THE HYPERPLANES OF $DW(5, q)$ WITH NO OVOIDAL QUAD

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Abstract. Let $\Delta$ be one of the dual polar spaces $DW(5, q)$ or $DH(5, q^2)$. We consider a class of subspaces of $\Delta$, each member of which carries the structure of a near hexagon, and classify all these subspaces. Using this classification, we determine all hyperplanes of $DW(5, q)$ without ovoidal quads.

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1. Introduction. Let $\Delta$ be a finite thick dual polar space of rank 3. It is the dual geometry of the singular subspaces of a non-degenerate finite polar space $\Pi$ of rank 3 with at least 3 points on every line and at least 3 planes through every line. By Tits' classification of polar spaces ([12]), $\Pi$ is either a symplectic polar space $W(5, q)$, one of the orthogonal polar spaces $Q(6, q)$ and $Q^{-}(7, q)$, or one of the hermitian polar spaces $H(5, q^2)$ or $H(6, q^2)$, for some prime power $q$. Accordingly, one denotes $\Delta$ by $DW(5, q)$, $DQ(6, q)$, $DQ^{-}(7, q)$, $DH(5, q^2)$ or $DH(6, q^2)$.

The elements of type 1, 2 and 3 of $\Delta$ are the planes, lines and points, respectively, of $\Pi$. The point-line residue of a type 3-element $Q$ of $\Delta$ consists of the singular planes and lines of $\Pi$ through the corresponding point $Q$ of $\Pi$, whence they form a generalized quadrangle. Therefore, the elements of type 1, 2 and 3 of $\Delta$ are called points, lines and quads. We denote the collinearity of $\Delta$ by $\perp$. The dual polar space $\Delta$ is a near hexagon (Shult and Yanushka [11]) which means that for every point $p$ and every line $L$, there exists a unique point on $L$ nearest to $p$.

A hyperplane of $\Delta$ is a proper subspace meeting every line. If $H$ is a hyperplane of $\Delta$ then, for every quad $Q$ of $\Delta$, either $Q \subset H$ or $Q \cap H$ is a hyperplane of $Q$. Hence, one of the following possibilities occurs (see Payne and Thas [6, 2.3.1]).

- $Q \subset H$: in this case $Q$ is called a deep quad.
- $Q \cap H = p \perp \cap Q$ for some point $p$ of $Q$: in this case $Q$ is called a singular quad with deep point $p$.
- $Q \cap H$ is an ovoid: in this case $Q$ is called an ovoidal quad.
- $Q \cap H$ is a proper subquadrangle of $Q$: in this case $Q$ is called a subquadrangular quad.

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If $H$ is a hyperplane such that all quads not contained in $H$ are of the same kind, then $H$ is called uniform, otherwise nonuniform. A uniform hyperplane is called locally singular, locally subquadrangular or locally ovoidal if all its quads not contained in $H$ are singular, subquadrangular or ovoidal, respectively.

The locally singular hyperplanes of dual polar spaces of rank 3 have been classified in Shult [10] in the finite case and Pralle [8] without the finiteness hypothesis. One class of locally singular hyperplanes are the split Cayley hexagons in the orthogonal dual polar space $DQ(6, F)$ for a field $F$. The only other class of locally singular hyperplanes is the singular hyperplane with deepest point $p$ consisting of the points of the dual polar space at non-maximal distance from $p$.

For finite dual polar spaces, the locally subquadrangular hyperplanes have been classified in Pasini and Shpectorov [5]. The locally ovoidal hyperplanes are precisely the ovoids. Ovoids do not exist in the dual polar spaces $DH(5, q^2)$, $DQ(6, q)$ for $q$ odd, and $DH(6, 4)$, since their quads do not have ovoids. We refer to [6] for the nonexistence of ovoids in $Q'(5, q)$ and $W(q)$, $q$ odd; Brouwer showed by computer that $DH(4, 4)$ has no ovoid. The nonexistence of ovoids in $DW(5, q)$, $q$ even, readily follows from [6, 1.8.5], as has been noticed by Shult. See e.g. [5, Proposition 2.8]. The nonexistence of ovoids in $DW(5, q)$, $q$ odd, has been shown in Cooperstein and Pasini [2]. The existence of ovoids in the dual polar spaces $DH(6, q^2)$, $q \geq 3$, and $DQ^-(7, q)$ is still an open problem.

