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SIMPLY CONNECTED MANIFOLDS WITH LARGE HOMOTOPY STABLE CLASSES

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Abstract

For every $k \geq 2$ and $n \geq 2$, we construct n pairwise homotopically inequivalent simply connected, closed $4k$-dimensional manifolds, all of which are stably diffeomorphic to one another. Each of these manifolds has hyperbolic intersection form and is stably parallelisable. In dimension four, we exhibit an analogous phenomenon for spin^c structures on $S^2 \times S^2$. For $m \geq 1$, we also provide similar $(4m-1)$-connected $8m$-dimensional examples, where the number of homotopy types in a stable diffeomorphism class is related to the order of the image of the stable J-homomorphism $\pi_{4m-1}(SO) \to \pi^s_{4m-1}$.

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1. Introduction

Let q be a positive integer and let $W_g := \#_g(S^q \times S^q)$ be the g-fold connected sum of the manifold $S^q \times S^q$ with itself. Two compact, connected smooth $2q$-manifolds M_0 and M_1 with the same Euler characteristic are *stably diffeomorphic*, written $M_0 \cong_{\mathrm{st}} M_1$, if there exists a nonnegative integer g and a diffeomorphism

$$M_0 \# W_g \to M_1 \# W_g.$$

Note that $S^q \times S^q$ admits an orientation-reversing diffeomorphism. Hence, the same is true of W_g and it follows that when the M_i are orientable the diffeomorphism type of the connected sum does not depend on orientations.

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A paradigm of modified surgery, as developed by Kreck [Kre99], is that one first seeks to classify $2q$-manifolds up to stable diffeomorphism, and then for each M_0, one tries to understand its *stable class*:

$$\mathcal{S}^{\mathrm{st}}(M_0) := \{M_1 \mid M_1 \cong_{\mathrm{st}} M_0\}/\text{diffeomorphism}.$$

The efficacy of this method was first demonstrated by Hambleton and Kreck, who applied it to 4-manifolds with finite fundamental group in a series of papers [HK88a, HK88b, HK93a, HK93b].

However, the Browder–Novikov–Sullivan–Wall surgery exact sequence [Wal99] aims instead to classify manifolds within a fixed homotopy class. In general, there is no obvious relationship between homotopy equivalence and stable diffeomorphism, although in some cases, there are implications, for example, [Dav05]. To enable a comparison between the two approaches, we define the *homotopy stable class* of M_0 to be

$$\mathcal{S}^{\mathrm{st}}_{\mathrm{h}}(M_0) = \{M_1 \mid M_1 \cong_{\mathrm{st}} M_0\}/\text{homotopy equivalence}.$$

Our aim is to investigate the cardinality of $\mathcal{S}^{\mathrm{st}}_{\mathrm{h}}(M_0)$, and, in particular, we provide new examples of simply connected manifolds with arbitrarily large homotopy stable class.

Throughout this article, we consider closed, connected, simply connected, smooth manifolds. To define the intersection form and related invariants, we orient all manifolds. When necessary, to achieve unoriented results, we later factor out the effect of the choice of orientation.

When the dimension is $4k+2$, Kreck showed that the stable class of such manifolds is trivial [Kre99, Theorem D]. We therefore focus on dimensions $4k$ with $k > 1$ (dimension four is discussed separately below). Kreck also showed that for every such simply connected manifold M^{4k}, the stable class of $M^{4k}\#W_1$ is trivial. However, as pointed out by Kreck and Schafer [KS84, I], for $k>1$, examples of closed, simply connected $(2k-1)$-connected $4k$-manifolds M with arbitrarily large homotopy stable class have been implicit in the literature since Wall's classification of these manifolds up to the action of the group of homotopy spheres [Wal62]. These examples are distinguished by their intersection form

$$\lambda_M : H_{2k}(M;\mathbb{Z}) \times H_{2k}(M;\mathbb{Z}) \to \mathbb{Z},$$

which must be definite (to have inequivalent forms); moreover, to realise the forms by closed, almost-parallelisable manifolds, they must have signature divisible by $8|bP_{4k}|$, where $|bP_{4k}|$ is the order of the group of homotopy $(4k-1)$-spheres that bound parallelisable manifolds [MK60, Corollary on page 457].

In this paper, we consider examples where the intersection form is isomorphic to the standard hyperbolic form

$$H^+(\mathbb{Z}) = \left(\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right)$$

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and where there is an additional invariant, a homomorphism $f : \mathbb{Z}^2 \to \mathbb{Z}$. The pair $(H^+(\mathbb{Z}), f)$ is an example of an *extended symmetric form*; see Definition 3.5. The isometries of the rank two hyperbolic form are highly restricted: they are generated by switching the two basis vectors and multiplying both basis vectors by -1. As such, the unordered pair

$$\{a, b\} := \{f(1, 0), f(0, 1)\}/(\pm 1),$$

considered up to multiplication of both integers by -1, gives an invariant of the isometry class of the extended symmetric form $(H^+(\mathbb{Z}), f)$. However, in the Witt class, or stable equivalence class, only the divisibility $d := \gcd(a, b)$ and the product $A = ab$ are invariants. Since a fixed number A can often be factorised in many ways as a product of coprime integers a, b, if we can define a suitable f, this simple algebra has the chance to detect large stable classes. In the proof of our first main theorem, we define such an f using the cohomology ring of the manifolds we construct.

THEOREM 1.1. *Fix positive integers n and $k \geq 2$. There are infinitely many stable diffeomorphism classes of closed, smooth, simply connected $4k$-manifolds $\{[M_i]_{\mathrm{st}}\}_{i=1}^{\infty}$, such that $|\mathcal{S}_h^{\mathrm{st}}(M_i)| \geq n$. Moreover, $\mathcal{S}_h^{\mathrm{st}}(M_i)$ contains a subset $\{M_i^j\}_{j=1}^n$ of cardinality n, where $M_i^1 = M_i$, and each M_i^j is stably parallelisable and has hyperbolic intersection form.*

Here, *stably parallelisable* means that the tangent bundle becomes trivial after taking the Whitney sum with a trivial bundle of sufficiently high rank. More than one notion of stabilisation appears in this article, one for manifolds and one for vector bundles.

Kreck and Schafer [KS84] constructed examples of $4k$-manifolds M with nontrivial finite fundamental groups, such that the homotopy stable class of M contains distinct elements with hyperbolic intersection forms. However, as far as we know, our construction gives the first simply connected examples and the first for which the homotopy stable class has been shown to have arbitrary cardinality. In a companion paper [CCPS21], we investigate the homotopy stable class in more detail, also for manifolds with nontrivial fundamental group, and we relate the homotopy stable class to computations of the ℓ-monoid from [CS11].

The manifolds we construct to prove Theorem 1.1 are shown to be homotopically inequivalent using their cohomology rings. An alternative construction to obtain nontrivial homotopy stable class instead uses Pontryagin classes to define the homomorphism f in an extended symmetric form. This was alluded to in [KS84], but not carried through. Section 3 proves a theorem which implies the following result.

THEOREM 1.2. *For every $m \geq 1$, there exists a pair of closed, smooth, $(4m-1)$-connected $8m$-manifolds M_1 and M_2 with hyperbolic intersection forms that are stably diffeomorphic but not homotopy equivalent.*

Compared with the manifolds from Theorem 1.1 (for even k), the manifolds M_1 and M_2 from Theorem 1.2 are not stably parallelisable; however, since they are $(4m-1)$-connected and have the same intersection pairing, their cohomology rings are isomorphic. In particular, once again, the intersection form does not help.

To show that the manifolds in Theorem 1.2 are not homotopy equivalent, we use Wall's homotopy classification of $(4m-1)$-connected $8m$-manifolds [Wal62, Lemma 8], which makes use of an extended symmetric form $(H^+(\mathbb{Z}), f : \mathbb{Z}^2 \to \mathbb{Z}/j_m)$, where j_m is the order of the image of the stable J-homomorphism $J : \pi_{4m-1}(SO) \to \pi^s_{4m-1}$; see Section 3.

REMARK 1.3. The limiting factor preventing us from exhibiting arbitrarily large homotopy stable classes in Theorem 1.2 is that our lower bound on the cardinality of $\mathcal{S}^{\mathrm{st}}_{\mathrm{h}}(M_1)$ depends only on the number of primes dividing j_m. This grows with m, but in a fixed dimension, cannot be made arbitrarily large. However, if we instead count diffeomorphism classes, then we show in Theorem 3.3(2) that the stable class can be arbitrarily large for $(4m-1)$-connected $8m$-manifolds with hyperbolic intersection forms.

1.1. Dimension four. Dimension four was absent from the above discussion. This is because closed, smooth, simply connected 4-manifolds M and N are stably diffeomorphic if and only if they are homotopy equivalent. Here is an outline of why this holds. First, two such 4-manifolds are stably diffeomorphic if and only if there are orientations such that they have the same signatures, Euler characteristics, and w_2-types, that is, $\sigma(M) = \sigma(N)$, $\chi(M) = \chi(N)$, and their intersection forms have the same parity (even or odd). Thus homotopy equivalence implies stable diffeomorphism. For the other direction, $\sigma(M) = \sigma(N)$ and $\chi(M) = \chi(N)$ implies that the intersection forms are either both definite or both indefinite. In the definite case, the intersection forms must be diagonal by Donaldson's theorem [Don83], and so the intersection forms are isometric and therefore the manifolds are homotopy equivalent [Mil58a, Whi49]. In the indefinite case, the intersection form is determined up to isometry by its rank, parity and signature, and so again M and N are homotopy equivalent. Thus, the assumption that $k \geq 2$ was essential in Theorem 1.1.

One way in which an analogous phenomenon does occur in dimension four is by considering spin^c structures. Seiberg–Witten invariants of 4-manifolds and Heegaard–Floer cobordism maps are indexed by spin^c structures. The first Chern class c_1 of the spin^c structure then defines the map f in the extended symmetric forms. We illustrate this in Section 4, using the 4-manifold $S^2 \times S^2$.

THEOREM 1.4. *Let $C \in \mathbb{Z}$ with $|C| \geq 16$ and $8 \mid C$. Define $P(C)$ to be the number of distinct primes dividing $C/8$. There are $n := 2^{P(C)-1}$ stably equivalent spin^c structures $\mathfrak{s}_1, \ldots, \mathfrak{s}_n$ on $S^2 \times S^2$ with $c_1(\mathfrak{s}_i)^2 = C \in H^4(S^2 \times S^2) \cong \mathbb{Z}$ that are all pairwise inequivalent.*

1.2. Organisation. Section 2 proves Theorem 1.1, Section 3 proves Theorem 1.2 and Section 4 proves Theorem 1.4.

1.3. Conventions. Throughout this paper, all manifolds are compact, simply connected and smooth. As mentioned above, we also equip our manifolds with an orientation. For the remainder of this paper, all (co)homology groups have integral coefficients. We write $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

2. Simply connected 4*k*-manifolds with arbitrarily large stable class

We prove Theorem 1.1 by stating and proving Proposition 2.2 below. In the proposition, we construct a collection of $4k$-manifolds $N_{a,b}$ for each unordered pair of positive integers $\{a, b\}$ such that $(2k)!$ divides $2ab$. If $\{a, b\} \neq \{a', b'\}$, then $N_{a,b}$ and $N_{a',b'}$ are not homotopy equivalent. However, $ab = a'b'$ if and only if $N_{a,b}$ and $N_{a',b'}$ are stably diffeomorphic. Moreover, every manifold $N_{a,b}$ is closed, simply connected, has hyperbolic intersection form and is stably parallelisable. Thus, the proposition immediately implies Theorem 1.1.

First we have a lemma. To rule out orientation-reversing homotopy equivalences, we appeal to the following observation.

LEMMA 2.1. *Let N and N' be closed, oriented $4k$-manifolds. Suppose that a class z freely generates $H^2(N)$ and satisfies that $z^{2k} = n$ for some nonzero $n \in \mathbb{Z} = H^{4k}(N)$, and similarly for (N', z'). Then any homotopy equivalence $f : N \to N'$ must be orientation preserving.*

PROOF. Assume that f is of degree $\varepsilon = \pm 1$. Since f is a homotopy equivalence, N and N' have isomorphic cohomology rings. In particular, $H^2(N') \cong \mathbb{Z}$ is generated by $z' = \pm(f^*)^{-1}(z)$. Since $z'^{2k} = n$ in $H^{4k}(N') \cong \mathbb{Z}$ and $f^*(z'^{2k}) = z^{2k}$, properties of the cap and cup products show that

$$n = f_*(z^{2k} \cap [N]) = f_*(f^*(z'^{2k}) \cap [N]) = z'^{2k} \cap f_*([N]) = z'^{2k} \cap \varepsilon[N'] = \varepsilon n.$$

Since $n \neq 0$, this implies that f must be orientation preserving. □

Now we proceed with the construction of the promised manifolds.

PROPOSITION 2.2. *Fix $k > 1$. Given an unordered pair $\{a, b\}$ of positive coprime integers such that $(2k)!$ divides $2ab$, there exists a closed, oriented, $4k$-manifold $N^{4k}_{a,b}$ with the following properties.*

(i) *The manifold $N_{a,b}$ is simply connected and stably parallelisable.*
(ii) *The ring $H^*(N_{a,b})$ has generators w, x, y, z and 1 of degrees $2k{+}2$, $2k$, $2k$, 2 and 0, respectively, with $z^k = ax + by, x^2 = 0 = y^2, 2abw = z^{k+1}, xz = bw, yz = aw$ and xy generates $H^{4k}(N_{a,b})$.*

In particular, the intersection form of $N_{a,b}$ is hyperbolic and $z^{2k} = 2abxy$ is $2ab$ times a fundamental class of $N_{a,b}$. If $\{a, b\} \neq \{a', b'\}$, then $N_{a,b}$ and $N_{a',b'}$ have nonisomorphic integral cohomology rings and so are not homotopy equivalent. Moreover, $ab = a'b'$ if and only if $N_{a,b}$ and $N_{a',b'}$ are stably diffeomorphic.

