}_n(A)$ the set of Markov measures of memory less than or equal to n.

When M is a non-negative matrix compatible with A_n and r is a strictly positive right eigenvector of M corresponding to its maximum eigenvalue $\lambda > 0$ (as furnished by the Perron-Frobenius theorem), the matrix $\bar{M}$ defined by

$$\bar{M}(i,j) = M(i,j)r(j)/\lambda r(i)$$

is stochastic and compatible with A_n; $\bar{M}$ is called the *stochastic version* of M. This operation may be used to turn $\mathcal{M}_n(A)$ into a linear space: Let $p, p' \in \mathcal{M}_n(A)$ be described by stochastic matrices P, P' compatible with A_n. The sum $p + p'$ is defined to be the element of $\mathcal{M}_n(A)$ described by the stochastic version of the matrix M with

$$M(i,j) = P(i,j)P'(i,j).$$

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For $t \in \mathbb{R}$, the product tp is the element of $\mathcal{M}_n(A)$ described by the stochastic version of the matrix M compatible with A_n and having

$$M(i,j) = P(i,j)^t$$

whenever $P(i,j) \neq 0$. The inclusions $\mathcal{M}_n(A) \to \mathcal{M}_{n+1}(A)$ then become linear injections and, in the direct limit, we have the linear space $\mathcal{M}(A) = \bigcup_{1 \le n < \infty} \mathcal{M}_n(A)$ of all Markov measures on Σ_A. This viewpoint is due to W. Parry and S. Tuncel; further details may be found in [**3**], [**5**] and [**8**].

When should two subshifts of finite type Σ_A and Σ_B be considered equivalent with respect to their spaces of Markov measures, $\mathcal{M}(A)$ and $\mathcal{M}(B)$? If Σ_A and Σ_B are topologically conjugate, then the conjugacy sets up a linear isomorphism between $\mathcal{M}(A)$ and $\mathcal{M}(B)$. In general, however, topological conjugacy is too strict. Consider the case where Σ_B is the inverse (shift reversal) of Σ_A, the subshift of finite type $\Sigma_{A^{\mathrm{tr}}}$ defined by the transpose of A. Recall that Σ_A and $\Sigma_{A^{\mathrm{tr}}}$ are not, in general, topologically conjugate. (See, for example, [**9**].) Supposing A is $k \times k$, a homeomorphism $\varphi_1 : \Omega_A \to \Omega_{A^{\mathrm{tr}}}$ may be defined by requiring

$$\varphi_1(1\ i_1 i_2 \cdots i_l 1) = 1\ i_l i_{l-1} \cdots i_1 1,$$

for all words $1\ i_1 \cdots i_l 1$ with $i_1, \ldots, i_l \in \{2, \ldots, k\}$. In this well-known construction, φ_1 sends ones to ones (the symbol 1 is a 'marker') and φ_1 reverses the words between markers. It is not hard to see that φ_1 sends each measure $p \in \mathcal{M}_1(A)$, with defining matrix P, to its inverse (reversal) $p^* \in \mathcal{M}_1(A^{\mathrm{tr}})$ defined by the stochastic version of the transpose P^{tr}. Similarly, for each $n \in \mathbb{N}$, a Σ_A-word of length n may be used to define a homeomorphism $\varphi_n : \Omega_A \to \Omega_{A^{\mathrm{tr}}}$ which will send every $p \in \mathcal{M}_n(A)$ to its reversal $p^* \in \mathcal{M}_n(A^{\mathrm{tr}})$. (Apply the above construction of φ_1 to Ω_{A_n} and $\Omega_{A_n^{\mathrm{tr}}}$.) We are thus led to the following definition, which allows for the fact that $\mathcal{M}(A)$ and $\mathcal{M}(B)$ are direct limits, and is reminiscent of the equivalences used for direct limits of C^*-algebras [**4**].

Two subshifts of finite type Σ_A and Σ_B are called *Markov equivalent* if there exists a sequence of (shift-commuting) hyperbolic homeomorphisms between their doubly transitive points, $\varphi_n : \Omega_A \to \Omega_B$, such that the induced maps give, in the direct limit, an isomorphism of $\mathcal{M}(A)$ and $\mathcal{M}(B)$; that is, for each $p \in \mathcal{M}(A)$ there exists $q \in \mathcal{M}(B)$ with $p \circ \varphi_n^{-1} = q$ for all large n, and vice versa. The main result of this paper is the following.

