ON THE CO-DEDEKINDIANFINITE $p$-GROUPS WITH NON-CYCLIC ABELIAN SECOND CENTRE

ALI-REZA JAMALI

Institute of Mathematics, University for Teacher Education, Tehran 15614, Iran

and HAMID MOUSAVI

Institute for Studies in Theoretical Physics and Mathematics, University for Teacher Education, Tehran 15614, Iran

Abstract. A group $G$ is called co-Dedekindian if every subgroup of $G$ is invariant under all central automorphisms of $G$. In this paper we give some necessary conditions for certain finite $p$-groups with non-cyclic abelian second centre to be co-Dedekindian. We also classify 3-generator co-Dedekindian finite $p$-groups which are of class 3, having non-cyclic abelian second centre with $|\Omega_1(G^p)| = p$.

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1. Introduction. Let $G$ be a group, and let $Z(G)$ denote the centre of $G$. An automorphism $\alpha$ of $G$ is called central if $x^{-1}\alpha(x) \in Z(G)$ for each $x \in G$. The set of all central automorphisms of $G$, denoted by $Aut_c(G)$, is a normal subgroup of the full automorphism group of $G$. A group $G$ is called co-Dedekindian ($\varepsilon$-group for short) if every subgroup of $G$ is invariant under all central automorphisms of $G$. In [1], Deaconescu and Silberberg give a Dedekind-like structure theorem for the non-nilpotent $\varepsilon$-groups with trivial Frattini subgroup and by reducing the finite nilpotent $\varepsilon$-groups to the case of $p$-groups they obtain the following theorem.

**Theorem 1.1.** Let $G$ be a $p$-group. If $G$ is a non-abelian $\varepsilon$-group, then $Z_2(G)$ is a Dedekindian group. If $Z_2(G)$ is non-abelian, then $G \cong Q_8$. If $Z_2(G)$ is cyclic, then $G \cong Q_{2n}$, $n \geq 4$, where $Q_{2n}$ is the generalized quaternion group of order $2^n$.

In [1], the authors notice that non-abelian $p$-groups with abelian non-cyclic second centre and which are $\varepsilon$-groups do exist. They show that if $G$ is a non-abelian $\varepsilon$-group of order $p^4$, with $Z_2(G)$ abelian and non-cyclic, then $p = 3$ and

$$G = \langle a, b | a^9 = 1, \quad b^3 = a^6, \quad [a, b]^3 = 1, \quad [a, [a, b]] = a^3, \quad [b, [a, b]] = 1 \rangle.$$ 

The purpose of this paper is (1) to find some necessary conditions for certain $p$-groups with abelian non-cyclic second centre to be $\varepsilon$-groups and (2) to classify the

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3-generator $\varphi$-groups $G$ satisfying these conditions with the additional condition $\text{cl}(G) = 3$.

Finally we show that given any natural number $m \geq 3$, there is a 2-group with abelian non-cyclic second centre which is a $\varphi$-group of class $m$.

Our notation is standard. We refer in particular to [6].

### 2. General results

In this section we first give some results that will be used later. Throughout the paper $G$ will stand for a finite non-abelian $p$-group. If $\alpha \in \text{Aut}_c(G)$, we shall denote $F_\alpha = \{ x \in G | \alpha(x) = x \}$ and $K_\alpha = \langle x^{-1} \alpha(x) | x \in G \rangle$. Also we put $F = \bigcap_{\alpha \in A} F_\alpha$, where $A = \text{Aut}_c(G)$, and $K = \langle K_\alpha | \alpha \in \text{Aut}_c(G) \rangle$. We now collect some information about the subgroups $F$ and $K$ of $G$.

**Lemma 2.1.** Let $G$ be a $\varphi$-group.

(i) $\Omega_1(G) \leq F \leq \Phi(G)$;

(ii) if $|G : \Phi(G)| > |G^p \cap \Omega_1(G)|$, then $G$ is not regular.