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PROOF. Note that if we have a manifold $N_{a,b}$ and if we choose a stable normal framing on $N_{a,b}$, then the pair $(N_{a,b}, z)$ corresponds to a (normally) framed manifold over $\mathbb{C}P^\infty$ using the identification $H^2(N_{a,b}) \cong [N_{a,b}, \mathbb{C}P^\infty]$. This motivates the method we use, constructing $N_{a,b}$ by framed surgery on stably normally framed manifolds over $\mathbb{C}P^\infty$. It then follows automatically that the manifolds we obtain are stably parallelisable, since a manifold with trivial stable normal bundle has trivial stable tangent bundle too.

We start with S^2 together with the unique framing of its stable normal bundle corresponding to a choice of orientation, and consider the corresponding dual orientation class $\alpha \in H^2(S^2)$. Take the $2k$-fold product of S^2 with itself,

$$X_0 := S^2 \times \cdots \times S^2,$$

and define $\beta_0 \in H^2(X_0)$ to be the class that restricts to α in each S^2 factor. This means that under the inclusion

$$\iota_j : \{*\} \times \cdots \times S^2 \times \cdots \{*\} \to S^2 \times \cdots \times S^2$$

in the jth factor, $\iota_j^*(\beta_0) = \alpha$. Equivalently, let $p_i : S^2 \times \cdots \times S^2 \to S^2$ be the ith projection. Then $\beta_0 = \sum_{i=1}^{2k} p_i^*(\alpha)$. An elementary calculation shows that

$$\beta_0^{2k} = (2k)!\,[X_0]^* \in H^{4k}(X_0).$$

Here we write $[X_0]^* \in H^{4k}(X_0)$ for the dual of the fundamental class $[X_0] \in H_{4k}(X_0)$. To make this calculation, use $\beta_0 = \sum_{i=1}^{2k} p_i^*(\alpha)$ and note that:

(i) $p_i^*(\alpha) \cup p_j^*(\alpha) = p_j^*(\alpha) \cup p_i^*(\alpha)$ for $i \neq j$;
(ii) $p_i^*(\alpha) \cup p_i^*(\alpha) = p_i^*(\alpha \cup \alpha) = p_i^*(0) = 0$ and
(iii) $p_1^*(\alpha) \cup \cdots \cup p_{2k}^*(\alpha) = [X_0]^*$.

By assumption, there is a positive integer j such that $2ab = j(2k)!$. Take $X_1 := \#^j X_0$ to be the framed j-fold connected sum of X_0 and $\beta_1 \in H^2(X_1)$ to be the class that restricts to β_0 in each summand. That is, $H^2(X_1) \cong \bigoplus^j H^2(X_0)$ and $\beta_1 = (\beta_0, \ldots, \beta_0)$. Then,

$$\beta_1^{2k} = j\beta_0^{2k} = j(2k)!\,[X_1]^* = 2ab[X_1]^* \in H^{4k}(X_1).$$

The element $\beta_1 \in H^2(X_1)$ and the normal framing on X_1 define a normal map

$$(\beta_1, \overline{\beta}_1) : X_1 \to \mathbb{C}P^\infty,$$

where we take the trivial bundle over $\mathbb{C}P^\infty$. By surgery below the middle dimension, the normal map $(\beta_1, \overline{\beta}_1)$ is normally bordant to a $2k$-connected map $(\beta_2, \overline{\beta}_2) : X_2 \to \mathbb{C}P^\infty$. Since X_0 has signature zero, the same holds for X_1 and X_2. Since the stable normal bundle of X_2 is framed, so is the stable tangent bundle. Therefore, the stable tangent bundle has the trivial $2k$th Wu class vanishing and so the intersection form on X_2 is even. Let $z_\infty \in H^2(\mathbb{C}P^\infty)$ be the generator restricting to $\alpha \in H^2(\mathbb{C}P^1) = H^2(S^2)$ via the inclusion $\mathbb{C}P^1 \to \mathbb{C}P^\infty$, and consider the Poincaré dual of $\beta_2^*(z_\infty^k)$,

$$u := \mathrm{PD}(\beta_2^*(z_\infty^k)) \in H_{2k}(X_2).$$

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Since $\beta_2 : X_2 \to \mathbb{C}P^\infty$ is $2k$-connected, $H_{2k}(X_2) \to H_{2k}(\mathbb{C}P^\infty) \cong \mathbb{Z}$ is onto and therefore splits since $\mathbb{Z}$ is free. Since all homology groups are torsion-free, the dual map can be identified with the map $\beta_2^* : H^{2k}(\mathbb{C}P^\infty) \to H^{2k}(X_2)$ on cohomology. The splitting for β_2 dualises to a splitting for β_2^*, so the image of a generator $\beta_2^*(z_\infty^k)$ generates a summand. Applying Poincaré duality, we see that $u \in H_{2k}(X_2)$ is a primitive element; that is, u generates a summand of $H_{2k}(X_2)$.

We take the connected sum with an additional copy of $S^{2k} \times S^{2k}$ with null-bordant framing and trivial map to $\mathbb{C}P^\infty$ to obtain

$$X_3 := X_2 \#(S^{2k} \times S^{2k})$$

and a normal map $(\beta_3, \overline{\beta}_3) : X_3 \to \mathbb{C}P^\infty$. Note that up until this point, we have only used the product ab, rather than the data of the pair $\{a, b\}$. This changes for the upcoming construction of $X_4 = N_{a,b}$.

The intersection form λ_{X_3} of X_3 has an orthogonal decomposition corresponding to the connected sum decomposition of X_3:

$$(H_{2k}(X_3), \lambda_{X_3}) = (H_{2k}(X_2), \lambda_{X_2}) \oplus H^+(\mathbb{Z}),$$

where $H^+(\mathbb{Z})$ is the standard symmetric hyperbolic form. Note that since λ_{X_2} is even, so is λ_{X_3}. Let $\{e, f\}$ be a standard basis for $H^+(\mathbb{Z})$. Since a and b are coprime, we may and shall choose integers c, d such that $ad - bc = 1$. We also write $u = \mathrm{PD}(\beta_3^*(z_\infty^k))$. Here note that u is essentially the same element as the element $u \in H_2(X_2)$ that we defined above thinking of $H_2(X_2)$ as a subgroup of $H_2(X_3)$. Keeping this in mind, we have that

$$\lambda_{X_3}(u, u) = \langle \beta_3^*(z_\infty^k) \cup \beta_3^*(z_\infty^k), [X_3]\rangle = \langle \beta_3^*(z_\infty^{2k}), [X_3]\rangle = \langle z_\infty^{2k}, (\beta_3)_*[X_3]\rangle = 2ab,$$

since z_∞^{2k} generates $H^{4k}(\mathbb{C}P^\infty)$ and since $(\beta_3)_*$ sends $[X_3]$ to the same multiple of the generator of $H_{4k}(\mathbb{C}P^\infty)$ as that to which $(\beta_1)_*$ sends $[X_1]$. Since $u \in H_{2k}(X_2) \subseteq H_{2k}(X_3)$ is primitive and since λ_{X_2} is nonsingular, there is an element $v'' \in H_{2k}(X_2) \subseteq H_{2k}(X_3)$ such that

$$\lambda_{X_3}(u, v'') = \lambda_{X_2}(u, v'') = 1.$$

Now set $v' := (ad + bc)v''$ as well as

$$v := v' + e + \frac{2cd - \lambda_{X_3}(v', v')}{2} f.$$

Here, we use that the form λ_{X_3} is even. Since $u \in H_2(X_2)$ and $e, f \in H^+(\mathbb{Z})$, we observe that $\lambda_{X_3}(u, e) = \lambda_{X_3}(u, f) = 0$. As a consequence, the elements u, v span a subspace $H_{u,v} \subseteq H_{2k}(X_3)$, where λ_{X_3} restricted to $H_{u,v}$ has matrix

$$A = \begin{pmatrix} 2ab & ad + bc \\ ad + bc & 2cd \end{pmatrix},$$

which has determinant $4abcd - (ad + bc)^2 = -(ad - bc)^2 = -1$. Hence, $H_{u,v}$ is an orthogonal summand of $(H_{2k}(X_3), \lambda_{X_3})$ and a calculation shows that $H_{u,v}$ is hyperbolic

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with standard basis $\{e_1, f_1\}$, where $u = ae_1 + bf_1$ and $v = ce : 1 + df_1$. To see this, let $P := \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and note that $P\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)P^T = A$.

The orthogonal complement of $H_{u,v}$, namely $H_{u,v}^{\perp}$, has signature equal to the signature of X_3, which is zero; and hence since the intersection form is even, $H_{u,v}^{\perp}$ is stably hyperbolic.

We assert that $H_{u,v}^{\perp}$ maps trivially to $H_{2k}(\mathbb{C}P^{\infty})$ under β_{3*}. To see this, first note that $H^{2k}(\mathbb{C}P^{\infty}) \cong \mathbb{Z}$, generated by z_{∞}^{k}. We have an isomorphism

$$z_{\infty}^{k} \cap - : H_{2k}(\mathbb{C}P^{\infty}) \xrightarrow{\cong} H_0(\mathbb{C}\mathbb{P}^{\infty}) \cong \mathbb{Z}.$$

Recall that now $u = \mathrm{PD}(\beta_3^*(z_{\infty}^{k})) \in H_{u,v}$ and let $x \in H_{u,v}^{\perp}$. Then,

$$0 = \lambda_{X_3}(u, x) = \mathrm{PD}^{-1}(u) \cap x = \beta_3^*(z_{\infty}^{k}) \cap x = z_{\infty}^{k} \cap (\beta_3)_*(x).$$

Since $z_{\infty}^{k} \cap -$ is an isomorphism, this implies that $(\beta_3)_*(x) = 0$, which proves the assertion.

Now, since $\beta_3 : X_3 \to \mathbb{C}P^{\infty}$ is $2k$-connected and since $H_{u,v}^{\perp}$ maps trivially to $H_{2k}(\mathbb{C}P^{\infty})$, the Hurewicz theorem and the linked long exact sequences

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \pi_{2k+1}(\mathbb{C}P^{\infty}, X_3) & \longrightarrow & \pi_{2k}(X_3) & \longrightarrow & \pi_{2k}(\mathbb{C}P^{\infty}) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle\cong} & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & H_{2k+1}(\mathbb{C}P^{\infty}, X_3) & \longrightarrow & H_{2k}(X_3) & \xrightarrow{(\beta_3)_*} & H_{2k}(\mathbb{C}P^{\infty}) & \longrightarrow & \cdots
\end{array}$$

show that every element of $H_{u,v}^{\perp}$ is represented by a map from a $2k$-sphere in $\pi_{2k}(X_3)$. Hence, standard surgery arguments allow us to perform framed surgery on $(\beta_3, \overline{\beta}_3) : X_3 \to \mathbb{C}P^{\infty}$ to kill $H_{u,v}^{\perp}$. We obtain a normal map $(\beta_4, \overline{\beta}_4) : X_4 \to \mathbb{C}P^{\infty}$, with intersection form isomorphic to $(H_{u,v}, \lambda_{X_3}|_{H_{u,v}})$. The manifold

$$N_{a,b} := X_4$$

is the required manifold, as we verify next. We use the orientation corresponding to the fundamental class $[N_{a,b}]$ induced from tracking $[X_0]$ through the construction.

We have already noted at the beginning of the proof that the construction via normally framed surgery implies that $N_{a,b}$ is stably parallelisable. As the map $\beta_4 : N_{a,b} \to \mathbb{C}P^{\infty}$ is $2k$-connected and since there is an isomorphism $\theta : H_{2k}(N_{a,b}) \to H_{u,v} \cong \mathbb{Z}^2$, the manifold $N_{a,b}$ is simply connected and has the correct integral (co)homology groups. By construction of $N_{a,b}$, we can assume that θ is an isometry. To verify that $N_{a,b}$ has the required cohomology ring, we set

$$z := \beta_4^*(z_{\infty}), \quad x := \mathrm{PD}^{-1}(\theta^{-1}(e_1)), \quad y := \mathrm{PD}^{-1}(\theta^{-1}(f_1)).$$

Since $u = ae_1 + bf_1$, it follows that $z^k = ax + by$. Since $\theta^{-1}(e_1), \theta^{-1}(f_1)$ form a standard hyperbolic basis for $(H_{2k}(N_{a,b}), \lambda_{N_{a,b}})$, it follows that xy generates $H^{4k}(N_{a,b})$ and $z^{2k} \cap [N_{a,b}] > 0$. Finally, since z^{k-1} generates $H^{2k-2}(N_{a,b}) \cong \mathbb{Z}$, there is a generator $w \in H^{2k+2}(N_{a,b})$ such that $z^{k-1}w = xy$. The remaining properties of $H^*(N_{a,b})$ follow from Poincaré duality.

Finally, let $\langle z^k\rangle \subseteq H^{2k}(N_{a,b})$ be the subgroup generated by z^k and consider the isomorphism class of the pair $(H^{2k}(N_{a,b}), \langle z^k\rangle)$. This pair, modulo the action of the self-equivalences of $N_{a,b}$ on $H^{2k}(N_{a,b})$, is a homotopy invariant of $N_{a,b}$. Since $z^{2k} \neq 0$ and since $z^{2k} \cap [N_{a,b}] > 0$, every self-homotopy equivalence of $N_{a,b}$ is orientation preserving by Lemma 2.1.