THEOREM. *If the subshifts of finite type Σ_A and Σ_B are Markov equivalent, then $\zeta_A = \zeta_B$.*

Further motivation for Markov equivalence comes from the recent work of K. Schmidt ([**11**], [**12**]). This work extends earlier results on finitary isomorphisms with finite expected code lengths ([**6**], [**10**]). From a categorical viewpoint and for coding purposes, it establishes hyperbolic finitary isomorphisms as a more general and natural setting than finitary isomorphisms with finite expected code lengths. By showing that a number of invariants extend to this setting, it also leads one to ask if the classifications of Markov chains by hyperbolic finitary isomorphism and by almost block isomorphism are identical. That is, can a hyperbolic finitary isomorphism of Markov chains always be replaced by a measure-preserving almost topological conjugacy? Such a replacement would in particular provide, between doubly transitive points, a hyperbolic homeomorphism and induce an isomorphism of the linear spans of the Markov measures. This (intermediate) situation is related to Markov equivalence. In the final section we shall specialize to almost topological conjugacies.

I should also point out that it is not known if, in the above definition of Markov equivalence, the hyperbolicity condition follows from the other requirements on the maps φ_n. (Meir Smorodinsky has shown that a homeomorphism $\varphi: \Omega_A \to \Omega_B$ need not be hyperbolic.)

2. *Markers and hyperbolicity*

Let Σ_A, Σ_B be subshifts of finite type, and let $\varphi: \Omega_A \to \Omega_B$ be a shift-commuting continuous surjection. Let $m, a: \Omega_A \to \mathbb{N}$ be the memory and anticipation of φ, and let m^*, a^* be as defined as in the previous section. A Σ_A-word $u = i_{-l} \cdots i_0 \cdots i_l$, of length $2l+1$, is called a *marker* (for φ) if

$$C(u) = \{x \in \Omega_A: x_{-l} = i_{-l}, \ldots, x_l = i_l\} \subset \{x \in \Omega_A: m^*(x), a^*(x) \leq l\}.$$

When u is a marker and $x \in C(u)$, the coordinates $\{(\varphi x)_n: n \in \mathbb{N}\}$ are determined by $(x_n)_{-l \leq n < \infty}$, and $\{(\varphi x)_{-n}: n \in \mathbb{N}\}$ are determined by $(x)_{-\infty < n \leq l}$. Hyperbolicity is characterized by the existence of a marker:

PROPOSITION. *A continuous surjection $\varphi: \Omega_A \to \Omega_B$ is hyperbolic if and only if φ has a marker.*

Proof. It is easy to see that if there is a marker for φ then φ is hyperbolic. Conversely, suppose that φ is hyperbolic, so that $m^*(x)$, $a^*(x) < \infty$ for all $x \in \Omega_A$. Since Ω_A is a G_δ in Σ_A, the topology of Ω_A may be given by a metric so that, with respect to this metric, Ω_A is complete. Furthermore, m^* and a^* being lower semi-continuous functions, the sets $\{x \in \Omega_A: m^*(x) \leq n\}$ and $\{x \in \Omega_A: a^*(x) \leq n\}$ are closed. Clearly

$$\Omega_A = \bigcup_{1 \leq n < \infty} \{x \in \Omega_A: m^*(x), a^*(x) \leq n\},$$

and the Baire category theorem shows that one of the sets in this union, say $\{x \in \Omega_A: m^*(x), a^*(x) \leq N\}$, has non-empty interior. Hence, there exists a word $u = i_{-l} \cdots i_0 \cdots i_l$ such that $l \geq N$ and

$$C(u) = \{x \in \Omega_A: x_{-l} = i_{-l}, \ldots, x_l = i_l\} \subset \{x \in \Omega_A: m^*(x), a^*(x) \leq N\}.$$

As the referee has pointed out, this proposition and its proof are valid in the more general setting of transitive subshifts (not necessarily of finite type).