**Proof.** (i) By [1, Lemma 3.1], we have $\Omega_1(G) \leq F$. Now let $M$ be any maximal subgroup of $G$, and let $z$ be an element of order $p$ in $M \cap Z(G)$. Let $x \notin M$, and define $\alpha : G \to G$ by $\alpha(x^m) = x^m z^i$, where $i \in \{ 0, 1, ..., p - 1 \}$ and $m \in M$. It is easy to see that $\alpha \in \text{Aut}_c(G)$ and $F_\alpha = M$. Hence $F \leq \Phi(G)$.

(ii) Since $\Omega_1(G) \leq \Phi(G)$, we have $\Omega_1(G)G^p \leq \Phi(G)G^p$. This shows that $|\Omega_1(G)|G^p| \leq |\Phi(G)||G^p \cap \Omega_1(G)| < |G|$. Hence $G$ is not regular by [6, Chapter 4, Theorem 3.14(iv)]

**Proposition 2.2.** Let $G$ be a finite non-abelian $p$-group. If $G$ is a $\varphi$-group, then $Z(G)$ is cyclic and $Z(G) \leq \Phi(G)$.

**Proof.** Let $M$ be any maximal subgroup of $G$ and let $u$ be an element of order $p$ in $Z(G)$ and $x \notin M$. By considering the central automorphism $\alpha$ defined in the proof of Lemma 2.1(i), we have $\alpha(x) = ux$. Since $G$ is a $\varphi$-group, $u \in \langle x \rangle$. Hence $|\Omega_1(Z(G))| = p$, from which we conclude that $Z(G)$ is cyclic. Next we let $g$ be an element of $G \langle F \cup Z(G) \rangle$. Since $G$ is a $\varphi$-group and $g \notin F$, there is an $l \in \mathbb{N}$ such that $g^p \neq 1$ and $g^p \in Z(G)$. We define $l_g$ to be the least positive integer such that $g^{l_g} \in Z(G)$. We then have $g^{l_g} = z^{l_g}$, where $z$ is a generator of $Z(G)$ and $k_g$ is a non-negative integer. We claim that $l_g > k_g$ for some element $g$ of $G \langle F \cup Z(G) \rangle$. Denying this assertion, we may write $(g^{-1}z^{l_g})^{l_g} = 1$. Now as $g^{-1}z^{l_g} \notin Z(G)$, we must have $g^{-1}z^{l_g} \in F$; for, put $a = g^{-1}z^{l_g}$ and assume $a \notin F$; this implies $a^{l_g} \in Z(G)$ and so $g^{l_g} \in Z(G)$, which is contrary to the minimality assumption. Hence $g \in Z(G)F$, showing that $G = Z(G)F$. It follows that $G/F$ is cyclic; so is $G/\Phi(G)$, giving a contradiction. Now $l_g > k_g$ leads to $z^{-1}g^{l_g - k_g} \in F$, by a similar argument. However, $g^{l_g - k_g} \in \Phi(G)$, which implies that $z \notin \Phi(G)$.

**Lemma 2.3.** Let $G$ be a finite non-abelian $p$-group with $|\Omega_1(G^p)| = p$. If $G$ is a $\varphi$-group, then $G^p$ is cyclic. Moreover, if $Z_2(G)$ is non-cyclic and abelian, then $p$ is odd.

**Proof.** It is clear that if $p$ odd, then $G^p$ is cyclic. Now suppose that $p = 2$. We have $\Phi(G) = G^2$ and hence $\Omega_1(G) \leq G^2$, by Lemma 2.1(i). It follows that $G \cong \mathbb{Z}_2^n$, as
If $G$ is a finite non-abelian $p$-group with a non-cyclic abelian second centre $Z_2(G)$. Suppose that $|\Omega_1(G^p)| = p$. If $G$ is a $\nu$-group, then $G^p = Z(G)$ and $Z_2(G) \leq \Phi(G)$.