Pralle [7] showed that every nonuniform hyperplane of $\Delta$ must contain at least one singular quad. All nonuniform hyperplanes without subquadrangular quads have been determined in Pralle [8]. Among the finite thick dual polar spaces of rank 3 only the quads of $DW(5, q)$ and $DH(5, q^2)$ have subquadrangles as hyperplanes. All hyperplanes of $DH(5, q^2)$ have been classified in De Bruyn and Pralle [3] and [4]. In Proposition 4.2, we determine all nonuniform hyperplanes of $DW(5, q)$ without ovoidal quads. From [2], [5], [7], [8], [10] and Proposition 4.2 of the present paper, our main result follows.

**Theorem 1.1.** If $H$ is a hyperplane of $DW(5, q)$, then precisely one of the following holds.

- $H$ is a singular hyperplane.
- There exist a quad $Q$ and a subgrid $G$ of $Q$ such that $H = \bigcup_{x \in G} x^\perp$.
- There exist a quad $\bar{Q}$ and an ovoid $O$ in $Q$ such that $H = \bigcup_{x \in O} x^\perp$.
- The order $q$ is even and the points and lines of $H$ build a split Cayley hexagon.
- $H$ is the (up to isomorphism) unique locally subquadrangular hyperplane of $DW(5, 2)$.
- There are a point $p$ and a set $O$ of points at distance 2 from $p$ which meets every line at distance 2 from $p$, such that $H = p^\perp \cup O$.
- We have $q = 2$ and $H$ is a hyperplane on 81 points mentioned in (e) of Proposition 4.2.
- There exist a singular, a subquadrangular and an ovoidal quad.

In Section 2, we define a class $C$ of subspaces of $DW(5, q)$ and $DH(5, q^2)$. Each member $S$ of $C$ satisfies the following properties: (i) $S$ contains two disjoint quads, (ii) $S$ is the union of a certain family of quads, (iii) the points and lines in $S$ define a near hexagon. Section 3 is devoted to the classification of the subspaces of $C$ and several new examples of near hexagons are developed. Using this classification in Section 4, we determine all hyperplanes of $DW(5, q)$ without ovoidal quads.
2. A class of subspaces in $DW(5, q)$ and $DH(5, q^2)$ containing two disjoint quads.

The following lemma is straightforward. See Brouwer et al. [1, Lemma 3.1].

**Lemma 2.1.** Let $S$ be a subspace of a dual polar space $\Delta$ of rank 3 with the property that, for every point $x \in S$, there exists a quad $Q_x \subseteq S$ through $x$. Then the points and lines contained in $S$ define a near polygon.

Several nice classes of near hexagons arise in the way described in Lemma 2.1. See [1]. In the present paper, we determine all subspaces of $DW(5, q)$ and $DH(5, q^2)$ belonging to a certain class $C$ of subspaces. All of them satisfy the conditions of Lemma 2.1 and give rise to near hexagons.

If $S$ is a subspace of a dual polar space $\Delta$ of rank 3 such that for every point $x \in S$, there exists a quad $Q_x \subseteq S$ through $x$, then one of the following two possibilities occurs.

- There exists a point in $\Delta$ that is contained in every quad $Q \subseteq S$.
- There exist two disjoint quads $Q_1$ and $Q_2$ contained in $S$.

The union of a number of quads through a given point is always a subspace. Lemmas 2.2, 2.3 and Proposition 3.1 determine all subspaces of $DW(5, q)$ containing two disjoint quads. All of these subspaces belong to the afore-mentioned class $C$.