Thus, $\langle z^k\rangle$ modulo the action of $\mathrm{Aut}(H^+(\mathbb{Z}))$ is a homotopy invariant. We claim that the pair $\{a, b\}$ is an invariant of this action. To see this, from the form of the matrix A above, it is easy to see that the automorphisms of the hyperbolic form are

$$\pm\,\mathrm{Id} \text{ and } \pm\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

So automorphisms can change the sign of both a and b simultaneously, and they can switch a and b. Then since we always take $a, b > 0$, the unordered pair of positive integers $\{a, b\}$ is an invariant of the homotopy type. Hence, if there is a homotopy equivalence $N_{a,b} \to N_{a',b'}$, then we have $\{a, b\} = \{a', b'\}$.

Now we address the final statement of the proposition, which concerns stable diffeomorphism. Observe that $\mathbb{Z} \cong H^2(N_{a,b}) \cong H^2(N_{a,b}\#S^{2k} \times S^{2k})$, and that the image of z, which we call $z_{\mathrm{st}} \in H^2(N_{a,b}\#S^{2k} \times S^{2k})$, satisfies $z_{\mathrm{st}}^{2k} = 2ab[N_{a,b}\#S^{2k} \times S^{2k}]$. Since this property of z_{st} and the fundamental class are preserved under diffeomorphism, it follows that if $N_{a,b}$ and $N_{a',b'}$ are stably diffeomorphic, then $ab = a'b'$.

However, for a fixed product $ab = a'b'$, the manifolds $N_{a,b}$ and $N_{a',b'}$ are obtained from the $4k$-manifold X_3 by performing surgery on a stably hyperbolic form $H^{\perp}_{u,v}$. Recall that u and v depend on a, b, so in particular, we may need to stabilise a different number of times for $H^{\perp}_{u,v}$ versus $H^{\perp}_{u',v'}$ to make them hyperbolic. Let $h(u, v)$ and $h(u', v')$ be the number of stabilisations required and let $h := \max\{h(u, v), h(u', v')\}$. Then for some g,

$$N_{a,b}\#W_g \cong X_3\#W_h \cong N_{a',b'}\#W_g,$$

as desired. So indeed, $ab = a'b'$ if and only if $N_{a,b} \cong_{\mathrm{st}} N_{a',b'}$. □

3. $(4m-1)$-connected $8m$-manifolds with nontrivial homotopy stable class

In this section, for every $m \geq 1$, we construct $(4m-1)$-connected $8m$-manifolds with hyperbolic intersection form and with nontrivial homotopy stable class. Specifically, we describe certain $8m$-manifolds $M_{a,b}$, for positive integers a and b, and we give bounds from above and below on the size of the homotopy stable class of $M_{a,b}$ in terms of a, b and m. In particular, for each m, there are infinitely many choices of a, b such that $|\mathcal{S}^{\mathrm{st}}_{\mathrm{h}}(M_{a,b})| > 1$.

In contrast to the manifolds in the previous section, the homotopically inequivalent manifolds constructed here have isomorphic integral cohomology rings, but are not stably parallelisable. We detect that our manifolds are not homotopy equivalent using a refinement of the mth Pontryagin class.

This section is organised as follows. In Section 3.1, we recall some facts about exotic spheres and the J homomorphism, which we need for the statement and the proof of Theorem 3.3. We state this theorem in Section 3.2. In Section 3.3, we recall Wall's classification of $(4m-1)$-connected $8m$-manifolds up to the action of the group of homotopy $8m$-spheres, then in Section 3.4, we determine the stable classification of such manifolds, again up to the action of the homotopy spheres. Next, in Section 3.5, we construct the manifolds $M_{a,b}$ appearing in Theorem 3.3 and we prove this theorem in Section 3.6.

3.1. Exotic spheres and the *J*-homomorphism. Let Θ_n denote the group of h-cobordism classes of *homotopy n-spheres*, that is closed, connected, oriented n-manifolds that are homotopy equivalent to S^n, with the group operation given by connected sum. By [KM63], these are finite abelian groups. We briefly recall some of what is known about them, focussing on dimensions $n = 8m$ and $n = 8m-1$, for $m \geq 1$.

Recall that $bP_{n+1} \subseteq \Theta_n$ is the subgroup of h-cobordism classes of homotopy n-spheres that bound parallelisable $(n+1)$-manifolds. Kervaire and Milnor showed that this is a finite cyclic group and, for $n+1 = 4\ell > 4$, the order of bP_{n+1} is given by a formula in terms of Bernoulli numbers and the image of the J-homomorphism [KM63]. Following results of Adams [Ada66] and Quillen [Qui71] on the J-homomorphism, this formula led to the computation of $|bP_{4\ell}|$; we give more details shortly. The group $bP_{4\ell}$ is generated by the boundary of Milnor's E_8 plumbing [Bro72, V], a 4ℓ-manifold obtained from plumbing disc bundles according to the E_8 lattice.

Let

$$J_n : \pi_n(SO) \to \pi_n^s$$

be the stable J-homomorphism [Whi42, Section 3], where π_n^s is the stable n-stem. Kervaire and Milnor [KM63] showed that $\Theta_{8m} \cong \operatorname{coker} J_{8m}$ and that there is a short exact sequence

$$0 \to bP_{8m} \to \Theta_{8m-1} \to \operatorname{coker} J_{8m-1} \to 0.$$

Later Brumfiel [Bru68] defined a splitting $\Theta_{8m-1} \to bP_{8m}$ and so proved that

$$\Theta_{8m-1} \cong bP_{8m} \oplus \operatorname{coker} J_{8m-1}.$$

Consider a $(4m-1)$-connected $8m$-manifold W with boundary $\partial W \in \Theta_{8m-1}$. Extending work of Stolz [Sto85] and Burklund *et al.* [BHS19], Burklund and Senger [BS20, Theorem 1.2] proved that $[\partial W] \in bP_{8m}$, except possibly when $m = 3$, when they also showed that $2[\partial W] \in bP_{24}$. For our purposes later in this section, we also assume that W has signature 0. This ensures that ∂W is a multiple of the homotopy sphere denoted Σ_Q by Krannich and Reinhold [KR20, Section 2]; see just below Lemma 3.9 for the definition of Σ_Q.

DEFINITION 3.1. Let $\mathfrak{bp}_m$ be the order of Σ_Q in Θ_{8m-1}.

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REMARK 3.2. The precise value of $\mathfrak{bp}_m$ can be calculated, assuming knowledge of the relevant Bernoulli numbers, from [KR20, Lemma 2.7]. In particular, $\mathfrak{bp}_m \mid |bP_{8m}|$. This is clear when $m \neq 3$, since $\Sigma_Q \in bP_{8m}$. It follows from a direct calculation when $m = 3$, given that the projection of Σ_Q to bP_{24} has order divisible by 2.

We now recall some facts about the J-homomorphism for context and later use. We start with the stable J-homomorphism $J_{4m-1} : \pi_{4m-1}(SO) \to \pi^s_{4m-1}$ and write

$$j_m := |\mathrm{Im}(J_{4m-1})|.$$

For example,

$$j_1 = 24, \quad j_2 = 240, \quad \text{and} \quad j_3 = 504.$$

Later, we use the fact that $4 \mid j_m$, for $m = 1, 2$, as we see here. Since the stable homotopy groups of spheres are finite, so is j_m. Since $\pi_{4m-1}(SO) \cong \mathbb{Z}$, in fact $\mathrm{Im}(J_{4m-1}) \cong \mathbb{Z}/j_m$. By [Ada66] (see, for example, [Lüc02, Theorem 6.26]), j_m can be computed using the denominator of the rational number $B_m/4m$, where $B_m \in \mathbb{Q}$ is the mth Bernoulli number, defined by the generating function

$$\frac{e^t}{e^t - 1} = 1 - \frac{t}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} B_n}{(2n)!} t^{2n}.$$

By [KM63, Section 7], $|bP_{8m}|/(2^{4m-2}(2^{4m-1} - 1))$ equals the numerator of the rational number $2B_{2m}/m$, from which one can compute $|bP_{8m}|$.

Next we consider the unstable J-homomorphism, $J_{4m-1,4m} : \pi_{4m-1}(SO_{4m}) \to \pi_{8m-1}(S^{4m})$, which, along with the stable J-homomorphism, the Euler class e and the Hopf-invariant H, fits into the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_{4m-1}(SO_{4m}) & \xrightarrow{\mathrm{e}\oplus S} & \mathbb{Z} \oplus \pi_{4m-1}(SO) & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle J_{4m-1,4m}} & & \downarrow{\scriptstyle \mathrm{Id}\oplus J_{4m-1}} & & \downarrow{\scriptstyle =} & & \\
0 & \longrightarrow & \pi_{8m-1}(S^{4m}) & \xrightarrow{H\oplus S} & \mathbb{Z} \oplus \pi^s_{4m-1} & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0
\end{array} \qquad (*)$$

The commutativity of the left-hand square in (*) is equivalent to the classical statements that $\mathrm{e} = H \circ J_{4m-1,4m}$ and that the J-homomorphism commutes with stabilisation [JW54, 1.2 and 1.3]. That $\mathrm{e} \oplus S$ is injective with index 2 is reviewed in [Wal62, page 171]. That the same statements hold for $H \oplus S$ follows from Toda's calculations in the exceptional cases $m = 1, 2$ [Tod62, V, (iii) and (vii)] and from Adams' solution of the Hopf invariant 1 problem for $m > 2$ [Ada60]. For $m > 2$, both $\mathrm{e}(\pi_{4m-1}(SO_{4m})) \subseteq \mathbb{Z}$ and $H(\pi_{8m-1}(S^{4m})) \subseteq \mathbb{Z}$ are index 2 subgroups and stabilisation is a split surjection, [Ada60, BM58]. In particular, this means that for $m > 2$, the Euler class e is always even for rank $4m$ oriented vector bundles over S^{4m}. When $m = 1, 2$, the maps e and H are both onto and $\mathrm{e} = H \circ J_{4m-1,4m} \equiv S \bmod 2$ [Wal62, page 171] and $H \equiv S \bmod 2$ by Toda's computations mentioned above. These computations show that for $m = 1$, $H \oplus S : \pi_7(S^4) \cong \mathbb{Z} \oplus \mathbb{Z}/12 \to \mathbb{Z} \oplus \mathbb{Z}/24$ sends $(x, y) \mapsto (x, x + 2y)$. For $m = 2$, the map $H \oplus S :$

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$\pi_{15}(S^8) \cong \mathbb{Z} \oplus \mathbb{Z}/120 \to \mathbb{Z} \oplus \mathbb{Z}/240$ is also given by $(x, y) \mapsto (x, x + 2y)$. It follows that $H \equiv S \bmod 2$ as asserted.

3.2. Estimating $\mathcal{S}_{\mathrm{h}}^{\mathrm{st}}(M)$. In this section, we give upper and lower bounds for the homotopy stable class of certain $(4m-1)$-connected $8m$-manifolds. To state these bounds, we require a certain amount of notation.

Let m be a positive integer and let $\{a, b\}$ be a pair of positive integers. Since the dimensions 8 and 16 are exceptional, we introduce the factor

$$c_m := \begin{cases} 2 & m = 1 \text{ or } 2, \\ 1 & m > 2, \end{cases}$$

to handle the exceptional dimensions. We define

$$d := \gcd(a, b)c_m$$

and write

$$ac_m = da' \quad \text{and} \quad bc_m = db'$$

for some coprime a', b'. Set

$$A := a'b' = abc_m^2/d^2.$$

For a positive integer n, we let $\mathcal{P}_n$ be the set of prime factors of n:

$$\mathcal{P}_n := \{p \in \mathbb{N} : p \text{ prime}, \ p \mid n\}.$$

We set $\bar{j}_m = j_m/\gcd(j_m, d)$ and consider the sets $\mathcal{P}_A$, $\mathcal{P}_{\bar{j}_m}$ and their intersection

$$\mathcal{P}_{A,m} := \mathcal{P}_A \cap \mathcal{P}_{\bar{j}_m},$$

the set of primes dividing both $\bar{j}_m$ and A. We define the nonnegative integers

$$q_A := |\mathcal{P}_A| - 1 \text{ and } q_{A,m} := |\mathcal{P}_{A,m}| - 1.$$

Now we can state the main theorem of this section. Its proof occupies the remainder of the section.

THEOREM 3.3. *Let m be a positive integer and let $\{a, b\}$ be a pair of positive integers such that $\mathrm{bp}_m \mid ab$. If $d = \gcd(a, b)$ and $\bar{j}_m = j_m/\gcd(j_m, d)$, then the closed, $(4m-1)$-connected $8m$-manifolds $M_{a,b}$ constructed in Section 3.5 satisfy the following:*

(1) *$M_{a,b}$ has hyperbolic intersection form;*
(2) *$|\mathcal{S}^{\mathrm{st}}(M_{a,b})/\Theta_{8m}| = 2^{q_A}$ and*
(3) *$2^{q_{A,m}} \leq |\mathcal{S}_{\mathrm{h}}^{\mathrm{st}}(M_{a,b})| \leq \lfloor ((\bar{j}_m)^2 + 2\bar{j}_m + 4)/4 \rfloor$.*

Adams' work on j_m [Ada66], a theorem of von Staudt and Clausen (see [IR90, Theorem 3, page 233]) on the denominator of B_m, and a result of von Staudt on the

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numerator of B_m (see [Mil58b, Lemma 2]) combine to show that

$$\mathcal{P}_{j_m} = \{p \text{ prime} : (p-1) \mid 2m\}.$$

Since 2 and 3 certainly lie in the latter set, $|\mathcal{P}_{j_m}| \geq 2$. Now define

$$q_m := |\mathcal{P}_{j_m}| - 1 \geq 1.$$

By choosing a and b with some care, we obtain the following corollary, which implies Theorem 1.2.