A Σ_A-word of the form $j_0 j_1 \cdots j_{k-1} j_0$ is called a Σ_A-*cycle*; the length of this cycle is k. Σ_A-cycles are in one-to-one correspondence with periodic orbits of Σ_A. Fix a Σ_A-cycle $j_0 j_1 \cdots j_{k-1} j_0$, and put $v = j_0 \cdots j_{k-1}$. We write v^n for the concatenation $vv \cdots v$ of n copies of v. For a Markov measure $p \in \mathcal{M}(A)$, the *weight* of $j_0 \cdots j_{k-1} j_0$ with respect to p is given by the conditional probability

$$w_p(j_0 \cdots j_{k-1} j_0) = p\{x_{kn} \cdots x_{kn+k-1} = v \mid x_0 x_1 \cdots x_{kn-1} = v^n\},$$

where n is large enough for the length kn of v^n to exceed the memory of p.

Now assume that $\varphi: \Omega_A \to \Omega_B$ is a hyperbolic homeomorphism and the Markov measures $p \in \mathcal{M}(A)$, $q \in \mathcal{M}(B)$ are such that $p \circ \varphi^{-1} = q$. Let $u = i_{-l} \cdots i_0 \cdots i_l$ be a marker for φ, and put $u_1 = i_{-l} \cdots i_0$, $u_2 = i_0 \cdots i_l$. If $j_0 j_1 \cdots j_{k-1} j_0$ is a Σ_A-cycle with

$k \geq 2l+1$, $j_0 j_1 \cdots j_l = u_2$ and $j_{k-l} \cdots j_{k-1} j_0 = u_1$, then there exists a Σ_B-cycle b (of length k) such that $\varphi(x)_0 \varphi(x)_1 \cdots \varphi(x)_k = b$ whenever $x \in \Omega_A$ has $x_{-l} \cdots x_{k+l} = u_1 j_1 \cdots j_{k-1} u_2$; we write $\varphi(j_0 j_1 \cdots j_{k-1} j_0) = \varphi(u_1 j_1 \cdots j_{k-1} u_2) = b$. The following is an immediate consequence of Proposition (4.4) of [**6**]: If $j_0 j_1 \cdots j_{k-1} j_0$ is a Σ_A-cycle with $k \geq 2l+1$, $j_0 j_1 \cdots j_l = u_2$ and $j_{k-l} \cdots j_{k-1} j_0 = u_1$, then

$$w_q(\varphi(j_0 j_1 \cdots j_{k-1} j_0)) = w_p(j_0 j_1 \cdots j_{k-1} j_0).$$

In particular, it follows from this that φ gives a linear map on the subspace of $\mathcal{M}(A)$ it sends into $\mathcal{M}(B)$.

I would like to thank Bruce Kitchens and Klaus Schmidt for discussions; the proposition above was worked out with K. Schmidt.

3. *Proof of theorem*

To establish the zeta function as an invariant of Markov equivalence, we first consider a special case. Letting Σ_A, Σ_B be subshifts of finite type, suppose that a Markov equivalence between them is effected by a single shift-commuting hyperbolic homeomorphism $\varphi : \Omega_A \to \Omega_B$, so that $p \to p \circ \varphi^{-1}$ is an isomorphism between $\mathcal{M}(A)$ and $\mathcal{M}(B)$. (Note that the existence of a shift-commuting homeomorphism $\varphi : \Omega_A \to \Omega_B$ and consideration of σ-invariant measures are enough to guarantee $h(\Sigma_A) = h(\Sigma_B)$.)