Let $\Pi$ be a polar space isomorphic to either $W(5, q)$ or $H(5, q^2)$ embedded in a 5-dimensional projective space $\mathbb{P}$ and let $\Delta$ denote the corresponding dual polar space. For every quad $Q$ of $\Delta$ and every point $x$ not in $Q$, let $\pi_Q(x)$ denote the unique point of $Q$ collinear with $x$. Let $Q_1$ and $Q_2$ denote two disjoint quads of $\Delta$. The quad $Q_i$, $i \in \{1, 2\}$, corresponds with a point $x_i$ of the polar space $\Pi$. Let $x_1, \ldots, x_{q+1}$ denote the $q + 1$ points of $\Pi$ on the line $x_1x_2$ of $\mathbb{P}$ and let $Q_i$, $i \in \{3, \ldots, q + 1\}$, denote the quad of $\Delta$ corresponding with $x_i$. Every line meeting $Q_1$ and $Q_2$ also meets every $Q_i$, $i \in \{3, \ldots, q + 1\}$.

Let $A$ denote the set of all quads meeting $Q_1$ and $Q_2$. It corresponds to the set of points of $\Pi$ contained in $(x_1x_2)^c$ where $\zeta$ denotes the polarity defining $\Pi$. If $R \in A$, then $R$ meets each $Q_i$, $i \in \{1, \ldots, q + 1\}$, in a line and $R \cap (Q_1 \cup \cdots \cup Q_{q+1})$ is a subgrid $G_R$ of $R$. If $x$ is a point of $\Delta$ not contained in $Q_1 \cup \cdots \cup Q_{q+1}$, then $x$ is contained in the unique element of $A$ that is the quad containing $x$ and the line through $\pi_Q(x)$ meeting $Q_2$. We have seen that the quads of $A$ partition the set of points of $\Delta$ not contained in $Q_1 \cup \cdots \cup Q_{q+1}$. Every element $R$ of $A$ corresponds with a point $x_R$ of $\Pi$ that is contained in the three dimensional subspace $\mathbb{P}' = (x_1x_2)^c$ of $\mathbb{P}$. Two quads $R_1$ and $R_2$ of $A$ meet if and only if $x_{R_1} \in x_{R_2}^c$. For a subset $B$ of $A$, we define

$$S_B := (Q_1 \cup Q_2 \cup \cdots \cup Q_{q+1}) \cup \bigcup_{R \in B} R,$$

and $X_B$ denotes the set of all points $x_R$, where $R$ is an element of $B$.

**Lemma 2.2.** If $\Delta \cong DW(5, q)$, then $S_B$ is a subspace of $\Delta$ if and only if $x_1x_2 \subseteq X_B$, for all $x_1, x_2 \in X_B$ with $x_1 \neq x_2$. Let $\Delta \cong DH(5, q^2)$. Then $S_B$ is a subspace of $\Delta$ if and only if $x_1x_2 \cap H(5, q^2) \subseteq X_B$, for all $x_1, x_2 \in X_B$ with $x_1 \neq x_2$.

**Proof.** Let $L$ denote an arbitrary line of $\Delta$. There are four possibilities.

- $L$ is contained in $Q_1 \cup \cdots \cup Q_{q+1}$ and so in $S_B$.
- $L$ intersects $Q_1 \cup \cdots \cup Q_{q+1}$ in a unique point $x$, and there exists a unique quad $R$ of $A$ containing $L$. If $R$ is contained in $B$, then $L \subseteq S_B$. If $R$ is not contained in $B$, then $L \cap S_B = \{x\}$.  

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L is disjoint from $Q_1 \cup \cdots \cup Q_{q+1}$ and contained in a certain quad $R$ of $A$. This case can only occur if $\Delta \cong DH(5, q^2)$. If $R$ is contained in $B$, then $L \subseteq S_B$. If $R$ is not contained in $B$, then $L$ is disjoint from $S_B$.