**Proof.** By Lemma 2.3, $G^p$ is cyclic and $p$ is odd. Let $G^p = \langle a \rangle$. We first show that $G^p = Z(G)$. The proof is divided into three steps.

**Step 1.** If $g \in G\setminus \Phi(G)$, then $G^p = \langle a^p \rangle$.

Suppose that $g^p = a^i$, where $(i, p) = 1$ and $i$ is a positive integer. Since $[a^i, g^{-1}] \in [G^p, g^{-1}]$ and $[G^p, g^{-1}]$ is properly contained in $G^p$, we have $[a^i, g^{-1}] \in \langle a^p \rangle$ and, consequently, $[a^{i(p-1)}g^{-1}] \in \langle a^p \rangle \leq Z(\langle a, g^{-1} \rangle)$. Thus $(a^{i(p-1)}g^{-1})^p = a^{ip}g^{-p}[a^{ip}, g^{-1}]^{(p-1)/2} = 1$. We now have $a^{ip}g^{-1} \in \Omega_1(G)$, from which we get $g \in \Phi(G)$, a contradiction.

**Step 2.** $G^p \leq Z(G)$.

Suppose that $G^p$ is not contained in $Z(G)$, so that $aZ(G) \neq Z(G)$. For any minimal generating set $\{y_iZ(G)\}$ of $G/Z(G)$, we have $y_i \not\in \Phi(G)$ for each $i$. Hence, by Step 1, $y_i^{p_n} = a$ for some positive integer $n_i$. Thus for each $i$, $y_i^{p_n}Z(G) = aZ(G)$, contrary to [5, 3.2.10]. Hence $G^p \leq Z(G)$.

**Step 3.** If $g \in G\setminus \Phi(G)$ then $Z(G) = \langle g \rangle$, and hence $G^p = Z(G)$.

If $g^p = z^p$ for some $z \in Z(G)$, then $gz^{-1}$ has order $p$ and so $gz^{-1} \in \Phi(G)$. It follows, by Proposition 2.2, that $g \in \Phi(G)$, a contradiction. Hence $G^p = Z(G)$.

To prove the second part of the theorem, we assume that $x \in Z_2(G)\setminus \Phi(G)$. Thus $y^p = x^p$ for each $y \in G\setminus \Phi(G)$, where $(l, p) = 1$ (because in view of Step 1, $x^p$ and $y^p$ are generators of $G^p$). Hence

$$(yx^{-l})^p = y^p x^{-lp}[x^{-l}, y]^{p(p-1)/2} = [x^{-l}, y^p]^{(p-1)/2} = 1.$$ 

Therefore $yx^{-l} \in \Phi(G)$, whence $G/\Phi(G)$ is cyclic, a contradiction. 

The following result will be used throughout the sequel.

**Lemma 2.5.** Let $G$ be a metabelian group. If $x, y$ are elements of $G$ and $n \in \mathbb{N}$, then

$$(xy)^n = x^n y^n[y, x]^{n(n-1)/2}[\eta_2, x][\eta_1, y],$$

for some $\eta_1, \eta_2 \in G'$.

**Proof.** This is a special case of P. Hall’s formula and is easily proved by using the identity $xy = yx[x, y]$. 

**Theorem 2.6.** Let $G$ be a finite metabelian $p$-group with a non-cyclic abelian second centre $Z_2(G)$. Suppose that $|\Omega_1(G^p)| = p$. If $G$ is a $\nu$-group, then $|Z(G)| = p$. Hence $\Phi(G) = G'$ and $Z_2(G)$ is elementary abelian.