As before, let $u = i_{-l} \cdots i_0 \cdots i_l$ be a marker for φ, and write $u_1 = i_{-l} \cdots i_0$, $u_2 = i_0 \cdots i_l$. Fix Σ_A-cycle $a = j_0 j_1 \cdots j_{k-1} j_0$, and suppose the least period of a is given by k. Fixing Σ_A-words $i_l k_1 \cdots k_{l'} j_0$ and $j_0 k'_1 k'_2 \cdots k'_{l''} i_{-l}$, define Σ_A-cycles a_n by

$$a_n = u_2 k_1 \cdots k_{l'} (j_0 j_1 \cdots j_{k-1})^n j_0 k'_1 k'_2 \cdots k'_{l''} u_1.$$

For $n \in \mathbb{N}$, let $b_n = \varphi(a_n)$. The Σ_B-cycles b_n all start and end with the same symbol of Σ_B. For $p \in \mathcal{M}(A)$ we have $w_p(a_n) = w_{p\circ\varphi^{-1}}(b_n)$. Moreover:

(*) The ratio

$$(w_p(a_{n+1}))/(w_p(a_n)) = (w_{p\circ\varphi^{-1}}(b_{n+1}))/(w_{p\circ\varphi^{-1}}(b_n))$$

is independent of n, provided n exceeds the memory of p.

Consider the multiplicative free Abelian group generated by Σ_B-transitions (Σ_B-words of length 2). Regard the ratio b_{n+1}/b_n as an element of this group. Thus, b_{n+1}/b_n is regarded as a monomial in the transitions - a product of (possibly negative) integral powers of the transitions.

Let us say that a Σ_B-transition ii' is forced if $B(i,j) = 0$ for all Σ_B-symbols $j \neq i'$; otherwise the transition is unforced. In b_{n+1}/b_n, identify the forced transitions, by putting them all equal to the same indeterminate s. Ignoring powers of s, and letting N be such that $q \circ \varphi \in \mathcal{M}_N(A)$ for all $q \in \mathcal{M}_1(B)$, the unforced parts of b_{n+1}/b_n must be identical for all $n \geq N$; otherwise we could find $q \in \mathcal{M}_1(B)$ that would assign different values $w_q(b_{n+1})/w_q(b_n)$ to them, and violate (*). Thus there exist $\alpha \in \mathbb{Z}$ and a (reduced) monomial π in unforced transitions such that

$$b_{n+1}/b_n = s^\alpha \pi$$

for $n \geq N$. Furthermore, as $b_{N+n}/b_N = s^{n\alpha}\pi^n$ and b_N has finite length, π does not contain any negative powers of transitions. Letting $\beta \geq 1$ be the degree of π and using the fact that the degree of b_{n+1}/b_n is given by the length k of $j_0j_1\cdots j_{k-1}$, we have $\alpha+\beta=k$. Denoting by γ_n the number of forced transitions in b_n and using $b_{N+n}/b_N=s^{n\alpha}\pi^n$ again,

$$kn=\gamma_{N+n}-\gamma_N+\beta n.$$

Hence,

$$n(\beta-k)=\gamma_N-\gamma_{N+n}\le\gamma_N$$

for all $n\in\mathbb{N}$. It follows that $\beta-k\le 0$. Therefore, $1\le\beta\le k$, $\alpha+\beta=k$, and $\alpha\ge 0$.

We now know that b_{n+1}/b_n consists of k transitions, and the unforced transitions appearing in this ratio are independent of $n\ge N$. But, as forced transitions follow unforced ones, these too must be independent of $n\ge N$. Thus there exists a product ρ of k transitions such that $b_{n+1}/b_n=\rho$ for all $n\ge N$. Suppose, for the moment, that ρ corresponds to a well-defined Σ_B-cycle b of least period k. We then obtain a map Φ from Σ_A-cycles to Σ_B-cycles by putting $\Phi(a)=b$. It is easy to see, using weights, that Φ does not depend on the choice of the marker u or the connecting words $k_1\cdots k_{l'}$ and $k_1'k_2'\cdots k_{l''}'$. The map Φ is such that a and $\Phi(a)$ have the same least period and

$$w_p(a)=w_{p\circ\varphi^{-1}}(\Phi(a))$$

for all $p\in\mathcal{M}(A)$. Since an inverse Φ^{-1} of Φ may be constructed by using φ^{-1} in place of φ, Φ is bijective.

The remaining task is to establish the following:

(**) The product ρ corresponds to a well-defined Σ_B-cycle of least period k.