- $L$ is disjoint from $Q_1 \cup \cdots \cup Q_{q+1}$ and not contained in any quad of $A$. Then the points of $L$ are contained in mutually disjoint elements of $A$ that correspond with a hyperbolic line of $P'$. Conversely, every hyperbolic line of $P'$ corresponds with a set of $q + 1$ mutually disjoint quads of $A$ and there always exists a line outside $Q_1 \cup \cdots \cup Q_{q+1}$ meeting these quads.

Thus $S_B$ is a subspace if and only if every hyperbolic line of $P'$ has either 0, 1 or $q + 1$ points in common with $X_B$. This proves the lemma.

**Definition.** Let $C$ denote the set of all subspaces of the form $S_B$, where $B$ is a subset of $A$.

**Lemma 2.3.** Suppose that $\Delta \cong DW(5, q)$. If $S$ is a subspace of $\Delta$ containing the disjoint quads $Q_1$ and $Q_2$, then $S$ belongs to $C$.

**Proof.** Let $R$ denote an arbitrary element of $A$. The subgrid $G_R$ of $R$ is contained in $S$ and $R \cap S$ is a subspace of $R$. If $R \setminus G_R$ contains a point of $S$, then $R$ is contained in $S$, and $S$ is of the form $S_B$, for some subset $B$ of $A$.

In the following section, we determine all subspaces of $C$, or equivalently, all sets $X_B$ satisfying the conditions of Lemma 2.2.

**Remark.** If $S$ is a subspace of $\Delta \cong DH(5, q^2)$ containing the disjoint quads $Q_1$ and $Q_2$, then the set $X_B$ with $B$ the set of all quads that are contained in $S$ and intersect $Q_1$ and $Q_2$ in lines, still satisfies the conditions of Lemma 2.2.

### 3. The classification of the subspaces of $C$.

The classification follows from Lemma 2.2 and Propositions 3.1 and 3.2.

**Proposition 3.1.** Let $\zeta$ denote a symplectic polarity of $PG(3, q)$ and let $X$ be a set of points of $PG(3, q)$ with the property that $x_1 x_2 \subseteq X$ (1) for every two points $x_1$ and $x_2$ of $X$ with $x_1 \notin x_2^\zeta$. Then one of the following cases occurs:

- (a) $X \subseteq L$, for some totally isotropic line $L$;
- (b) $X = L$, for some hyperbolic line $L$;
- (c) $X = L \cup \{x\}$, for some hyperbolic line $L$ and some point $x \in L^\zeta$;
- (d) $X = L \cup L^\zeta$, for some hyperbolic line $L$;
- (e) $X = p^\zeta \setminus \{p\}$, for some point $p$ of $PG(3, q)$;
- (f) $X = p^\zeta$, for some point $p$ of $PG(3, q)$;
- (g) $X = PG(3, q)$;
- (h) $q = 2$ and $X$ is the complement of an ovoid of the generalized quadrangle $Q_{\zeta} \cong W(2)$ associated with $\zeta$.

**Proof.** If $x_1 \in x_2^\zeta$, for all points $x_1$ and $x_2$ of $X$, then $X$ is as in (a). Suppose that there exist points $x_1, x_2 \in X$ such that $x_1 \notin x_2^\zeta$, and let $L$ denote the line through $x_1$ and $x_2$. If $X$ has no points outside $L \cup L^\zeta$, then one of the cases (b), (c) or (d) occurs.