**Proof.** According to Theorem 2.4, $Z(G) = G^p$. We first suppose that there exists an element $x$ in $\Phi(G)$ such that $Z(G) = \langle x^p \rangle$. Then we may choose an element $y$ in $G\setminus \Phi(G)$ with $x^p = y^p$. Since $x \in \Phi(G)$ and $\Phi(G)$ is abelian, we have
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\[ (y x^{-1})^p = y^p x^{-p} [x^{-1}, y]^{p(p-1)/2} \eta, y = [\eta, y], \]
where \( \eta \in G' \). It follows that \((y x^{-1})^p = 1 \). Now since \( y x^{-1} \not\in \Phi(G) \), we see that \((y x^{-1})^p \neq 1 \) and \( Z(G) = (y x^{-1})^p \). Hence \(|Z(G)| = p \).

Now suppose that for each \( g \in \Phi(G) \), \( (g^p) \) is a proper subgroup of \( Z(G) \). Then, by choosing \( x, y \) in \( G \setminus \Phi(G) \) with \( x^p = y^p \) and \( xy^{-1} \not\in \Phi(G) \), we have
\[ (y x^{-1})^p = [x^{-1}, y]^{p(p-1)/2} [\eta_1, x^{-1}] [\eta_2, y], \]
where \( \eta_1, \eta_2 \in G' \). By Theorem 2.4, \((y x^{-1})^p \) is a generator of \( G^p \); put \( a = (y x^{-1})^p \). By our assumption, \([x^{-1}, y]^{p(p-1)/2} = a^{pl} \) for some \( l \in \mathbb{N} \). Now since \([\eta_2, x^{-1}] \) and \([\eta_1, y] \) are of order \( p \), we get \( a^p = d^p \), and so \( a^p = 1 \). Hence, \(|Z(G)| = p \). Obviously \( \Phi(G) = G' \).

For the final part of Theorem, we let \( x \in Z_2(G) \). If \( x^p \neq 1 \), then \( x^p \) is a generator of \( Z(G) \) and, as before, there is an element \( y \) in \( G \setminus \Phi(G) \) such that \( x^p = y^p \), and so \((y x^{-1})^p = y^p x^{-p} [x^{-1}, y]^{p(p-1)/2} = 1 \), because \([x^{-1}, y] \in Z(G) \). Consequently, \( yx^{-1} \in \Phi(G) \) and we have \( y \in \Phi(G) \), a contradiction.

**Corollary 2.7.** Let \( G \) be a finite \( p \)-group of class 3 with non-cyclic abelian second centre \( Z_2(G) \). Suppose that \(|\Omega_1(G^p)| = p \). If \( G \) is a \( \varphi \)-group, then
(i) \( Z(G) = G^p \) and \(|Z(G)| = p \);
(ii) \( \Phi(G) = G' = Z_2(G) \), and \( \exp(\Phi(G)) = p \);
(iii) \( p = 3 \).

**Proof.** In view of Theorem 2.4, \( Z(G) = G^p \), and \( Z_2(G) \leq \Phi(G) \). Since \( G' \leq Z_2(G) \), we have \( G' = \Phi(G) = Z_2(G) \) and \(|Z(G)| = p \). Now \( G \) is not regular, by Lemma 2.1(ii) and so \( p \leq c\ell(G) = 3 \) using [6, Chapter 4, 3.13(ii)]. Hence \( p = 3 \). □

**3. An application.** In this section we classify the finite 3-generator \( p \)-groups \( G \) that are \( \varphi \)-groups with the following properties:
(i) \( Z_2(G) \) is abelian and non-cyclic,
(ii) \( |\Omega_1(G^p)| = p \),
(iii) \( c\ell(G) = 3 \).

There is one family of such groups consisting of four non-isomorphic groups.

We also give an example of a 2-group with abelian non-cyclic second centre and arbitrarily large nilpotency class that is a \( \varphi \)-group.

From now on \( G \) will stand for a finite \( p \)-group in \( \varphi \) satisfying the conditions (i)–(iii).

**Lemma 3.1.** If \( a, b \) and \( c \) belong to a minimal generating set of \( G \), then
(i) \( \{a, b\} \not\subseteq \varphi G([a, b]) \),
(ii) \( Z(G) \) intersects \( \langle[a, b], [a, c], [b, c] \rangle \) trivially.