To start with, note that the transitions appearing in ρ yield exactly the same collection of initial and terminal symbols, because the cycles b_n, b_{n+1} have this property. This means that ρ may be decomposed into a product of cycles. It is, therefore, easy to see that (**) holds when the least period k of the Σ_A-cycle a is equal to one. We use induction on k. Assume that (**) holds for Σ_A-cycles of least period lower than k, so that we already have the bijection Φ between cycles of period less than k. Decompose ρ into a product $\rho=\rho_1\rho_2\cdots\rho_J$ of Σ_B-cycles $\rho_1,\ldots,\rho_J$. If $J>1$ then the least periods of $\rho_1,\ldots,\rho_J$ are lower than k and we find that

$$w_p(a)=\prod_{1\le j\le J}w_p(\Phi^{-1}(\rho_j))$$

for all $p\in\mathcal{M}(A)$. Since this is impossible, we conclude that $J=1$ and that (**) holds for cycles of least period k also.

We have thus dealt with the case where a Markov equivalence between Σ_A and Σ_B is established by a single homeomorphism $\varphi:\Omega_A\to\Omega_B$. It should, however, be clear that the above argument may be extended to the general case: Suppose that a Markov equivalence between Σ_A and Σ_B is established by a sequence of shift-commuting hyperbolic homeomorphisms $\varphi_i:\Omega_A\to\Omega_B$. There is then a bijection Φ between Σ_A-cycles and Σ_B-cycles, sending each Σ_A-cycle to a Σ_B-cycle of the same least period, which is determined by the sequence φ_i in the following way. If $j_0j_1\cdots j_{k-1}j_0$ is a Σ_A-cycle, there exists $I\in\mathbb{N}$ such that, for $i\ge I$, a marker $u=$

$i_{-l} \cdots i_0 \cdots i_l$ for φ_i, Σ_A-cycles a_n of the form

$$a_n = i_0 \cdots i_l k_1 \cdots k_{l'} (j_0 j_1 \cdots j_{k-1})^n j_0 k'_1 \cdots k'_{l''} i_{-l} \cdots i_0,$$

and large n, the cycle $\varphi_i(a_{n+1})$ differs from $\varphi_i(a_n)$ by $\Phi(j_0 j_1 \cdots j_{k-1} j_0)$.

4. *Almost topological conjugacies*

Almost topological conjugacies, which were first considered by Adler and Marcus [1], give an important finite way of constructing hyperbolic homeomorphisms between doubly transitive points of subshifts of finite type. In this section we concentrate on Markov equivalences established through almost topological conjugacies, and find conditions on the periodic points. These conditions, for this special case, straightforwardly lead to the correspondences of the previous section. For convenience, we assume throughout the section that Σ_A, Σ_B are aperiodic subshifts of finite type. We shall use the following combinatorial lemma.

LEMMA 1. *If a is a Σ_A-cycle of least period α and $l < \alpha$, then the number of distinct Σ_A-words of length l appearing in a is at least $l/2$.*

In other words, when a is regarded as a Σ_{A_l}-cycle, it contains at least $l/2$ symbols of Σ_{A_l}. We omit the proof of this lemma.

Again, we first consider the case where a Markov equivalence between Σ_A and Σ_B is set up by a single almost topological conjugacy. We assume that we have a subshift of finite type Σ_C and continuous shift-commuting surjections $\theta: \Sigma_C \to \Sigma_A$, $\psi: \Sigma_C \to \Sigma_B$ so that θ, ψ are one-to-one when restricted to Ω_C and the resulting homeomorphism $\varphi = \psi \circ \theta^{-1}: \Omega_A \to \Omega_B$ induces a bijection between $\mathcal{M}(A)$ and $\mathcal{M}(B)$. We have the following picture.

$$\begin{tikzcd} & \Sigma_C \arrow[dl, "\theta"'] \arrow[dr, "\psi"] & \\ \Sigma_A & & \Sigma_B \end{tikzcd}$$

LEMMA 2. *If $z \in \Sigma_C$ is a periodic point, then $x = \theta(z)$ and $y = \psi(z)$ have the same least period.*