If there exists a point $x_3 \in X$ not contained in $L \cup L^\zeta$, then let $p$ be the point of $PG(3, q)$ such that $p^\zeta = \{x_1, x_2, x_3\}$. Property (1) implies that every point of $p^\zeta \setminus \{p\}$ is contained in $X$. If all points of $X$ are contained in $p^\zeta$, then either case (e) or (f) occurs. Now, suppose that there exists a point $x_4$ in $X$ not contained in $p^\zeta$. If $p \in X$, then
every point of \( \text{PG}(3, q) \setminus x_4^5 \) belongs to \( X \) by property \((*)\). Also every point of \( x_4^5 \) is contained in \( X \) by property \((*)\), since through every point \( y \in x_4^5 \) there exists a line not contained in \( x_4^5 \cup y^5 \). Hence, if \( p \in X \), then \( X = \text{PG}(3, q) \). Suppose therefore that \( p \notin X \) and consider the complement \( X^C := \text{PG}(3, q) \setminus X \) of \( X \). By property \((*)\), every point of \( X \) is contained in \( X^C \) and consider the complement \( X^C := \text{PG}(3, q) \setminus X \) of \( X \). By property \((*)\), every point of \( X^C \) is contained in either the line \( x_4^p \) or the plane \( x_4^5 \). Now, let \( K \) be a line through \( p \) different from \( x_4^p \) and not contained in \( p^5 \). Then \( K \) contains at most two points of \( X \) (namely \( p \) and \( K \cap x_4^5 \)). By property \((*)\), it then follows that (i) \( q + 1 \leq 3 \) or \( q = 2 \), and (ii) \( K \cap x_4^5 \) is contained in \( X^C \). One easily verifies that \( X^C = (x_4^5 \setminus ((x_4^5 \cap p^5) \cup \{x_4\})) \cup (px_4 \setminus \{x_4\}) \) and that the set \( X^C \) is an ovoid of \( Q_5 \cong W(2) \).

**PROPOSITION 3.2.** Let \( H \) denote a hermitian variety in \( \text{PG}(3, q^2) \) and let \( \xi \) denote the hermitian polarity of \( \text{PG}(3, q^2) \) associated with \( H \). Let \( X \) be a set of points of \( H \) with the property that \( x_1 x_2 \cap H \subseteq X \), for every two points \( x_1 \) and \( x_2 \) of \( X \) with \( x_1 \notin x_2^5 \). The lines and points lying in \( H(3, q^2) \) define a generalized quadrangle \( Q(5, q) \) and \( X \) corresponds with a set \( X' \) of lines of \( Q(5, q) \). One of the following cases occurs.

(a) \( X \) is a (possibly empty) set of points on a line of \( H \). \( X' \) is a (possibly empty) set of lines through a given point of \( Q(5, q) \).

(b) \( X = L \cap H \) for some secant line \( L \) (i.e. \( |L \cap H| = q + 1 \)). \( X' \) is a regulus of \( Q(5, q) \).

(c) \( X = (L \cap H) \cup \{x\} \), for some secant line \( L \) and some point \( x \in L^5 \cap H \). \( X' \) consists of a regulus of \( Q(5, q) \) together with a line of its opposite regulus.

(d) \( X = (L \cup L^5) \cap H \), for some secant line \( L \). \( X' \) consists of the lines contained in a subgrid of order \((q, 1)\) of \( Q(5, q) \).

(e) \( X = \alpha \cap H \), for some nontangent plane \( \alpha \). \( X' \) is a regular spread of \( Q(5, q) \).

(f) \( X = \alpha \cap H \), for some tangent plane \( \alpha \). \( X' \) is the set of lines of \( Q(5, q) \) having nonempty intersection with a given line of \( Q(5, q) \).

(g) \( X = (p^5 \cap H) \setminus \{p\} \), for some point \( p \) of \( H \). \( X' \) is the set of lines of \( Q(5, q) \) intersecting a given line in a unique point.

(h) \( X = B \) where \( B \) is a Baer-subplane which is contained in \( p^5 \cap H \), for some point \( p \) of \( B \). There exists a subquadrangle \( Q \cong Q(4, q) \) in \( Q(5, q) \), and \( X' \) consists of all lines of \( Q \) having nonempty intersection with a given line of \( Q \).

(i) \( X = B \setminus \{p\} \) with \( p \) a point of \( H \) and \( B \) a Baer-subplane of \( p^5 \) through \( p \) completely contained in \( H \). There exists a subquadrangle \( Q \cong Q(4, q) \) in \( Q(5, q) \) and \( X' \) consists of all lines of \( Q \) intersecting a given line of \( Q \) in a unique point.