COROLLARY 3.4. *Let m be a positive integer and let $\{a, b\}$ be a pair of positive, coprime integers such that $\mathfrak{bp}_m \mid ab$ and $j_m/c_m \mid A = abc_m^2$. Then the closed, $(4m-1)$-connected $8m$-manifolds $M_{a,b}$ constructed in Section 3.5 have hyperbolic intersection form and satisfy that $2 \leq 2^{q_m} \leq |\mathcal{S}^{\mathrm{st}}_{\mathrm{h}}(M_{a,b})|$.*

In particular, any coprime, positive a, b such that $\mathfrak{bp}_m \cdot j_m/c_m$ divides $A = abc_m^2$ satisfy the hypotheses of the corollary. Note that changing A does not alter the lower bound, which is purely a function of m.

PROOF. Since a and b are coprime, $d = c_m$, $\bar{j}_m = j_m/c_m$ and $\mathcal{P}_{\bar{j}_m} = \mathcal{P}_{j_m}$ (using $4 \mid j_m$ for $m = 1, 2$). Since $j_m/c_m = \bar{j}_m \mid A$, we see that $\mathcal{P}_{\bar{j}_m} \subseteq \mathcal{P}_A$ and therefore $\mathcal{P}_{A,m} = \mathcal{P}_{\bar{j}_m} = \mathcal{P}_{j_m}$, so that $q_{A,m} = q_m$. Since $q_m \geq 1$, the corollary follows from the lower bound in Theorem 3.3(3). □

3.3. The almost-diffeomorphism classification of $(4m-1)$-connected $8m$-manifolds. In this section, we recall the relevant part of Wall's classification results for closed, $(4m-1)$-connected $8m$-manifolds. Recall that two closed manifolds M_0 and M_1 are *almost diffeomorphic* if there is a homotopy sphere Σ and a diffeomorphism $f : M_0 \# \Sigma \to M_1$.

Let M be a closed, $(4m-1)$-connected $8m$-manifold, and equip M with an orientation. The *intersection form of M* is a symmetric bilinear form

$$\lambda_M : H_{4m}(M) \times H_{4m}(M) \to \mathbb{Z}.$$

The *obstruction class* of M is the homomorphism

$$S\alpha_M : H_{4m}(M) \to \pi_{4m-1}(SO) \cong \mathbb{Z}$$

defined by representing a homology class x by a smoothly embedded sphere $S^{4m}_x \hookrightarrow M$, whose existence is ensured by Hurewicz theorem and [Hae61, Theorem 1(a)], and then taking the homotopy class of the clutching map of the stable normal bundle of S^{4m}_x. The map $S\alpha_M$ is the stabilisation of a map α_M defined by taking the normal bundle of S^{4m}_x. This is important in the proof of Theorem 3.7 below. As shown by Wall [Wal62, page 171 and Lemma 2], if $m = 1, 2$, then the existence of rank $4m$ vector bundles over S^{4m} with odd Euler class implies that the obstruction class is characteristic for the intersection form; that is, if $m = 1$ or 2, then for all $x \in H_{4m}(M)$,

$$\lambda_M(x, x) \equiv S\alpha_M(x) \bmod 2. \qquad (\dagger)$$

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For $m > 2$, by Wall [Wal62, page 171], there is no relation between $S\alpha_M$ and λ_M. As also shown in [Wal62, page 171 and Lemma 2], since $\mathrm{e} = H \circ J_{4m-1,4m}$ and since for $m > 2$ we have that $H \circ J_{4m-1,4m}$ is even, the Euler number is always even and therefore $\lambda_M(x, x) \equiv 0 \bmod 2$ for all $x \in H_{4m}(M)$.

For the homotopy classification, we consider the stable J-homomorphism

$$J_{4m-1} : \pi_{4m-1}(SO) \to \mathbb{Z}/j_m \subseteq \pi^s_{4m-1}.$$

The *homotopy obstruction class* of M, $S\alpha^{\mathrm{h}}_M$, is the composition of $S\alpha_M$ with J_{4m-1},

$$S\alpha^{\mathrm{h}}_M := J_{4m-1} \circ S\alpha_M : H_{4m}(M) \to \mathbb{Z}/j_m.$$

Since j_1 and j_2 are divisible by 2, the congruence of (†) implies that if $m = 1, 2$, then

$$\lambda_M(x, x) \equiv S\alpha^{\mathrm{h}}_M(x) \bmod 2.$$

We now define the invariants we use to classify $(4m-1)$-connected $8m$-manifolds up to almost diffeomorphism and homotopy equivalence.

DEFINITION 3.5 (Extended symmetric form). Fix a homomorphism $v : G \to \mathbb{Z}/2$ from an abelian group G to $\mathbb{Z}/2$. An *extended symmetric form over* v consists of a triple (H, λ, p) where:

(1) H is a finitely generated free $\mathbb{Z}$-module;
(2) $\lambda : H \times H \to \mathbb{Z}$ is a symmetric, bilinear form and
(3) $f : H \to G$ is a homomorphism such that $\lambda(x, x) \equiv v \circ f(x) \bmod 2$.

Two extended symmetric forms (H, λ, f) and (H', λ', f') are *equivalent* if there is an isometry $h : (H, \lambda) \to (H', \lambda')$ such that $f' \circ h = f : H \to G$.

In our applications to $8m$-manifolds, the group G is either the infinite cyclic group $\pi_{4m-1}(SO) \cong \mathbb{Z}$ or the finite cyclic group $\mathrm{Im}(J_{4m-1}) \cong \mathbb{Z}/j_m$. Due to the existence of rank $4m$ bundles over S^{4m} with odd Euler number when $m = 1, 2$, and the nonexistence of such bundles for $m \geq 3$, we set v to be nonzero for $m = 1, 2$ (recall $2 \mid j_1$ and $2 \mid j_2$) and zero for $m > 2$. Hence for $m > 2$, (3) is just the requirement that λ_M be even. With these conventions on v, the following assignments define extended symmetric forms.

DEFINITION 3.6 (The extended symmetric forms of M). Let M be an oriented $(4m-1)$-connected $8m$-manifold.

(1) The *smooth extended symmetric form* of M is the triple

$$(H_{4m}(M), \lambda_M, S\alpha_M)$$

with $G \cong \mathbb{Z}$.

(2) The *homotopy extended symmetric form* of M is the triple

$$(H_{4m}(M), \lambda_M, S\alpha^{\mathrm{h}}_M)$$

with $G \cong \mathbb{Z}/j_m$.

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The following result is a direct consequence of classification results of Wall [Wal62, page 170 and Lemma 8].

THEOREM 3.7 (Wall). *Let M_1 and M_2 be closed, oriented, $(4m-1)$-connected $8m$-manifolds. The manifolds M_1 and M_2 are:*

(1) *almost diffeomorphic, via an orientation-preserving diffeomorphism, if and only if their smooth extended symmetric forms are equivalent;*
(2) *homotopy equivalent, via a degree one homotopy equivalence, if and only if their homotopy extended symmetric forms are equivalent.*

When applying these classifications, we have to factor out the effect of the orientation choice on the extended symmetric forms.

PROOF. We start with the almost-diffeomorphism classification (1). As mentioned above, the homomorphism $S\alpha_M$ is the stabilisation of a certain quadratic form, the extended quadratic form of M, which is the map

$$\alpha_M : H_{4m}(M) \to \pi_{4m-1}(SO_{4m}),$$

defined by representing a homology class by a smoothly embedded sphere $S^{4m} \hookrightarrow M$, and then taking the classifying map in $\pi_{4m-1}(SO_{4m})$ of the normal bundle of the embedded sphere. For all $x, y \in H_{4m}(M)$, [Wal62, Lemma 2] and the fact that $\mathrm{e} = H \circ J_{4m-1,4m}$ prove that α_M relates to the intersection form of M by the equations

$$\lambda_M(x,x) = \mathrm{e}(\alpha_M(x)) \quad \text{and} \quad \alpha_M(x+y) = \alpha_M(x) + \alpha_M(y) + \lambda(x,y)\tau.$$

Here the map $\mathrm{e} : \pi_{4m-1}(SO_{4m}) \to \mathbb{Z}$ is the Euler number of the corresponding bundle and $\tau \in \pi_{4m-1}(SO_{4m})$ is the clutching function of the tangent bundle of S^{4m}. Wall also proved [Wal62, page 170] that the triple $(H_{4m}(M), \lambda_M, \alpha_M)$ is a complete almost-diffeomorphism invariant of M. In fact, Wall stated his classification in terms of *almost-closed* manifolds: compact manifolds with boundary a homotopy sphere. However, this also yields the almost-diffeomorphism classification, as follows. If the extended symmetric forms of two closed $(4m-1)$-connected $8m$-manifolds are equivalent, then by the almost-closed classification, the manifolds are diffeomorphic after removing a ball D^{8m} from each. Gluing the balls back in compatibly with the diffeomorphism might change one of the manifolds by connected sum with a homotopy sphere, but nonetheless, the two closed manifolds are almost diffeomorphic. However, almost-diffeomorphic manifolds are diffeomorphic after removing a ball from each and then by the classification the extended symmetric forms are equivalent.

As mentioned above, $S\alpha_M := S \circ \alpha_M$, where $S : \pi_{4m-1}(SO_{4m}) \to \pi_{4m-1}(SO)$ is the stabilisation homomorphism. The homotopy exact sequence of the fibration $SO_{4m} \to SO_{4m+1} \to S^{4m}$ shows that the kernel of S is generated by τ [Lev85, Lemma 1.3 and Theorem 1.4] and since

$$\mathrm{e} \oplus S : \pi_{4m-1}(SO_{4m}) \to \mathbb{Z} \oplus \pi_{4m-1}(SO)$$

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is injective by (*), it follows that the pair $(\lambda_M(x,x), S\alpha_M(x)) = (\mathrm{e}(\alpha_M(x)), S(\alpha_M(x))) \in \mathbb{Z} \oplus \pi_{4m-1}(SO) \cong \mathbb{Z} \oplus \mathbb{Z}$ determines $\alpha_M(x)$ for all $x \in H_{4m}(M)$. The theorem now follows from Wall's almost-diffeomorphism classification.

The proof of the homotopy classification is similar. By Wall [Wal62, Lemma 8], the triple $(H_{4m}(M), \lambda_M, \alpha_M^h := J_{4m-1,4m} \circ \alpha_M)$ is a complete homotopy invariant of the manifolds under consideration. Since $\mathrm{e} = HJ$ and

$$H \oplus S : \pi_{8m-1}(S^{4m}) \to \mathbb{Z} \oplus \pi^s_{4m-1}$$

is injective by (*), it follows that the pair $(\lambda_M(x,x), S\alpha^{\mathrm{h}}_M(x)) = (H(\alpha^{\mathrm{h}}_M(x)), S(\alpha^{\mathrm{h}}_M(x))) \in \mathbb{Z} \oplus \pi^s_{4m-1}$ determines $\alpha^{\mathrm{h}}_M(x)$ for all $x \in H_{4m}(M)$. The theorem now follows from Wall's homotopy classification. □

3.4. Stable almost-diffeomorphism classification of $(4m-1)$-connected $8m$-manifolds. In this section, we give the stable classification of closed $(4m-1)$-connected $8m$-manifolds up to connected sum with homotopy $8m$-spheres. Define the nonnegative integer d_M by the equation

$$S\alpha_M(H_{4m}(M)) = d_M\mathbb{Z}.$$

Equivalently, d_M is the divisibility of $S\alpha_M \in H^{4m}(M)$, where, since M is $(4m-1)$-connected, we may regard $S\alpha_M$ as an element of the group $H^{4m}(M)$ via the inverse of the evaluation map $\mathrm{ev} : H^{4m}(M) \to \mathrm{Hom}(H_{4m}(M), \mathbb{Z})$, which is an isomorphism. In particular, it makes sense to consider the class $(S\alpha_M)^2 \in H^{8m}(M) \cong \mathbb{Z}$.

THEOREM 3.8. *Two closed, oriented, $(4m-1)$-connected $8m$-manifolds M and N with the same Euler characteristic are almost-stably diffeomorphic, via an orientation-preserving diffeomorphism, if and only if the following hold:*

(1) $d_M = d_N$;
(2) $\sigma(M) = \sigma(N)$;
(3) $\langle (S\alpha_M)^2, [M]\rangle = \langle (S\alpha_N)^2, [N]\rangle$.

PROOF. First, we note that d_M, the signature and $(S\alpha_M)^2$ are invariants of orientation-preserving almost-stable diffeomorphisms, so one implication holds.