Proof. Let α, β, γ be the least periods of x, y, z. Suppose $\alpha \neq \beta$; say $\alpha > \beta$. Choose n so large that the Σ_{A_n}-cycle corresponding to x has (at least) one transition which occurs in it only once. Assign a prime number π to this transition and 1 to every other Σ_{A_n}-transition to obtain a non-negative integral matrix S compatible with A_n. Let $\lambda > 0$ be the maximum eigenvalue of S, and $p \in \mathcal{M}(A)$ the measure defined by the stochastic version of S. Writing $\Gamma(p)$ for the multiplicative subgroup of the positive reals generated by the weights (with respect to p) of Σ_A-cycles, it is easy to see that

$$\Gamma(p) \subset \{\pi^k \lambda^l : k, l \in \mathbb{Z}\}.$$

Now let $q = p \circ \varphi^{-1} \in \mathcal{M}(B)$. According to [6], we then have $\Gamma(q) = \Gamma(p)$. In particular, $w_q(y) \in \Gamma(p)$. Using $w_p(x) = \pi \lambda^{-\alpha}$ and the fact that weights are preserved by θ and ψ (see [5]), we find

$$w_q(y) = \pi^{\beta/\alpha}\lambda^{-\beta}.$$

This means that there exist $k, l \in \mathbb{Z}$ with

$$\pi^{\beta - k\alpha} = \lambda^{\alpha(l+\beta)}.$$

Since λ is the maximum eigenvalue of an aperiodic matrix, this equation implies that $\lambda \in \mathbb{N}$. Moreover, as π is prime, λ must be a power of π, and we conclude that α divides β, which is impossible if $\alpha > \beta$.

LEMMA 3. *Two periodic orbits of Σ_C are mapped to the same Σ_A-orbit by θ if and only if they are mapped to the same Σ_B-orbit of ψ.*

Proof. If, for example, two periodic orbits of Σ_C were sent to distinct Σ_A-orbits by θ while being identified by ψ, we could find $p \in \mathcal{M}(A)$ such that their common image in Σ_B would not have a well-defined weight with respect to $p \circ \varphi^{-1} \in \mathcal{M}(B)$.

Lemmas 2 and 3 show that, in the case under consideration, we can find a bijection Φ from the periodic orbits of Σ_A to the periodic orbits of Σ_B such that Φ has the following property. For each periodic orbit a of Σ_A, the orbits a and $\Phi(a)$ have the same least period and the sets $\theta^{-1}(a)$ and $\psi^{-1}(\Phi(a))$ are identical. Put another way, $\zeta_A = \zeta_B$ and θ and ψ restrict to the same map on the periodic points of Σ_C.

More generally, we would have a sequence of almost topological conjugacies

$$\begin{tikzcd} & \Sigma_{C_n} \arrow[dl, "\theta_n"'] \arrow[dr, "\psi_n"] & \\ \Sigma_A & & \Sigma_B \end{tikzcd}$$

and a Markov equivalence between Σ_A and Σ_B would be established by the homeomorphisms $\varphi_n = \psi_n \circ \theta_n^{-1} : \Omega_A \to \Omega_B$. The behaviour of periodic orbits is now given by the following.

PROPOSITION. *There exists a bijection Φ from the periodic orbits of Σ_A to the periodic orbits of Σ_B with the following property. For each periodic orbit a of Σ_A, the orbits a and $\Phi(a)$ have the same least period and there exists $N \in \mathbb{N}$ such that the sets $\theta_n^{-1}(a)$ and $\psi_n^{-1}(\Phi(a))$ are identical for every $n \geq N$.*

Proof. The proposition is established by using lemma 1 to generalize lemmas 2 and 3. We do this in detail for lemma 2. Let b be a periodic Σ_B-orbit of least period β, and find Σ_{C_n}-orbits c_n with $\psi_n(c_n) = b$. Let α_n denote the least period of $a_n = \theta_n(c_n)$. We claim that $\alpha_n = \beta$ for large n. To show this, we assume that the sequence α_n is unbounded. (Otherwise, there exists a Σ_A-orbit a with $a_n = a$ for infinitely many n and the proof of lemma 2 may be used directly.) By passing to a subsequence, we take $\alpha_n > 2\beta$. Using lemma 1, there are more than β words of length $2\beta + 1$ appearing in the Σ_A-cycle corresponding to a_n. Hence, passing to a subsequence if necessary, we find a Σ_A-word $i_1 i_2 \cdots i_{2\beta+1}$ and $\gamma_n \in \mathbb{N}$ such that $1 \leq \gamma_n < \alpha_n/\beta$ and $i_1 i_2 \cdots i_{2\beta+1}$ appears γ_n times in the Σ_A-cycle corresponding to a_n. Assign a prime number π to the $\Sigma_{A_{2\beta}}$-transition given by $i_1 i_2 \cdots i_{2\beta+1}$ and 1 to every other $\Sigma_{A_{2\beta}}$-transition to obtain a non-negative matrix S compatible with $A_{2\beta}$. Let $\lambda > 0$ be the maximum eigenvalue of S and $p \in \mathcal{M}(A)$ the measure defined by the stochastic version of S. Find $q \in \mathcal{M}(B)$ with $p \circ \varphi_n^{-1} = q$ for large n. Then