(j) \( X \) is a 3-dimensional Baer-subspace of \( \text{PG}(3, q^2) \) contained in \( H \). \( X' \) consists of all lines contained in a subquadrangle \( Q \cong Q(4, q) \) of \( Q(5, q) \).

(k) \( X = H \). \( X' \) is the whole set of lines of \( Q(5, q) \).

(l) \( q = 2 \) and \( X' \) consists of all 18 lines which are contained in three mutually disjoint grids of \( Q(5, 2) \).

(m) \( q = 2 \) and there exist a subquadrangle \( Q \cong Q(4, 2) \) of \( Q(5, 2) \) and a spread \( S \) in \( Q \) such that \( X' \) consists of all lines of \( Q \) that are not contained in \( S \).

In the remainder of this section, we prove Proposition 3.2.

**LEMMA 3.3.** If \( \alpha \) is a nontangent plane such that \( \alpha \cap H \) contains three noncollinear points of \( X \), then \( \alpha \cap H \) is completely contained in \( X \).

**Proof.** This follows from property \((*)\) and the fact that every subspace of a Steiner system \( S(2, q + 1, q^3 + 1) \) is a point, a line or the whole Steiner system. For, if \( U \) is a proper subspace of \( S(2, q + 1, q^3 + 1) \) containing a line \( L \) and a point not contained in
Lemma 3.4. If $\alpha$ is a tangent plane such that $\alpha \cap X = \alpha \cap H$, then $X$ is of type (f) or (k).

Proof. If all points of $X$ are contained in $\alpha$, then $X$ is of type (f). Suppose that there exists a point $x \in X$ not contained in $\alpha$. Let $x'$ denote an arbitrary point of $H$ not contained in $x^\xi$ and let $x''$ denote the unique point of $\alpha$ on the line $xx'$. If $x'' \in H$, then by property (\star), also $x' \in H$. Suppose $x'' \notin H$. There are at least $q^2 - q - 1 \geq 1$ nontangent planes through $xx'$ not containing the point $\alpha^x$. If $\gamma$ is such a plane, then every point of $\gamma \cap H$ is contained in $X$, by Lemma 3.3. In particular, also $x'$ belongs to $X$, and every point of $H$ outside $x^\xi$ belongs to $X$. Now, let $y$ denote an arbitrary point of $x^\xi \cap H$. There exists a line through $y$ not contained in $x^\xi \cup y^\xi$ and, by property (\star), it follows that $y \in X$. Hence $X = H$.

Lemma 3.5. If $\alpha$ is a tangent plane such that $\alpha \cap X = (\alpha \cap H) \setminus \{\alpha^x\}$, then $X$ is of type (g).

Proof. Suppose the contrary. Then there exists a point $x \in X$ not contained in $\alpha$. The same argument, as in the proof of Lemma 3.4, shows that every point of $H$ not contained in $xx\alpha^x \cup xx'$ is contained in $X$. Let $L$ denote a line through $\alpha^x$ not contained in $\alpha$ such that $L \cap x^\xi \notin H$. The line $L$ is a secant line and property (\star) implies that $\alpha^x \in X$, a contradiction.

Lemma 3.6. Suppose that $q = 2$. If $\alpha$ is a nontangent plane such that $\alpha \cap H = \alpha \cap X$, then $X$ is of type (e), (k) or (l).

Proof. The points of $\alpha \cap H$ correspond with a regular spread $S$ of $Q(5, 2)$. The lines and reguli of $S$ define an affine plane of order 3. If $X' = S$, then $X$ is of type (e). Suppose now that there exists a line $L$ in $X' \setminus S$. Then there exists a unique partition $\{G_1, G_2, G_3\}$ of $Q(5, 2)$ in three subgrids such that each line of $S \cup \{L\}$ is contained in some grid of the partition. Let $T$ denote the set of 18 lines contained in one of the grids of the partition. By property (\star) it follows that $T \subseteq X'$. If $X' = T$, then $X$ is of type (l). If $T$ is a proper subset of $X'$, then property (\star) implies that $X'$ is the whole set of lines of $Q(5, 2)$. In this case $X$ is of type (k).