For the other implication, we assume that M and N are such that $d_M = d_N$, $\sigma(M) = \sigma(N)$, and $(S\alpha_M)^2 = (S\alpha_N)^2$ and we show that M and N are stably diffeomorphic. The normal $(4m-1)$-type of M and N is determined by $d = d_M = d_N$ and is described as follows. Let d be a nonnegative integer. Let $BO\langle 4m-1\rangle \to BO$ be the $(4m-1)$-connected cover of BO and let $p \in H^{4m}(BO\langle 4m-1\rangle) \cong \mathbb{Z}$ be a generator. We regard $\rho_d(p)$, the mod d reduction of p, as a map $\rho_d(p) : BO\langle 4m-1\rangle \to K(\mathbb{Z}/d, 4m)$ and define $BO\langle 4m-1, d_M\rangle$ to be the homotopy fibre of $\rho_d(p)$. The normal $(4m-1)$-type of M and N is represented by the fibration given by the composition

$$BO\langle 4m-1, d\rangle \to BO\langle 4m-1\rangle \to BO.$$

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For brevity, use (B_d, η_d) to denote the fibration $\eta_d : BO\langle 4m-1, d\rangle \to BO$. We assert that M and N admit unique normal $(4m-1)$-smoothings $\overline{\nu}_M : M \to B_d$ and $\overline{\nu}_N : N \to B_d$. We prove the assertion for M, as the proof for N is identical. Since M is $(4m-1)$-connected, its stable normal bundle $\nu_M : M \to BO$ lifts (up to homotopy) uniquely to $\nu_{4m-1} : M \to BO\langle 4m-1\rangle$. To lift ν_{4m-1} to B_d, we consider the long exact sequence (of pointed sets) of the fibration

$$0 = H^{4m-1}(M; \mathbb{Z}/d) \to [M, B_d] \to [M, BO\langle 4m-1\rangle] \to H^{4m}(M; \mathbb{Z}/d) \to \cdots,$$

where on the left, we use $[M, \Omega K(\mathbb{Z}/d, 4m)] = [M, K(\mathbb{Z}/d, 4m-1)] = H^{4m-1}(M; \mathbb{Z}/d) = 0$, because M is $(4m-1)$-connected. The assertion is now proved by noting that $\nu_{4m-1} \in [M, BO\langle 4m-1\rangle]$ maps to $S\alpha_M \in H^{4m}(M; \mathbb{Z}/d)$, which is zero by definition of the divisibility d_M.

By [Kre99, Theorem 2], M and N are orientation preserving stably diffeomorphic if

$$[M, \overline{\nu}_M] = [N, \overline{\nu}_N] \in \Omega_{8m}(B_d, \eta_d).$$

Since homotopy $8m$-spheres have a unique (B_d, η_d)-structure, there is a well-defined homomorphism $\Theta_{8m} \to \Omega_{8m}(B_d, \eta_d)$. Now the arguments in Wall's computation of the Grothendieck groups of almost-closed $(4m-1)$-connected $8m$-manifolds [Wal62, Theorem 2] show that there is an exact sequence

$$\Theta_{8m} \to \Omega_{8m}(B_d, \eta_d) \xrightarrow{(\sigma, (S\alpha)^2)} \mathbb{Z}^2, \qquad (\Omega)$$

where $\sigma([M, \overline{\nu}_M]) = \sigma_M$ and $S\alpha^2([M, \overline{\nu}_M]) = (S\alpha_M)^2([M])$. It follows that there is a homotopy $8m$-sphere Σ such that $[M\#\Sigma, \overline{\nu}_{M\#\Sigma}] = [N, \overline{\nu}_N] \in \Omega_{8m}(B_d, \eta)$. Hence, $M\#\Sigma$ and N are stably diffeomorphic and so M and N are almost-stably diffeomorphic. □

3.5. Construction of the manifolds $M_{a,b}$. In this section, we construct the manifolds $M_{a,b}$ appearing in Theorem 3.3. Let a and b be positive integers such that $\mathfrak{bp}_m \mid ab$. We build simply connected, closed $8m$-manifolds $M_{a,b}$ with the cohomology ring of $S^{4m} \times S^{4m}$ by attaching handles to an $8m$-ball. We attach to D^{8m} two $4m$-handles h_x and h_y, diffeomorphic to $D^{4m} \times D^{4m}$, using attaching maps $\phi_x, \phi_y : S^{4m-1} \times \{0\} \to S^{8m-1}$ with linking number 1. Note that for $m \geq 1$, 2-component links $S^{4m-1} \sqcup S^{4m-1} \hookrightarrow S^{8m-1}$ are classified up to smooth isotopy by the linking number, an integer [Hae62, Theorem in Section 5]. There is more data needed for the attaching maps, which for each $4m$-handle corresponds to a choice of framing for the attaching sphere $S^{4m-1} \subseteq S^{8m-1}$. The framings that induce a given orientation are in one to one correspondence with homotopy classes of maps $[S^{4m-1}, SO_{4m}]$, where the class of the constant map corresponds to the framing that extends over an embedded $4m$-disc $D^{4m} \subseteq S^{8m-1}$. Recall from (*) that $\pi_{4m-1}(SO_{4m}) \cong \mathbb{Z} \oplus \mathbb{Z}$, detected by $e \oplus S$ (although this map is not an isomorphism). We are attaching $4m$-handles h_x h_y; let x and y denote the corresponding classes in $(4m)$th homology and let $\xi_x, \xi_y \in \pi_{4m-1}(SO_{4m})$ be the framings for the attaching maps.

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Since we want $\lambda(x,x)=0$, we require that $\mathrm{e}(\xi_x)=0$ but we are otherwise free to choose ξ_x. Recall that $c_m=2$ if $m=1,2$ and $c_m=1$ if $m>2$, fix an isomorphism $\pi_{4m-1}(SO)=\mathbb{Z}$ and choose ξ_x such that $S(\xi_x)=ac_m$. By the discussion following (*), we can find such a ξ_x for any choice of a. Similarly, we attach the handle h_y with $\mathrm{e}(\xi_y)=0$ and $S(\xi_y)=bc_m$. Again, we can find such a ξ_y for any b. After attaching the pair of $4m$-handles, we write $W:=W_{a,b}$ for the resulting compact $8m$-manifold with boundary. Note that there is a homotopy equivalence $W\simeq S^{4m}\vee S^{4m}$. As above, let x and y be generators of $\mathbb{Z}^2\cong H_{4m}(W)$ and let $\{x^*,y^*\}$ be the dual basis for $\mathbb{Z}^2\cong H^{4m}(W)=H_{4m}(W)^*$. The manifold $W=W_{a,b}$ has smooth extended symmetric form given by

$$(H_{4m}(W),\lambda_W,S\alpha_W)=\left(\mathbb{Z}^2,\begin{pmatrix}0&1\\1&0\end{pmatrix},\begin{pmatrix}ac_m\\bc_m\end{pmatrix}:\mathbb{Z}^2\to\mathbb{Z}\right),$$

where the notation for $S\alpha_W$ means that $S\alpha_W(x)=ac_m$ and $S\alpha_W(y)=bc_m$.

Alternatively, the construction thus far can be achieved by taking the two D^{4m}-bundles over S^{4m} determined by ξ_x and ξ_y, and plumbing them together once.

The boundary of $W_{a,b}$ is a homotopy $(8m-1)$-sphere, which we denote by $\Sigma_{a,b}$. In particular, $\partial W_{1,1}$ is by definition the homotopy sphere Σ_Q from [KR20, Section 2]. More generally, $\partial W_{a,b}$ is given as follows.

Lemma 3.9. $[\partial W_{a,b}]=[ab\Sigma_Q]\in\Theta_{8m-1}$.

Proof. Recall from [Wal67, Section 17] the group $A_{8m}^{\langle 4m\rangle}$ of bordism classes of $(4m-1)$-connected $8m$-manifolds with boundary a homotopy sphere, where the bordisms are required to be h-cobordisms on the boundary. Addition is via boundary connected sum. Taking the boundary defines a homomorphism $A_{8m}^{\langle 4m\rangle}\to\Theta_{8m-1}$, and the characteristic numbers σ and $(S\alpha)^2$ of (Ω) are also well defined on $A_{8m}^{\langle 4m\rangle}$. Indeed, Wall [Wal62, Theorems 2 and 3] proved that $\sigma\oplus(S\alpha)^2:A_{8m}^{\langle 4m\rangle}\to\mathbb{Z}^2$ is an injective homomorphism. Since $W_{a,b}$ satisfies $\sigma(W_{a,b})=0$ and $(S\alpha_{W_{a,b}})^2=2abc_m^2$, we have that $(S\alpha_{W_{1,1}})^2=2c_m^2$ and so

$$(S\alpha_{W_{a,b}})^2=2abc_m^2=ab(S\alpha_{W_{1,1}})^2=(S\alpha_{\natural^{ab}W_{1,1}})^2,$$

where the last equality used that $(S\alpha)^2$ is a homomorphism. Since $\sigma\oplus(S\alpha)^2$ is injective, $W_{a,b}=\natural^{ab}W_{1,1}\in A_{8m}^{\langle 4m\rangle}$. So $\partial W_{a,b}$ and $\partial(\natural^{ab}W_{1,1})=ab\Sigma_Q$ are h-cobordant and therefore diffeomorphic. □

From Lemma 3.9 and our assumption that $\mathfrak{bp}_m\mid ab$, it follows that $[\Sigma_{a,b}]=0\in\Theta_{8m-1}$, so that there is a choice of diffeomorphism $f:\Sigma_{a,b}\to S^{8m-1}$. We write $M_{a,b,f}=W_{a,b}\bigcup_f D^{8m}$ for the closure of $W_{a,b}$ built using a diffeomorphism $f:\Sigma_{a,b}\to S^{8m-1}$. We also use $M_{a,b}$ to ambiguously denote any $M_{a,b,f}$. For any other choice of diffeomorphism f', $M_{a,b,f}$ and $M_{a,b,f'}$ are almost diffeomorphic.

Let us record the values of the key invariants on $M_{a,b}$. The stable almost-diffeomorphism invariants of $M_{a,b}$ are $d_{M_{a,b}}=\gcd(a,b)c_m$, $\sigma(M_{a,b})=0$ and $(S\alpha_{M_{a,b}})^2=$

$2abc_m^2$. The extended symmetric form of $M := M_{a,b}$ is the same as that of W:

$$(H_{4m}(M), \lambda_M, S\alpha_M) = \left(\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} ac_m \\ bc_m \end{pmatrix} : \mathbb{Z}^2 \to \mathbb{Z}\right).$$

This completes the construction of the manifolds $M_{a,b}$.

3.6. The proof of Theorem 3.3. Now that we have constructed the $(4m-1)$-connected $8m$-manifolds $M_{a,b}$, we are ready to prove Theorem 3.3.

Proof of Theorem 3.3. Let a and b be positive integers such that $\mathfrak{bp}_m \mid ab$. By construction, the oriented manifolds $M_{a,b}$ have hyperbolic intersection form, so Theorem 3.3(1) is immediate.

As before, write $d := \gcd(a, b)c_m$ and define $A := abc_m^2/d^2$. Let $p_1, \ldots, p_{q_A+1}$ be the prime-power factors of A, which are powers of pairwise distinct primes. Then there are 2^{q_A} ways to express A as a product y_iz_i of coprime positive integers, counting unordered pairs $\{y_i, z_i\}$. We consider the $8m$-manifolds

$$\{M_i := M_{dy_i,dz_i}\}_{i=1}^{2^{q_A}}.$$

For each i, $d_{M_i} = d$, $\sigma(M_i) = 0$ and $\langle (S\alpha_{M_i})^2, [M_i]\rangle = 2dy_idz_i = 2d^2A = 2abc_m^2$. Therefore, the manifolds M_i are pairwise almost-stably diffeomorphic by Theorem 3.8. *A priori* they cannot all lie in $\mathcal{S}^{\mathrm{st}}(M_{a,b})$, but the ambiguity of whether they are actually stably diffeomorphic is removed by more carefully choosing the diffeomorphisms $f_i : \Sigma_i \to S^{8m-1}$ used to glue on D^{8m} in the construction of the M_i. By changing the choice of the identification f_i, we change M_i by connected sum with an exotic sphere of our choice. The M_i are only determined up to this choice in our construction, so let us assume we made this consistently so that $M_i \in \mathcal{S}^{\mathrm{st}}(M_{a,b})$ for every $i = 1, \ldots, 2^{q_A}$. In $\mathcal{S}^{\mathrm{st}}(M_{a,b})/\Theta_{8m}$, this choice of the f_i is in any case irrelevant.

When we discuss extended symmetric forms on $\mathbb{Z}^2$, we always mean with respect to a particular choice of basis. For M_i, with its fixed choice of fundamental class $[M_i]$, we use a basis with respect to which the intersection form is represented by $H^+(\mathbb{Z}) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. We have constructed the M_i so that, with respect to such a basis, $S\alpha_{M_i} : \mathbb{Z}^2 \to \mathbb{Z}$ is represented by $\left(\begin{smallmatrix} ac_m \\ bc_m \end{smallmatrix}\right)$ with $ab > 0$ and $c_m \in \{1, 2\}$.

The smooth extended symmetric forms of the M_i are pairwise distinct, since isometries of the rank two hyperbolic intersection form can only change the sign and permute the basis elements. The map $S\alpha_{M_i} : \mathbb{Z}^2 \to \mathbb{Z}$ is given by (dy_i, dz_i). Since the unordered pairs $\{dy_i, dz_i\}$ are pairwise distinct, by the almost-diffeomorphism classification of Theorem 3.7(1), the M_i are pairwise distinct up to orientation-preserving almost diffeomorphism. We are able to deduce that $|\mathcal{S}^{\mathrm{st}}(M_{a,b})/\Theta_{8m}| \geq 2^{q_A}$ once we have factored out the effect of the choice of orientation of the M_i. In other words, we must show that there are also no orientation-reversing almost diffeomorphisms from M_i to M_j, for $i \neq j$, or equivalently that there is no orientation-preserving diffeomorphism $M_i \cong -M_j$.

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Changing the orientation of M_j changes the smooth extended symmetric form, with respect to the same basis for $H_{4m}(M)$, by altering the sign of the intersection form, but does not affect $S\alpha_{M_j}$. To see this, note that while changing the orientation of M_j changes the induced orientation of the fibres of the normal bundle of an embedded sphere x, $S\alpha_{M_j}(x) \in \pi_{4m-1}(SO)$ is the clutching map of this normal bundle, and this is unaffected by the orientation of the fibres.