**Proof.** (i) Assume that \( \{a, b\} \not\subseteq \varphi G([a, b]) \). Then we have \((ab)^3 = a^3 b^3 \) and \((ab^{-1})^3 = a^3 b^{-3} \). Since \( a^3 \) and \( b^3 \) are generators of \( Z(G) \) and \(|Z(G)| = 3 \), \( a^3 = b^3 \) or \( a^3 = b^{-3} \). Thus either \((ab)^3 = 1 \) or \((ab^{-1})^3 = 1 \). Consequently either \( ab \in \Phi(G) \) or \( ab^{-1} \in \Phi(G) \), a contradiction.

(ii) Assume that \( Z(G) \) intersects \( \langle[a, b], [a, c], [b, c] \rangle \) non-trivially. Since \( G' \) is elementary abelian, we may suppose that
for some $\varepsilon_i \in \{0, \pm 1\}, \; i = 1, 2$. Clearly $(\varepsilon_1, \varepsilon_2) \neq (0, 0)$ by (i). If $\varepsilon_2 = 0$, then $[a, bc^{\varepsilon_1}] \in Z(G)$, because $Z_2(G) = G'$. This is impossible, since $a, bc^{\varepsilon_1}$ belong to a minimal generating set of $G$. Similarly $\varepsilon_1 = 0$ is impossible. We now suppose that $\varepsilon_1 \neq 0$ and $\varepsilon_2 \neq 0$. If $\varepsilon_1 = \varepsilon_2$, then $[ab, bc^{\varepsilon_1}] \in Z(G)$. But $ab, b$ and $bc^{\varepsilon_1}$ belong to a minimal generating set, contrary to (i). Also if $\varepsilon_1 = -\varepsilon_2$, then $[ab^{-1}, bc^{\varepsilon_1}] \in Z(G)$, again a contradiction. □

In what follows $d(G)$ denotes the minimal number of generators of $G$.

**Corollary 3.2.** $|G| = 3^4$ if $d(G) = 2$, and $|G| = 3^7$ if $d(G) = 3$.

**Proof.** We prove the second part of the Corollary; the first part is established similarly. Let $d(G) = 3$ and $G = \langle a, b, c \rangle$. Since $G'$ is elementary abelian, we have

$$G' = \langle [a, b], [a, c], [b, c] \rangle \times Z(G),$$

by Lemma 3.1(ii), so that $|G'| = 3^4$. Now $G' = \Phi(G)$ shows that $|G| = 3^7$. □

**Lemma 3.3.** Let $a, b$ and $c$ be elements of $G$.

(i) $(ab)^3 = a^3 b^3 [a, [a, b]] [b, [a, b]]^{-1}$.

(ii) $(abc)^3 = a^3 b^3 c^3 [a, x][a, y][b, y]^{-1}[b, z][c, x]^{-1}[c, y]^{-1}[c, z]^{-1}$, where $x = [a, b], y = [a, c]$ and $z = [b, c]$.

(iii) $[b, [a, c]] = [a, [b, c]] [c, [a, b]]$.

(iv) If $a$ and $b$ are elements of a minimal generating set of $G$ such that $[b, [a, b]] = 1$, then $b^6 = [a, [a, b]]$.

**Proof.** The first two parts are easily checked. (iii) is most conveniently proved by using the identity $((ab)c)^3 = (a(bc))^3$. To prove (iv), we observe that $(ab^{-1})^3 = a^3 b^{-3} [a, [a, b]]^{-1}$, by (i). Now since $(ab)^3$ and $(ab^{-1})^3$ are generators of $Z(G)$, $(ab)^3(ab^{-1})^3 = 1$ or $(ab)^3 = (ab^{-1})^3$. The former shows that $a^3 = 1$, which is impossible. The result is now settled by using the latter. □

**Proposition 3.4.** If $d(G) \geq 3$, then $G$ has a minimal generating set containing three elements $a, b$ and $c$ such that

(i) $[a, [a, b]] = a^3, \; [b, [a, b]] = 1$,

(ii) $[b, [b, c]] = b^3, \; [c, [b, c]] = 1$.