Lemma 3.7. Suppose that $q \neq 2$. If $\alpha$ is a nontangent plane such that $\alpha \cap H = \alpha \cap X$, then $X$ is of type (e) or (k).

Proof. If all points of $X$ are contained in $\alpha$, then $X$ is of type (e). Now, suppose that there exists a point $x \in X$ not contained in $\alpha$. Let $x'$ denote an arbitrary point of $H$ not contained in $x^\xi$ and let $x''$ denote the unique point of $\alpha$ on the line $xx'$. If $x'' \in H$, then, by property (\star), also $x' \in H$. Suppose that $x'' \notin H$. There are at least $q^2 + 1 - 2(q + 1) \geq 1$ nontangent planes through $xx'$ that intersect $\alpha$ in a secant line. If $\gamma$ is such a plane, then every point of $\gamma \cap H$ is contained in $X$, by Lemma 3.3. In particular, also $x'$ belongs to $X$, so that every point of $H$ outside $x^\xi$ is contained in $X$. A similar argument as in Lemma 3.4 shows that also every point of $x^\xi \cap H$ belongs to $X$. Hence $X$ is of type (k).

In the sequel, we suppose that $X$ is not of type (a), (b), (c), (d), (e), (f), (g), (k) or (l). Then there exist three points $x_1$, $x_2$ and $x_3$ such that $x_2 \notin x_1^x$, $x_3 \notin x_1^x x_2$ and $x_3 \notin (x_1 x_2)^x$. By Lemmas 3.3, 3.6 and 3.7, the plane $\langle x_1, x_2, x_3 \rangle$ must be a tangent plane.
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plane $p^\delta$. The plane $p^\delta$ has a unique Baer-subplane $B$ through $p$, $x_3$ and $x_1x_2 \cap H$, and $B$ is completely contained in $H$. By property ($\ast$), every point of $B \setminus \{p\}$ is contained in $X$. The set $B \setminus \{p\}$ corresponds with a set $Y'$ of lines of a subquadrangle $Q \cong Q(4, q)$ of $Q(5, q)$ intersecting a given line $L$ of $Q$ in a unique point. If all lines of $X'$ are contained in $Q$, then $X$ is of type (h), (i), (j) or (m) corresponding with the respective cases (f), (e), (g) and (h) of Proposition 3.1. Suppose therefore that $X'$ contains a line $L'$ not contained in $Q$. If $L'$ meets $L$, then property ($\ast$) implies that every point of $p^\delta \setminus \{p\}$ belongs to $X$ (use e.g. similar countings as in the proof of Lemma 3.3), contradicting Lemmas 3.4 and 3.5. If $L'$ intersects $Q$ in a unique point not contained in $L'$, then we can choose disjoint lines $M$ and $N$ in $Y'$ such that $L'$ is disjoint from any line of $\{M, N\}^\perp$. Since there is no line in $Q(5, q)$ meeting $L'$, $M$ and $N$, they correspond with a set of three points on $H(3, q^2)$ generating a nontangent plane. This contradicts Lemmas 3.3, 3.6 and 3.7 and we have proved Proposition 3.2.

4. Application to hyperplanes of dual polar spaces.

PROPOSITION 4.1. If $H$ is a hyperplane of $DW(5, q)$ containing two disjoint deep quads, then one of the following cases occurs.

(a) There exists a quad $Q$ and a subgrid $G$ of order $(q, 1)$ in $Q$ such that $H = \bigcup_{x \in G} x^\perp$.

(b) $q = 2$ and $H$ is a locally subquadrangular hyperplane of $DW(5, 2)$.

(c) $q = 2$ and $H$ is a hyperplane of $DW(5, 2)$ with 81 points corresponding with possibility (d) of Proposition 3.1.