$$w_q(b) \in \Gamma(q) = \Gamma(p) \subset \{\pi^k \lambda^l \in \mathbb{Z}\}.$$

For large n,

$$w_q(b) = (\pi^{\gamma_n} \lambda^{-\alpha_n})^{\beta/\alpha_n} = \pi^{\gamma_n \beta/\alpha_n} \lambda^{-\beta},$$

so that there exist $k, l \in \mathbb{Z}$ with

$$\pi^{\gamma_n \beta/\alpha_n} \lambda^{-\beta} = \pi^k \lambda^l.$$

As before, we find that

$$\pi^{(\gamma_n \beta - k\alpha_n)} = \lambda^{\alpha_n (l+\beta)},$$

λ is a power of π and α_n divides $\gamma_n \beta$, which is impossible since $\gamma_n < \alpha_n/\beta$. This proves our claim. Now note that lemma 1 may be used to also establish the following. Let c_n, c'_n be periodic orbits of Σ_{C_n}. There exists a Σ_A-orbit a with $\theta_n(c_n) = \theta_n(c'_n) = a$ for large n if and only if there exists a Σ_B-orbit b with $\psi_n(c_n) = \psi_n(c'_n) = b$ for large n.

REFERENCES

[1] R. L. Adler & B. Marcus. Topological entropy and equivalence of dynamical systems. *Mem. Amer. Math. Soc.* **219** (1979).

[2] R. Bowen & O. E. Lanford. Zeta functions of the restrictions of the shift transformation. *Proc. Symp. Pure. Math. Amer. Math. Soc.* **14** (1970), 43–50.

[3] M. Boyle & S. Tuncel. Infinite-to-one codes and Markov measures. *Trans. Amer. Math. Soc.* **285** (1984), 657–684.

[4] E. G. Effros. *Dimensions and C*-algebras.* C.B.M.S. Regional Conference Seires 46, Amer. Math. Soc., Providence, 1981.

[5] B. Kitchens. Linear algebra and subshifts of finite type. In *Proceedings of the Conference on Modern Analysis and Probability, Contemp. Math.* **26**, Amer. Math. Soc., Providence, 1984.

[6] W. Parry & K. Schmidt. Natural coefficients and invariants for Markov shifts. *Invent. Math.* **76** (1984), 15–32.

[7] W. Parry & S. Tuncel. On the classification of Markov chains by finite equivalence. *Ergod. Th. & Dynam. Sys.* **1** (1981), 303–335.

[8] W. Parry & S. Tuncel. On the stochastic and topological structure of Markov chains. *Bull. London Math. Soc.* **14** (1982), 16–27.

[9] W. Parry & S. Tuncel. *Classification Problems in Ergodic Theory.* London Math. Soc. Lecture Notes 67, Cambridge Univ. Press, 1982.

[10] K. Schmidt. Invariants for finitary isomorphisms with finite expected code lengths. *Invent. Math.* **76** (1984), 33–40.

[11] K. Schmidt. Hyperbolic structure preserving isomorphisms of Markov shifts. To appear in *Israel J. Math.*

[12] K. Schmidt. Hyperbolic structure preserving isomorphisms of Markov shifts, II. Preprint.

[13] P. Walters. *An Introduction to Ergodic Theory.* Graduate Texts in Math. 79, Springer, New York, 1982.