**Proof.** Suppose that $a, b$ and $c$ are elements of a minimal generating set of $G$. Without loss of generality, we may assume that $[a, [a, b]] \neq 1$, by Lemma 3.1(i). Since $|Z(G)| = 3$ and $a^3 \in Z(G)$, it follows that $[a, [a, b]] = a^3$ or $a^6$. In the latter case, if we replace $b$ by $b^2$, we get $[a, [a, b]] = a^3$, as required. Now if $[b, [a, b]] \neq 1$, then we have

$$[a, [a, b]]^\varepsilon [b, [a, b]] = 1,$$

for some $\varepsilon = \pm 1$. Therefore, by setting $b' = a^\varepsilon b$, we find that $[b', [a, b']] = 1$. Here we still have $[a, [a, b']] = a^3$ and consequently (i) holds.
Now if \([b, [b, c]] \neq 1\), then we may repeat the above process to obtain the relations \([b, [b, c]] = b^3\) and \([c, [b, c]] = 1\) for a suitable \(c\). We suppose that \([b, [b, c]] = 1\), which implies that \([b, [c^\varepsilon a, b]] = 1\) for every \(\varepsilon \in \{-1, 0, 1\}\). Therefore, \([c^\varepsilon a, [c^\varepsilon a, b]] \neq 1\) by Lemma 3.1(i) which, together with the assumption \([c, [c, a]] \neq 1\), enables us to perform the above process with \(a' = c\), \(b' = c^\varepsilon a\) and \(c' = b\) in order to obtain the desired generators. Hence it suffices to show that \([c, [c, a]] \neq 1\). To see this, we consider the central elements \((abc)^3\) and \((abc)^{-1}\). If \((abc)^3(abc)^{-1} = 1\) then it follows, from the relations of (i) and Lemma 3.3, that \(a^n[c, [a, c]] = 1\), which gives us \([c, [c, a]] \neq 1\). Now we assume that \((abc)^3 = (abc)^{-1}\). Then \(c^3[a, y] = [b, y][c, x]\), and hence \((ab^{-1}c)^3 = c^6[a, y][c, x]^{-1}\). If \((ab^{-1}c)^3(abc)^3 = 1\), then \(a^3[c, [a, c]] = c^3[a, [a, c]]\), and hence \([c, [a, c]] = (ac)^{-1} \neq 1\). Also \((ab^{-1}c)^3 = (abc)^3\) leads to \(a^3c^3[a, [a, c]] = 1\), which shows that \([c, [a, c]] = (ac)^{-3} \neq 1\), completing the proof. 

**Theorem 3.5.** Let \(G\) be a 3-generator finite \(p\)-group of class 3 with non-cyclic abelian second centre \(Z_2(G)\) and let \(|\Omega_1(G^p)| = p\). If \(G\) is a 
\(_{-}\)-group, then \(G\) is generated by the elements \(a, b, c, x, y\) and \(z\), subject to the following defining relations:
\[
\begin{align*}
a^3 &= b^3 = c^3 = x^3 = y^3 = z^3 = 1, \\
[a^3, b] &= [a^3, c] = [x, y] = [x, z] = [y, z] = [b, x] = [c, z] = 1, \\
x &= [a, b], \\
y &= [a, c], \\
z &= [b, c], \\
[a, y] &= a^6(x^{-1})^{-1}, \\
[b, y] &= a^{3(m-n)}, \\
[a, z] &= a^{3(m+n)}, \\
[c, x] &= a^{3n},
\end{align*}
\]
where \(m, n \in \{0, 1, 2\}\). Furthermore, if we denote the above group \(G\) by \(G(m, n)\) then \(G(0, 0) \cong G(2, 0)\), \(G(0, 1) \cong G(2, 2)\), \(G(1, 1) \cong G(2, 1)\) and \(G(0, 2) \cong G(1, 0) \cong G(1, 2)\).