Proof. We use the same notations as in Section 2. Let $Q_1$ and $Q_2$ denote two disjoint quads contained in $H$. Since $H$ is a subspace, it is of the form $S_B$, for some subset $B$ of $S$; see Lemma 2.3. The set $X_B$ must correspond with one of the 8 possibilities mentioned in Proposition 3.1. But there exist additional restrictions on the set $X_B$. Since $H$ is a hyperplane, each line of $\Delta$ meets $H$, which implies that every hyperbolic line of $\text{PG}(3, q)$ meets $X_B$. (See the proof of Lemma 2.2.) Possibility (f) of Proposition 3.1 gives rise to a hyperplane of type (a). Possibility (h) of Proposition 3.1 gives rise to a hyperplane of type (b). See also Pasini and Shpectorov [5]. Possibility (d) of Proposition 3.1 gives rise to a hyperplane only when $q$ is equal to 2. All the remaining possibilities do not give rise to hyperplanes.

 Remark. The example mentioned in (c) of Proposition 4.1 is Example 6 of [9].

PROPOSITION 4.2. If $H$ is a hyperplane of $DW(5, q)$ not containing ovoidal quads, then one of the following cases occurs.

(a) $H$ is a singular hyperplane.

(b) There exist a quad $Q$ and a subgrid $G$ of order $(q, 1)$ in $Q$ such that $H = \bigcup_{x \in G} x^\perp$.

(c) $q$ is even and the points and lines contained in $H$ define a split Cayley hexagon.

(d) $q = 2$ and $H$ is a locally subquadrangular hyperplane of $DW(5, 2)$.

(e) $q = 2$ and $H$ is a hyperplane of $DW(5, 2)$ with 81 points corresponding with possibility (d) in Proposition 3.1.

Proof. If there exists no subquadrangular quad, then each quad is either deep or singular and either case (a) or (c) occurs by Shult [10]. If there exist two disjoint deep quads, then either case (b), (d) or (e) occurs by Proposition 4.1. Now, suppose that there exists a subquadrangular quad $Q$ and that any two deep quads meet. Let $G$ denote the subquadrangle $Q \cap H$. For every point $x$ of $G$, let $A_x$ denote the number
of lines through \( x \) that are contained in \( H \) but not in \( Q \). For every point \( x \) of \( G \), we choose a line \( L_x \subset Q \) through \( x \) not contained in \( G \). There are \( q \) (nonovoidal) quads through \( L_x \) different from \( Q \) in each of which there exists a line through \( x \) contained in \( H \). Hence \( A_x \geq q \), for every point \( x \) of \( G \). Now, consider two disjoint lines \( K_1 \) and \( K_2 \) in \( G \). It is impossible that there exist deep quads through both \( K_1 \) and \( K_2 \), for otherwise we should have two disjoint deep quads. Without loss of generality, we may suppose that there exist no deep quads through \( K_1 \). Now, consider the number \( N = \sum_{x \in K_1} A_x \) which is at least equal to \( q(q+1) \). A quad through \( K_1 \) different from \( Q \) contributes \( q \) to \( N \) if it is singular and \( q+1 \) if it is subquadrangular. Since there are precisely \( q \) quads through \( K_1 \) different from \( Q \), it follows that \( A_x \leq q(q+1) \), and we conclude the following statements.

- Every quad through \( K_1 \) is subquadrangular.
- Every quad through \( L_x \), where \( x \in K_1 \), that is different from \( Q \) is singular.

Now, let \( Q' \) denote an arbitrary subquadrangular quad through \( K_1 \) different from \( Q \). Let \( x_1 \) and \( x_2 \) denote two different points of \( K_1 \) and let \( K'_i \), \( i \in \{1, 2\} \), denote the unique line of \( Q' \cap H \) through \( x_i \) different from \( K_1 \). As before there exists an \( i \in \{1, 2\} \) such that every quad through \( K'_i \) is subquadrangular. But this is impossible since the quad \( \langle L_{x_i}, K'_i \rangle \) is singular. □

REFERENCES