The isometries from the rank two hyperbolic form $H^+(\mathbb{Z}) = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ to its negative $-H^+(\mathbb{Z})$ consist of the self-isometries of the hyperbolic form, namely $\pm\,\mathrm{Id}$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, composed with either $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Thus an orientation-reversing almost diffeomorphism could identify the smooth extended symmetric form characterised by $(H^+(\mathbb{Z}), \pm\{v, w\})$ with one of the extended symmetric forms $(-H^+(\mathbb{Z}), \pm\{-v, w\})$ or $(-H^+(\mathbb{Z}), \pm\{v, -w\})$. However, for both M_i and $-M_i$, the corresponding pair of integers is $\pm\{v, w\} = \pm\{dy_i, dz_i\}$, where both elements have the same sign. So our manifolds $\{M_i\}$ are indeed distinct up to almost diffeomorphism. This proves that $|\mathcal{S}^{\mathrm{st}}(M_{a,b})/\Theta_{8m}| \geq 2^{q_A}$.

Next we prove that $|\mathcal{S}^{\mathrm{st}}(M_{a,b})/\Theta_{8m}| \leq 2^{q_A}$. Any closed $8m$-manifold M that is almost-stably diffeomorphic to $M_{a,b}$ is also necessarily $(4m-1)$-connected, the divisibility of $S\alpha_M$ is d and the intersection form is rank two, indefinite and even, and therefore either hyperbolic or $-H^+(\mathbb{Z})$. If M and $M_{a,b}$ are almost-stably diffeomorphic, then there is an orientation on M such that M and $M_{a,b}$ are almost-stably diffeomorphic via an orientation-preserving stable diffeomorphism. Use this orientation and choose a basis for $H_{4m}(M)$ with respect to which the intersection form of M is $H^+(\mathbb{Z})$. Observe that the manifolds M_i cover all possibilities for $S\alpha_M$ while keeping $(S\alpha_M)^2$ a fixed multiple of the dual fundamental class. (If $S\alpha_M = \left(\begin{smallmatrix} -ac_m \\ bc_m \end{smallmatrix}\right)$, for example, then $(S\alpha_M)^2 = -2abc_m^2 < 0$, whereas $(S\alpha_{M_{a,b}})^2 = 2abc_m^2 > 0$. This would contradict that $M_{a,b}$ and M are orientation-preserving almost-stably diffeomorphic.) It follows by Theorem 3.7(1) that every such M is almost-stably diffeomorphic to one of the M_i, and therefore $|\mathcal{S}^{\mathrm{st}}(M_{a,b})/\Theta_{8m}| \leq 2^{q_A}$ as desired. This completes the proof of Theorem 3.3(2).

To prove (3), we need to estimate the size of the homotopy stable class of $M_{a,b}$ from above and below. We begin with the upper bound. As above, every closed $8m$-manifold M stably diffeomorphic to $M_{a,b}$ has an orientation such that M has hyperbolic intersection form and $d_M = d$. The possibilities for $S\alpha_M^h$, up to equivalence of extended symmetric forms, are therefore given by an unordered pair of elements of $\mathbb{Z}/j_m$, both of which are divisible by d. Such an element of $\mathbb{Z}/j_m$ lies in the subgroup generated by $\gcd(j_m, d)$, and so there are $\bar{j}_m = j_m/\gcd(j_m, d)$ possibilities. We assert that there are

$$\frac{\bar{j}_m(\bar{j}_m + 1)}{2}$$

such pairs. To see this, there are $\binom{\bar{j}_m}{2} = \bar{j}_m(\bar{j}_m - 1)/2$ choices with distinct elements (x, y), and $\bar{j}_m$ choices of the form (x, x). Then $\bar{j}_m(\bar{j}_m - 1)/2 + \bar{j}_m = \bar{j}_m(\bar{j}_m + 1)/2$, which

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is the count asserted. Next, we also factor out the action of $\mathbb{Z}/2$ on our set of unordered pairs which multiplies both numbers by -1. In the case that $\bar{j}_m$ is even, there are $\bar{j}_m/2+1$ fixed points of this action of the form $(x,-x)$, and also $(0,\bar{j}_m/2)$ is a fixed point. Thus, there are precisely $\bar{j}_m/2+2$ fixed points of the $\mathbb{Z}/2$ action on the set with $\bar{j}_m(\bar{j}_m+1)/2$ elements. A short calculation then shows that there are

$$\frac{\bar{j}_m^2+2\bar{j}_m+4}{4}$$

orbits. A similar calculation for $\bar{j}_m$ odd gives

$$\frac{(\bar{j}_m+1)^2}{4}=\left\lfloor\frac{\bar{j}_m^2+2\bar{j}_m+4}{4}\right\rfloor$$

orbits. The right-hand side is equal for both parities of $\bar{j}_m$ and gives our desired upper bound. Note that this upper bound does not take into account the requirement for the product ab to be constant within a stable diffeomorphism class.

It remains to prove that $2^{q_{A,m}}\leq|\mathcal{S}^{\mathrm{st}}_{\mathrm{h}}(M_{a,b})|$. As above, let $p_1,\dots,p_{q_A+1}$ be the prime-power factors of A, which are powers of pairwise distinct primes. By reordering if necessary, assume that $p_1,\dots,p_{q_{A,m}+1}$ are the prime-powers of the form p^ℓ, where $p\mid\bar{j}_m$. (It could be that the highest exponent of p that divides $\bar{j}_m$ is less than the highest exponent of p that divides A.) Recall that $d=\gcd(a,b)c_m$ and write

$$d':=d\cdot\prod_{\iota=q_{A,m}+2}^{q_A+1}p_\iota \text{ and } A':=\prod_{\iota=1}^{q_{A,m}+1}p_\iota.$$

Note that $d'A'=dA$. There are $2^{q_{A,m}}$ essentially distinct ways to express A' as a product v_iw_i of coprime positive integers, counting unordered pairs $\{v_i,w_i\}$. We consider the $8m$-manifolds

$$\{M_i:=M_{dv_i,d'w_i}\}_{i=1}^{2^{q_{A,m}}}.$$

For each i, $d_{M_i}=d$, $\sigma(M_i)=0$ and $\langle(S\alpha_{M_i})^2,[M_i]\rangle=2dv_id'w_i=2dd'A'=2d^2A=2abc_m^2$. Therefore, the M_i are pairwise almost-stably diffeomorphic by Theorem 3.8; so, up to homotopy equivalence, they are all stably diffeomorphic. As above, the ambiguity of whether they are actually stably diffeomorphic is removed by more carefully choosing the diffeomorphisms $f_i:\Sigma\to S^{8m-1}$ used to glue on D^{8m} in the construction of the M_i. Let us assume once again that we made this choice consistently so that $M_i\in\mathcal{S}^{\mathrm{st}}_{\mathrm{h}}(M_{a,b})$ for every $i=1,\dots,2^{q_{A,m}}$.

Next we show that the M_i are distinct up to homotopy equivalence. For this, by Theorem 3.7, we need to distinguish their homotopy extended symmetric forms, by showing that the maps $S\alpha^h_{M_i}:\mathbb{Z}^2\to\mathbb{Z}/j_m$ are pairwise distinct, up to precomposing with an isometry of the hyperbolic form, or to allow for the possibility of an orientation-reversing homotopy equivalence, up to an isometry between the hyperbolic form and its negative. This means we have to show that the unordered pair of elements of $\mathbb{Z}/j_m$ determining $S\alpha^h_{M_i}$ and $S\alpha^h_{M_j}$ are distinct up to changing signs.

Let M_i and M_j be two of our manifolds, for $i \neq j$. We show that they are not homotopy equivalent. First, $\gcd(d, j_m)$ divides d, so divides dv_i and $d'w_i$. As above, write $\bar{j}_m := j_m/\gcd(d, j_m)$. The map $S\alpha^h_{M_i} : \mathbb{Z}^2 \to \mathbb{Z}/j_m$ factors as

$$S\alpha^h_{M_i} : \mathbb{Z}^2 \to \mathbb{Z}/\bar{j}_m \to \mathbb{Z}/j_m$$

for all i, where $\mathbb{Z}/\bar{j}_m \to \mathbb{Z}/j_m$ is the standard inclusion sending 1 to $\gcd(d, j_m)$. Define

$$\bar{d} := \frac{d}{\gcd(d, j_m)} \quad \text{and} \quad \bar{d}' := \frac{d'}{\gcd(d, j_m)} = \bar{d} \cdot \prod_{\iota = q_{A,m}+2}^{q_A+1} p_\iota.$$

We obtain

$$S\bar{\alpha}^h_{M_i} = \begin{pmatrix} \bar{d}v_i \\ \bar{d}' w_i \end{pmatrix} : \mathbb{Z}^2 \to \mathbb{Z}/\bar{j}_m.$$

It suffices to prove that for $i \neq j$, the resulting pairs $\{\bar{d}v_i, \bar{d}' w_i\}$ and $\{\bar{d}v_j, \bar{d}' w_j\}$ are distinct, up to signs and switching the orders. Note that $\gcd(\bar{d}, \bar{j}_m) = 1 = \gcd(\bar{d}', \bar{j}_m)$.

Let p be a prime dividing A'. Up to possibly changing the orders of v_i and w_i, and of v_j and w_j, assume that p divides v_i and v_j. If so, p does not divide w_i and w_j, since $\gcd(v_i, w_i) = 1 = \gcd(v_j, w_j)$.

Now let $q \neq p$ be a prime dividing A' such that either:

(i) q divides w_j but q does not divide w_i or
(ii) q divides w_i but q does not divide w_j.

Without loss of generality suppose that (i) holds. Then also q divides v_i but q does not divide v_j, since both pairs (v_i, w_i) and (v_j, w_j) are coprime. There exists such a q, unless $q_{A,m} = 0$, in which case $2^{q_{A,m}} = 1$ and we have nothing to prove anyway. So we can assume that $q_{A,m}$ is positive and that such a q exists. The idea is that the primes p and q are chosen so that they divide the same element of the unordered pair associated with the homotopy extended symmetric form for M_i, but divide different elements of the unordered pair for M_j. It is this distinction we want to detect.

We consider the images of the four elements $\bar{d}v_i$, $\bar{d}' w_i$, $\bar{d}v_j$ and $\bar{d}' w_j$ of $\mathbb{Z}/\bar{j}_m$ under the canonical surjections

$$\rho_p : \mathbb{Z}/\bar{j}_m \to \mathbb{Z}/p \quad \text{and} \quad \rho_q : \mathbb{Z}/\bar{j}_m \to \mathbb{Z}/q.$$

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Since p and q divide $\bar{j}_m$ and $\gcd(\overline{d},\bar{j}_m) = 1 = \gcd(\overline{d}',\bar{j}_m)$, we know that p and q do not divide $\overline{d}$ and do not divide $\overline{d}'$. Therefore, for the $\mathbb{Z}/p$ reductions, we have

$$\rho_p(\overline{d}v_i) = 0, \quad \rho_p(\overline{d}'w_i) \neq 0, \quad \rho_p(\overline{d}v_j) = 0 \quad \text{and} \quad \rho_p(\overline{d}'w_j) \neq 0,$$

while for the $\mathbb{Z}/q$ reductions, we have

$$\rho_q(\overline{d}v_i) = 0, \quad \rho_q(\overline{d}'w_i) \neq 0, \quad \rho_q(\overline{d}v_j) \neq 0 \quad \text{and} \quad \rho_q(\overline{d}'w_j) = 0.$$

We indicate one of these calculations briefly, that $\rho_p(\overline{d}'w_i) \neq 0$, to give the idea. If $\overline{d}'w_i$ are 0 modulo p, then for some $a, b \in \mathbb{Z}$, we would have $ap = \overline{d}'w_i + b\bar{j}_m \in \mathbb{Z}$. However, $p \mid \bar{j}_m$ and so p divides $\overline{d}'w_i$, which is a contradiction.

Note that switching the sign of an element in $\mathbb{Z}/\bar{j}_m$ preserves whether or not its image under ρ_p or ρ_q is zero. Let us summarise the calculations above. For $\{\overline{d}v_i, \overline{d}'w_i\}$, one element is zero under the reductions modulo p and q, while the other element is nonzero under both reductions. However, for the pair $\{\overline{d}v_j, \overline{d}'w_j\}$, we have shown that precisely one element is zero under each of the modulo p and modulo q reductions. Switching the orders of the elements and switching signs preserves these descriptions, and therefore M_i and M_j are not homotopy equivalent. It follows that $|\mathcal{S}^{\mathrm{st}}_{\mathrm{h}}(M_{a,b})|$ is at least $2^{q_{A,m}}$, as desired. □ □

4. Stably equivalent spin^c structures on 4-manifolds

As explained in the introduction, the homotopy stable class is trivial for every closed, simply connected 4-manifold. However, a parallel phenomenon occurs when one considers equivalence classes of spin^c structures on the tangent bundle. In this section, we illustrate this on $S^2 \times S^2$.

For all $n \geq 2$, the group Spin_n is the connected double cover of SO_n and the group $\mathbb{Z}/2$ acts by deck transformations. The group $\mathbb{Z}/2$ acts on $U_1 \cong S^1$ by $\{\pm 1\}$. We quotient out the diagonal action on the product to obtain:

$$\mathrm{Spin}^c_n := U_1 \times_{\mathbb{Z}/2} \mathrm{Spin}_n$$

There are well-defined maps

$$U_1 \xleftarrow{\mathrm{pr}_1} \mathrm{Spin}^c_n \xrightarrow{\mathrm{pr}_2} SO_n,$$

which are defined by $\mathrm{pr}_1([\lambda, A]) = \lambda$ and $\mathrm{pr}_2([\lambda, A]) = A$, where $(\lambda, A) \in U_1 \times \mathrm{Spin}_n$.