This can be extended in the following way:
\[
[a, [a, b]] = a^3, \quad [b, [a, b]] = 1, \quad [b, [b, c]] = b^3, \quad [c, [b, c]] = 1.
\]
By Lemma 3.3(iv), we have \(a^3 = b^3 = c^3\). For convenience, we set \(x = [a, b]\), \(y = [a, c]\) and \(z = [b, c]\). We now consider the central elements \((abc)^3\), \((abc)^{-1}\) of \(G\). We claim that \((abc)^3 \neq (abc)^{-1}\). If this is not the case, in view of Lemma 3.3(iii) and the above relations, we shall have \([a, y] = [b, y][c, x]\). Thus \((abc)^3 = a^3[c, y][c, x]^{-1}\), and so \([c, y] \neq a^3\). It follows that \((ac)^3 = (abc)^{-1}\), by Lemma 3.3(i), and hence \([a, y] = c^6 \neq 1\). Now since \((ab^{-1}c)^3 = c^6[a, y][c, x]^{-1}\), we find that \((ab^{-1}c)^3(abc)^3 = 1\), and so \([c, y] = 1\). But \((ab^{-1}c)^3 = [c, y]\), a contradiction. Therefore we must have \((abc)^3(abc)^{-1} = 1\). In this case, \([c, y] = a^3 \neq 1\) and hence \((ac)^3 = a^3[a, y]\). Now we obtain
\[
(ab^{-1}c)^3 = [a, y][b, y][c, x].
\]
We first suppose that \((ab^{-1}c)^3 = 1\). In this case, \([a, y] = 1\) and so \([b, y][c, x] \neq 1\). As before, exactly one of \([a, z]\), \([b, y]\), \([c, x]\) is the identity element (otherwise, \([b, y] = [c, x]^{-1}\) by Lemma 3.3(iv).) Therefore we may assume that
\[
[b, y] = a^{3(m-n)}, \quad [a, z] = a^{3(m+n)}, \quad [c, x] = a^{3n},
\]
where \(m \in \{1, 2\}\) and \(n \in \{0, 1, 2\}\).
We next suppose that \((a^{-1}bc)^3 = (abc)^3\). Then \([b, y][c, x] = 1\) and so we have \([a, z] = [c, x]\) and \([a, y] \neq 1\), which implies that \([a, y] = [c, y]\) (otherwise \((ac^{-1})^3 = 1\)). Therefore, in this case the following defining relations are obtained for \(G\):

\[
[b, y] = a^{-3n}, \quad [a, z] = a^{3n}, \quad [c, x] = a^{3n},
\]

where \(n \in \{0, 1, 2\}\).

We are now able to write down a single presentation for \(G\) in both cases. On the other hand by using GAP [4], one can easily check that each group \(G(m, n)\) is a \(\varphi\)-group of order \(3^7\) and that \(G(0, 0), G(0, 1), G(0, 2)\) and \(G(1, 1)\) are the only non-isomorphic groups among the groups \(G(m, n)\) where \(m, n \in \{0, 1, 2\}\), as required. □

Deaconescu and Silberberg [1] have proved that a finite \(p\)-group with non-abelian or cyclic second centre is a \(\varphi\)-group if and only if \(G \cong Q_{2^n}\) for some \(n\). It seems reasonable to ask whether there are finite \(2\)-groups with non-cyclic abelian second centre that are \(\varphi\)-groups. The following example shows that given any positive integer \(m \geq 3\), there exists a finite \(2\)-group \(G\) with non-cyclic abelian second centre that is a \(\varphi\)-group of class \(m\).

**Example.** Let \(n\) be a positive integer, and let

\[
G_n = \langle a, b \mid b^4 = 1, b^2 = a^{2n+1}, b^{-1}a^2b = a^{-2}, \ [a, b]^{2^n} = 1 \rangle.
\]

It is easy to check that the following relations hold in \(G_n\):

\[
[a, b]^b = [a, b]^{-1}, \quad [a, b]^a = a^{-4}[a, b]^{-1}, \quad [a^2, [a, b]] = 1.
\]

Taking \(x = a^2\), \(y = [a, b]\), and \(L = \langle x, y \rangle\), we observe that \(L\) is an abelian subgroup of \(G_n\) with \(|G_n : L| = 4\). Using the procedure described in [3], a presentation on the generators \(x\) and \(y\) is obtained for \(L\) as follows:

\[
L = \langle x, y \mid x^{2n+1} = y^{2^n} = [x, y] = 1 \rangle.
\]

Hence \(G_n\) is of order \(2^{2n+3}\), \(|a| = 2^{n+2}\) and \(|b| = 4\). Next we put \(H = \langle a^4, [a, b] \rangle\) and see that \(H\) is an abelian normal subgroup of \(G_n\) and that \(|G_n : H| = 8\). As \(G_n/H\) is abelian and \(|G_n/G_n'| = 8\), we have \(G_n' = H\). Now by considering the normal subgroup \(K = \langle a^2 \rangle\) of \(G_n\), we find that

\[
G_n/K = \langle \bar{a}, \bar{b} \mid \bar{a}^2 = \bar{b}^2 = 1, \ [\bar{a}, \bar{b}]^{2^n} = 1 \rangle \cong D_{2^{n+2}},
\]

where \(\bar{g} = Kg\) for any \(g \in G_n\).

Hence \(Z(G_n/K) = \langle K[a, b]^{2^{n-1}} \rangle\), and we see that if \(z \in Z(G_n)\) then \(z = k[a, b]^{2^{n-1}}\), where \(k \in K\) (because \(Z(G_n)K/K \leq Z(G_n/K)\)). Therefore,

\[
1 = [a, z] = [a, [a, b]^{2^{n-1}}] = [a, [a, b]]^{2^{n-1}} = a^{2^{n+1}}[a, b]^{2^n} = b^2.
\]

Since \(b^2 \neq 1\), we get \(Z(G_n) \leq K\). Now we suppose that \(z\) is a generator of \(Z(G_n)\), and \(z = (a^2)^i\). Then \((a^2)^i = (a^2)^{2i} = a^{-2i}\), and hence \((a^2)^i = 1\), which shows that \(i = 2^n\). It follows that \(z = a^{2^{n+1}} = b^2\), and \(Z(G_n) \leq G_n'\).
Finally we show that $c\ell(G_n) = n + 2$. Obviously $\Gamma_2(G_n) = H$. Now since $H$ is abelian, $\Gamma_3(G_n) = [G_n, H] = \langle a^3, [a, b]^2 \rangle$. Inductively one can show that $\Gamma_i(G_n) = \langle a^{2^{i-1}}, [a, b]^{2^{i-2}} \rangle$ for $i \geq 3$. Hence $\Gamma_{n+2}(G_n) = \langle b^2 \rangle$ and $\Gamma_{n+3}(G_n) = 1$, proving that $c\ell(G_n) = n + 2$. Also, since $\Gamma_{n+1}(G_n) \leq Z_2(G_n)$, we see that $Z_2(G_n)$ has a subgroup of type $Z_2 \times Z_4$. In fact an easy calculation within $G_n$ shows that $Z_2(G_n) \cong Z_2 \times Z_4$.

It is worth noting that $\text{Aut}_c(G_n) \cong Z_2 \times Z_2$ by [2]. In fact, $\text{Aut}_c(G_n) = \langle \alpha, \beta \rangle$, where $\alpha(a) = a$, $\alpha(b) = b^{-1}$ and $\beta(a) = ab^2$, $\beta(b) = b$.

REFERENCES