There are natural inclusions $\mathrm{Spin}^c_n \hookrightarrow \mathrm{Spin}^c_{n+1}$ and the stable spin^c group is defined by $\mathrm{Spin}^c := \mathrm{colim}_{n\to\infty} \mathrm{Spin}^c_n$. There are also stable projections

$$U_1 \xleftarrow{\mathrm{pr}_1} \mathrm{Spin}^c \xrightarrow{\mathrm{pr}_2} SO,$$

where SO is the stable special orthogonal group. We use the same notation $\mathrm{pr}_1, \mathrm{pr}_2$ for the induced maps on classifying spaces.

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DEFINITION 4.1. Let M be a closed, oriented n-manifold. A *spin^c structure* on M is a lift

$$\begin{array}{ccc} & & B\,\mathrm{Spin}^c \\ & \overset{\mathfrak{s}}{\nearrow} & \downarrow \mathrm{pr}_2 \\ M & \xrightarrow{\tau_M} & BSO \end{array}$$

of the stable tangent bundle's classifying map to $B\,\mathrm{Spin}^c$.

For more background on spin^c structures on 4-manifolds, we refer to, for example, [GS99, Section 2.4.1] and [Sco05, Sections 10.2 and 10.7].

LEMMA 4.2 [GS99, Proposition 2.4.16]. *Every oriented 4-manifold admits a spin^c structure.*

PROOF. In [GS99], spin^c structures on 4-manifolds are defined by using $B\,\mathrm{Spin}^c_4$ in place of $B\,\mathrm{Spin}^c$, and [GS99, Proposition 2.4.16] proves the existence of a lift of the classifying map of the (unstable) tangent bundle to $B\,\mathrm{Spin}^c_4$. Composing with the maps to the colimit, this implies the existence of a spin^c-structure in the sense of Definition 4.1. □

DEFINITION 4.3 (Equivalence of spin^c structures). Let M be a closed, oriented 4-manifold.

(1) Two spin^c structures $\mathfrak{s}_1$ and $\mathfrak{s}_2$ on M are *equivalent* if there is an orientation-preserving diffeomorphism $f : M \to M$ such that $\mathfrak{s}_1, \mathfrak{s}_2 \circ f : M \to B\,\mathrm{Spin}^c$ are homotopic over BSO; that is, there is homotopy K and a commutative diagram

$$\begin{array}{ccc} & & B\,\mathrm{Spin}^c\ , \\ & \overset{K}{\nearrow} & \downarrow \mathrm{pr}_2 \\ M \times I & \xrightarrow{\tau_{M\times I}} & BSO \end{array}$$

where K restricts to $\mathfrak{s}_1$ on $M \times \{0\}$ and $\mathfrak{s}_2 \circ f$ on $M \times \{1\}$.

(2) Two spin^c structures $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are *homotopic* if they are equivalent as in the previous item, with $f = \mathrm{Id}_M$.

Recall that the projection onto the first component gives a compatible collection of maps $\mathrm{pr}_1 : B\,\mathrm{Spin}^c_n \to BU_1$, $n \in \mathbb{N}$. Therefore, passing to the colimit and keeping the same notation, we obtain a map $\mathrm{pr}_1 : B\,\mathrm{Spin}^c \to BU_1$.

DEFINITION 4.4. Via the map $\mathrm{pr}_1 : B\,\mathrm{Spin}^c \to BU_1$, a spin^c structure $\mathfrak{s} : M \to B\,\mathrm{Spin}^c$ on a 4-manifold M determines a line bundle $\mathcal{L}_{\mathfrak{s}}$. The *first Chern class* of $\mathfrak{s}$ is defined by

$$c_1(\mathfrak{s}) := c_1(\mathcal{L}_{\mathfrak{s}}) \in H^2(M).$$

Noting that BU_1 is a $K(\mathbb{Z}, 2)$, $c_1(\mathfrak{s})$ corresponds to $\mathrm{pr}_1 \circ \mathfrak{s} : M \to BU_1$ under the isomorphism $H^2(M) \cong [M, BU_1]$. The map pr_1 can be interpreted as a determinant and

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$\mathcal{L}_{\mathfrak{s}}$ is called the *determinant line bundle of* $\mathfrak{s}$. The next lemma follows from [GS99, Proposition 2.4.16].

LEMMA 4.5. *Let M be a closed, oriented 4-manifold.*

(i) *For every spin^c structure $\mathfrak{s}$ on M, reduction modulo 2 is such that:*

$$\begin{aligned} H^2(M) &\to H^2(M;\mathbb{Z}/2) \\ c_1(\mathfrak{s}) &\mapsto w_2(M), \end{aligned}$$

where $w_2(M)$ is the second Stiefel–Whitney class.

(ii) *There is a transitive action of $H^2(M)$ on the set of homotopy classes of spin^c structures on M, such that for $x \in H^2(M)$,*

$$c_1(x \cdot \mathfrak{s}) = c_1(\mathfrak{s}) + 2x \in H^2(M).$$

(iii) *If $H_1(M)$ is 2-torsion free, then this action is free.*

PROOF. As mentioned during the proof of Lemma 4.2, in [GS99], spin^c structures are defined by using $B\,\mathrm{Spin}_4^c$ in place of $B\,\mathrm{Spin}^c$ and therefore the Chern class of a spin^c-structure is defined using $\mathrm{pr}_1 : B\,\mathrm{Spin}_4^c \to BU_1$. However, since the map $B\,\mathrm{Spin}_4^c \to BU_1$ factors through $B\,\mathrm{Spin}^c$, both definitions of the Chern class coincide and so the lemma follows from [GS99, Proposition 2.4.16]. □

As a consequence, every characteristic cohomology class $y \in H^2(M)$ can be realised as the first Chern class of some spin^c structure on M, and if $H_1(M)$ is 2-torsion free, then this spin^c structure is uniquely determined by y. Here recall that y being *characteristic* means that $\langle x \cup x, [M]\rangle \equiv \langle x \cup y, [M]\rangle \bmod 2$ for every $x \in H^2(M)$.

The next lemma is immediate from the fact that the Chern class is an invariant of a spin^c structure, and is natural.

LEMMA 4.6. *If two spin^c structures $\mathfrak{s}_1$ and $\mathfrak{s}_2$ on a closed, oriented 4-manifold M are equivalent, then there is an isometry of the intersection form on $H^2(M)$ sending $c_1(\mathfrak{s}_1)$ to $c_1(\mathfrak{s}_2)$.* □

To define stable equivalence of spin^c structures, fix once and for all the preferred spin^c structure $\mathfrak{s}_g$ on $W_g := \#^g S^2 \times S^2$, to be the spin^c structure with $c_1(\mathfrak{s}_g) = 0 \in H^2(W_g)$. Such a spin^c structure exists by Lemma 4.5.

DEFINITION 4.7 (Stable equivalence of spin^c structures). Two spin^c structures $\mathfrak{s}_1$ and $\mathfrak{s}_2$ on a closed, oriented 4-manifold M are *stably equivalent* if there exists $g \in \mathbb{N}_0$ such that the induced spin^c structures on $M\#W_g$, extending $\mathfrak{s}_1$ and $\mathfrak{s}_2$ using the fixed spin^c structure $\mathfrak{s}_g$ on W_g, are equivalent.

The stable classification of spin^c structures $\mathfrak{s}$ on simply connected 4-manifolds M is analogous to the almost-stable classification of $(4m-1)$-connected $8m$-manifolds from Theorem 3.8 and we want to apply Kreck's stable diffeomorphism theorem [Kre99, Corollary 3]. Since Kreck works with the stable normal bundle, while it is convenient for us to work with the stable tangent bundle, we make some brief general remarks about moving between tangential and normal bundle data in the modified surgery setting; for the analogous discussion in classical surgery, see, for example, [CLM, Lemma 6.39].

REMARK 4.8. Recall that given a fibration $\xi : B \to BSO$ and an oriented manifold M, a normal ξ-structure on M is a lift $\overline{\nu} : M \to B$ of the stable normal bundle $\nu_M : M \to BSO$ and $\overline{\nu}$ is called a *normal k-smoothing* if it is $(k+1)$-connected [Kre99, Section 2]. Note that the fibration ξ is not required to be $(k+1)$-coconnected so that, in Kreck's terminology, the fibration ξ need not be the normal k-type of M. Now let $i : BSO \to BSO$ be the map classifying the passage from a stable bundle to its stable inverse. Since the Whitney sum $\tau_M \oplus \nu_M$ of the stable tangent and normal bundles of M is canonically trivialised, we have $\nu_M = i \circ \tau_M$. Setting $-\xi := i \circ \xi$, it follows that if $\overline{\nu} : M \to B$ is a normal $(-\xi)$-structure on M, then $\overline{\tau} = \overline{\nu}$ defines a stable tangential ξ-structure on M, and *vice versa*. We call a stable tangential ξ-structure $\overline{\tau} : M \to B$ a *tangential k-smoothing* if it is $(k+1)$-connected. With these observations and definitions, we can pass freely between (stable) tangential ξ-manifolds and normal $(-\xi)$-manifolds, applying Kreck's results to the latter. In particular, if $(M_0, \overline{\tau}_0)$ and $(M_1, \overline{\tau}_1)$ are n-dimensional tangential k-smoothings in (B, ξ) that are tangentially (B, ξ)-bordant, then the corresponding normal k-smoothings in $(B, -\xi)$ are normally $(B, -\xi)$-bordant; and so when $n = 2q$ and $k = q-1$, we may conclude that $(M_0, \overline{\tau}_0)$ and $(M_1, \overline{\tau}_1)$ are stably (B, ξ)-diffeomorphic.

REMARK 4.9. We also note that the stable spin^c structure $\mathfrak{s}_g$ defined above on W_g corresponds to the normal structure $\overline{\nu}_c$ on W_g defined in [Kre99, page 721], which is used in Kreck's stable diffeomorphism results [Kre99, Theorem 2 and Corollary 3]. Moreover, the discussion just prior to [Kre99, Theorem 2] and its proof show that [Kre99, Theorem 2 and Corollary 3] can be strengthened to include the normal $(B, -\xi)$-structure; specifically, they yield respectively s-cobordisms and stable diffeomorphisms of normal $(B, -\xi)$-manifolds, where stabilisation is with the normal $(-\xi)$-manifold $(W_g, \overline{\nu}_c)$. As a consequence, below we obtain stable diffeomorphisms of tangential $B\,\mathrm{Spin}^c$ manifolds, where the stabilisation is with $(W_g, \mathfrak{s}_g)$.

Returning to 4-manifolds, note in particular that a 1-smoothing has to be 2-connected. While we work with simply connected 4-manifolds, so $\pi_1(M) = 1 = \pi_1(B\,\mathrm{Spin}^c)$, the map $M \to B\,\mathrm{Spin}^c$ classifying a spin^c structure need not be surjective on π_2. To mitigate this, we make the following definition.

Given $(M, \mathfrak{s})$, we define the *divisibility* $d(\mathfrak{s}) \in \mathbb{N}_0$ of $c_1(\mathfrak{s})$ by the equation

$$c_1(\mathfrak{s})(\pi_2(M)) = d(\mathfrak{s})\mathbb{Z}.$$

Let $B\,\mathrm{Spin}^c(d)$ be the homotopy fibre of the mod d spin^c first Chern class, so that there is a fibre sequence

$$B\,\mathrm{Spin}^c(d) \xrightarrow{\pi} B\,\mathrm{Spin}^c \to K(\mathbb{Z}/d, 2).$$

By construction, $\pi : B\,\mathrm{Spin}^c(d) \to B\,\mathrm{Spin}^c$ is a fibration, and the universal stable bundle over $B\,\mathrm{Spin}^c$ pulls back to a stable bundle over $B\,\mathrm{Spin}^c(d)$.

DEFINITION 4.10. Let M be a closed, oriented n-manifold. A *spin$^c(d)$ structure* on M is a lift

$$\begin{array}{ccc} & & B\,\mathrm{Spin}^c(d) \\ & \overset{\mathfrak{s}(d)}{\nearrow} & \big\downarrow {\scriptstyle \mathrm{pr}_2 \circ \pi} \\ M & \xrightarrow{\tau_M} & BSO \end{array}$$

of the stable tangent bundle's classifying map to $B\,\mathrm{Spin}^c(d)$. We denote a manifold with a spin$^c(d)$-structure by $(M, \mathfrak{s}(d))$ and the corresponding bordism groups by $\Omega_*^{\mathrm{Spin}^c(d)}$.

For example, the trivial spin^c structure $\mathfrak{s}_g$ on W_g used in Definition 4.7 lifts to the trivial spin$^c(d)$ structure $\mathfrak{s}_g(d)$ on W_g.

LEMMA 4.11. *The following assertions hold:*

(1) $\pi_2(B\,\mathrm{Spin}^c(d)) = \mathbb{Z}$ *for* $d \neq 0$ *and* $\pi_2(B\,\mathrm{Spin}^c(d)) = 0$ *for* $d = 0$*;*
(2) *if* $(M, \mathfrak{s})$ *is a simply connected* spin^c *manifold, then M admits a* spin$^c(d)$ *structure* $\mathfrak{s}(d)$ *for* $d := d(\mathfrak{s})$ *such that the map* $\mathfrak{s}(d) : M \to B\,\mathrm{Spin}^c(d)$ *is* 2*-connected. In particular,* $\mathfrak{s}(d)$ *defines a tangential* 1*-smoothing into* $B\,\mathrm{Spin}^c(d)$*.*

PROOF. We have $\pi_2(B\,\mathrm{Spin}^c) \cong \mathbb{Z}$, and so the long exact sequence of a fibration in homotopy groups yields

$$0 = \pi_3(K(\mathbb{Z}/d, 2)) \to \mathbb{Z} \to \mathbb{Z} \twoheadrightarrow \mathbb{Z}/d = \pi_2(K(\mathbb{Z}/d, 2)) \to 0.$$

Since also $\pi_1(B\,\mathrm{Spin}^c) = 1$, we have that $B\,\mathrm{Spin}^c(d)$ is 1-connected and $\pi_2(B\,\mathrm{Spin}^c(d)) \cong \mathbb{Z}$ for $d \neq 0$. A similar calculation shows that $\pi_2(B\,\mathrm{Spin}^c(0)) = \pi_2(B\,\mathrm{Spin}) = 0$. In fact, it then follows from Whitehead's theorem that the map $B\,\mathrm{Spin} \to B\,\mathrm{Spin}^c(0)$, obtained from factoring the canonical map $B\,\mathrm{Spin} \to B\,\mathrm{Spin}^c$ through $B\,\mathrm{Spin}^c(0)$, is a homotopy equivalence. This concludes the proof of the first assertion.

We now assume that $(M, \mathfrak{s})$ is a simply connected spin^c manifold and prove the second assertion. The first point follows by observing that in the exact sequence

$$[M, B\,\mathrm{Spin}^c(d)] \to [M, B\,\mathrm{Spin}^c] \to [M, K(\mathbb{Z}/d, 2)] = H^2(M; \mathbb{Z}/d),$$

the spin^c structure $\mathfrak{s} \in [M, B\,\mathrm{Spin}^c]$ is mapped to zero, by definition of $d(\mathfrak{s})$. It only remains to show that $\mathfrak{s}(d)$ is 2-connected. Since this is clear for $d = 0$, we assume that $d \neq 0$. As we know that $\pi_2(B\,\mathrm{Spin}^c) = \mathbb{Z}$ and $\pi_2(B\,\mathrm{Spin}^c(d)) = \mathbb{Z}$, the long exact sequence of the fibration and the definition of $d = d(\mathfrak{s})$ imply that $\mathrm{Im}(\mathfrak{s}_*) = d\mathbb{Z} \subseteq \mathbb{Z} = \pi_2(B\,\mathrm{Spin}^c)$ and therefore $\mathfrak{s}(d)$ is surjective on π_2, as required. □

Our aim is now to construct an injective map $\Omega_*^{\mathrm{Spin}^c(d)} \to \mathbb{Z} \oplus \mathbb{Z}$. The first component of this map is the signature, while the second arises as a characteristic number obtained from $\mathfrak{s}(d) : M \to B\,\mathrm{Spin}^c(d)$ by pulling back a universal class $c_1/d \in H^2(B\,\mathrm{Spin}^c(d))$ that we now define. For $d \neq 0$, Lemma 4.11 implies that $H^2(B\,\mathrm{Spin}^c(d))$ is an infinite cyclic group. It is generated by a class $c_1/d \in H^2(B\,\mathrm{Spin}^c(d))$ such that the pullback

$\pi^*(c_1)$ of the spin^c first Chern class, satisfies

$$d(c_1/d) = \pi^*(c_1) \in H^2(B\,\mathrm{Spin}^c(d)). \qquad (\varpi)$$

For $d = 0$, $H^2(B\,\mathrm{Spin}^c(0)) = H^2(B\,\mathrm{Spin}) = 0$ and we set $c_1/d = 0$. As is conventional for characteristic classes, given a spin$^c(d)$ structure $\mathfrak{s}(d) : M \to B\,\mathrm{Spin}^c(d)$, we write $c_1/d(\mathfrak{s}(d)) := \mathfrak{s}(d)^*(c_1/d) \in H^2(M)$. For example, the trivial spin$^c(d)$-structure $\mathfrak{s}_g(d)$ on W_g is characterised by $c_1/d(\mathfrak{s}_g(d)) = 0$.

LEMMA 4.12. *There is an injective homomorphism*

$$\Theta : \Omega_4^{\mathrm{Spin}^c(d)} \to \mathbb{Z} \oplus \mathbb{Z},$$
$$[N, \mathfrak{s}(d)] \mapsto (\sigma(N), \langle (c_1/d(\mathfrak{s}(d)))^2, [N]\rangle).$$

PROOF. The given map is a homomorphism, and is a bordism invariant because the signature is bordism invariant and because c_1^2 is a characteristic number and therefore so is $(c_1/d)^2$.

It remains to prove injectivity of Θ. Let $(M, \mathfrak{s}(d))$ be a spin$^c(d)$-manifold with vanishing signature and $(c_1/d(\mathfrak{s}(d)))^2 = 0$. Since $\pi_1(B\,\mathrm{Spin}^c(d)) = 1$, after preliminary surgeries over $B\,\mathrm{Spin}^c(d)$ we may assume that M is simply connected. Since $\sigma(M) = 0$, the homeomorphism classification of smooth simply connected 4-manifolds [Fre82] means that we can assume that M is homeomorphic to one of the following model manifolds:

$$M \cong_{\mathrm{TOP}} \begin{cases} W_g & d \text{ even}, \\ X_g & d \text{ odd}, \end{cases}$$

where $X_g := \#^g S^2 \widetilde{\times} S^2$. In other words, M is a possibly exotic W_g or X_g. Now, exotic pairs of simply connected 4-manifolds are h-cobordant [Wal64, Theorem 2], and the spin$^c(d)$-structure on M propagates along an h-cobordism to a spin$^c(d)$ structure on either W_g or X_g, as appropriate. Hence, we may assume that $(M, \mathfrak{s}(d))$ is diffeomorphic to either $(W_g, \mathfrak{s}'_g(d))$ or $(X_g, \mathfrak{s}''_g(d))$ for some spin$^c(d)$ structure $\mathfrak{s}'_g(d)$ or $\mathfrak{s}''_g(d)$. Now, M has a standard coboundary N, $\partial N = M$, where

$$N \cong \begin{cases} Y_g & d \text{ even}, \\ Z_g & d \text{ odd}. \end{cases}$$

Here $Y_g := \natural^g D^3 \times S^2$ and $Z_g := \natural^g D^3 \widetilde{\times} S^2$, where $D^3 \widetilde{\times} S^2 \to S^2$ is the nontrivial bundle. By assumption, $(c_1/d(\mathfrak{s}(d)))^2 = 0$ and it follows that $c_1/d(\mathfrak{s}(d)) \in L$, for some lagrangian $L \subseteq H^2(M)$. Now, the automorphisms of the intersection form act transitively on the set of lagrangians (see, for example, [Wal64, pages 144–145]); and Wall [Wal64, page 144] also showed that every isometry of the intersection form of $H^2(M)$ is realised by a diffeomorphism. Hence, we may assume that $c_1/d(\mathfrak{s}(d)) \in H^2(M)$ lies in the standard lagrangian of $H^2(M)$, and so is the restriction to the boundary of c for some $c \in H^2(N)$. Since $H^2(N) \to H^2(M)$ is onto a summand, it

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follows that N admits a spin$^c(d)$-structure $\mathfrak{s}_N(d)$ that restricts to $\mathfrak{s}(d)$. Hence, $(N, \mathfrak{s}_N(d))$ is a spin$^c(d)$ null-bordism of $(M, \mathfrak{s}(d))$, and so Θ is indeed injective. □

Next, using Lemma 4.12, we deduce the stable classification of spin^c structures on simply connected 4-manifolds.

The fibration sequence defining $B\,\mathrm{Spin}^c(d)$ gives rise to an exact sequence

$$[M, \Omega K(\mathbb{Z}/d, 2)] \to [M, B\,\mathrm{Spin}^c(d)] \to [M, B\,\mathrm{Spin}^c] \to [M, K(\mathbb{Z}/d, 2)].$$

If M is simply connected, then $[M, \Omega K(\mathbb{Z}/d, 2)] \cong [M, K(\mathbb{Z}/d, 1)] \cong H^1(M; \mathbb{Z}/d) = 0$, so if a lift of spin^c structure $\mathfrak{s}$ to a spin$^c(d)$ structure $\mathfrak{s}(d)$ exists, then it is essentially unique.

A spin$^c(d)$ structure $\mathfrak{s}(d)$ on M induces a spin$^c(d)$ structure on $M\#W_g$, for any g: as in Definition 4.7, we extend the associated spin$^c(d)$ structure on M by the spin$^c(d)$ structure on W_g with $c_1 = 0$. By the previous paragraph, since W_g is simply connected, there is an essentially unique such spin$^c(d)$ structure on W_g. Then a lift to a spin$^c(d)$ structure on M determines such a lift on $M\#W_g$. We can therefore define stable equivalence of spin$^c(d)$ structures. The definition is identical to the definition for spin$^c(d)$ structure, just replacing Spin^c with Spin$^c(d)$ throughout Definition 4.3(1) and Definition 4.7.

THEOREM 4.13. *Let M be a closed, oriented, simply connected 4-manifold. Two spin^c structures $\mathfrak{s}_1$ and $\mathfrak{s}_2$ on M are stably equivalent if and only if $d(\mathfrak{s}_1) = d(\mathfrak{s}_2) \in \mathbb{N}_0$ and $c_1(\mathfrak{s}_1)^2 = c_1(\mathfrak{s}_2)^2 \in H^4(M)$.*

PROOF. For the forward direction, the square of the Chern class and its divisibility are preserved by stable equivalence because we fixed the spin^c structure on W_g to be the structure with trivial first Chern class, and because equivalence of spin^c structures preserves Chern numbers and the divisibility.

For the reverse direction, by Lemma 4.12, for a fixed 4-manifold M, two spin^c structures $\mathfrak{s}_1$ and $\mathfrak{s}_2$ on M with $d(\mathfrak{s}_1) = d(\mathfrak{s}_2) = d$ determine spin$^c(d)$-structures $\mathfrak{s}_1(d)$ and $\mathfrak{s}_2(d)$, and therefore elements of $\Omega_4^{\mathrm{Spin}^c(d)}$. Since $c_1(\mathfrak{s}_1)^2 = c_1(\mathfrak{s}_2)^2$, it follows that $(c_1/d(\mathfrak{s}_1(d)))^2 = (c_1/d(\mathfrak{s}_2(d)))^2$: for $d = 0$, this is automatic; for $d \neq 0$, apply π^* to $c_1(\mathfrak{s}_1)^2 = c_1(\mathfrak{s}_2)^2$ and use (ϖ). Therefore, since $\sigma(M)$ is independent of tangential structures, by Lemma 4.12, $(M, \mathfrak{s}_1(d))$ and $(M, \mathfrak{s}_1(d))$ are bordant over $B\,\mathrm{Spin}^c(d)$. Using Remark 4.8, we transfer to stable normal structures and apply Kreck's stable diffeomorphism theorem [Kre99, proof of Theorem 2 and Corollary 3], which in the current situation implies that spin$^c(d)$ structures on M are stably equivalent if they are bordant. Here we use that the maps $M \to B\,\mathrm{Spin}^c(d)$ are 1-smoothings by Lemma 4.11 and that stabilisation is with $(W_g, \mathfrak{s}_g(d))$. Via the map $\pi : B\,\mathrm{Spin}^c(d) \to B\,\mathrm{Spin}^c$, a stable equivalence of spin$^c(d)$ structures determines a stable equivalence of spin^c structures. □

Now we are ready to prove the main result of this section. Let $C \in \mathbb{Z}$ be such that $|C| \geq 16$ and $8 \mid C$. Define $P(C) := |\mathcal{P}_{C/8}|$, namely the number of distinct primes

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dividing $C/8$. We consider $S^2 \times S^2$ with a fixed orientation. This determines an identification $H^4(S^2 \times S^2) = \mathbb{Z}$.

THEOREM 4.14. *For every $C \in \mathbb{Z}$ with $|C| \geq 16$ and $8 \mid C$, there are $n := 2^{P(C)-1}$ stably equivalent spin^c structures $\mathfrak{s}_1, \ldots, \mathfrak{s}_n$ on $S^2 \times S^2$ with $c_1(\mathfrak{s}_i)^2 = C \in H^4(S^2 \times S^2) = \mathbb{Z}$ that are all pairwise inequivalent.*

PROOF. Let $M := S^2 \times S^2$. Let $x, y \in H^2(M) \cong \mathbb{Z}^2$ be generators dual to $[\mathrm{pt} \times S^2]$ and $[S^2 \times \mathrm{pt}]$ respectively. So $xy = 1 \in H^4(M)$ while $x^2 = y^2 = 0$. Henceforth, we identify $H^4(M) \cong \mathbb{Z}$. Let $Q := C/8$. There are $P(C)$ prime powers dividing Q. Up to switching the order and multiplying both by -1, there are $2^{P(C)-1}$ ways to write Q as a product of coprime integers $Q = q_1 q_2$. For each such factorisation, let $\mathfrak{s}_i$ be a spin^c structure with

$$c_1(\mathfrak{s}_i) = 2q_1 x + 2q_2 y.$$

Such spin^c structures exist by Lemma 4.5: every characteristic element of $H^2(M)$ can be realised as the first Chern class of some spin^c structure. Note that

$$c_1(\mathfrak{s}_i)^2 = 8q_1q_2 = 8Q = C \in \mathbb{Z} = H^4(S^2 \times S^2)$$

and $d(\mathfrak{s}_i) = 2$ for every i. Thus by Theorem 4.13, all the $\mathfrak{s}_i$ are stably equivalent to one another. However, as we saw in the proof of Proposition 2.2, there is no isometry of the intersection pairing of M that sends $(2q_1, 2q_2)$ to $(2q_1', 2q_2')$ in $H^2(M) \cong \mathbb{Z}^2$ for distinct unordered pairs $\{q_1, q_2\}$ and $\{q_1', q_2'\}$. By Lemma 4.6, it follows that the $\{\mathfrak{s}_i\}$ are pairwise inequivalent spin^c structures. □

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