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PERMUTATION PROBLEMS AND SPECIAL FUNCTIONS

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1. Quasisymmetric events. Suppose we have n events $A_1, \ldots, A_n$ and let $p(A_{j1}, \ldots, A_{jk})$ be the probability that the events $A_{j1}, \ldots, A_{jk}$ occur jointly. The probability P_0 that none of $A_1, \ldots, A_n$ occur is given by Poincaré's formula, the probabilistic version of the principle of inclusion and exclusion:

$$P_0 = 1 - \sum_j P(A_j) + \sum_{j>k} P(A_jA_k) - \ldots$$

When these events are symmetric $p(A_{j1}, \ldots, A_{jk})$ depends only on k, say it is ϕ_k, and P_0 has the symbolic form $(1 - E)^n\phi_0$ with $E^k\phi_0 = \phi_k$. Kaplansky [**14**] introduced the concept of *quasisymmetry* where $p(A_{i1}, \ldots, A_{ik})$ is either ϕ_k or zero; see also Mendelsohn [**18**]. For quasisymmetric events we first suppress all the vanishing terms in P_0 then replace ϕ_k by $E^k\phi_0$ to get the symbolic formula

$$(1.1) \qquad P_0(n) = P_0 = f(E)\phi_0,$$

where $f(E)$ is a polynomial that we shall call "the fundamental polynomial." This symbolic device also gives for P_r, the probability that exactly r of the events occur, the symbolic representation

$$(1.2) \qquad P_r(n) = P_r = f(E)\psi_0,$$

where

$$(1.3) \qquad \psi_k = (-1)^r\binom{k}{r}\phi_k.$$

Let us write the fundamental polynomial $f(E)$ as

$$(1.4) \qquad f(x) = \sum_{k=0}^{n} a_{n,k}x^k.$$

By Boas' theorem [**5**] we can represent $\phi_0, \phi_1, \ldots, \phi_n$ as moments

$$(1.5) \qquad \phi_k = \int_0^\infty t^k d\alpha(t), \quad k = 0, 1, \ldots, n,$$

where $\alpha(t)$ is of bounded variation on $(0, \infty)$. The substitutions of (1.5) and

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(1.4) in (1.1) establishes the integral representation

$$(1.6) \qquad P_0(n) = \int_0^\infty f(t)d\alpha(t).$$

Similarly P_r has the integral representation

$$(1.7) \qquad P_r = (-1)^r \int_0^\infty \left\{\sum_{k=0}^n a_{n,k}\binom{k}{r}t^k\right\}d\alpha(t)$$

$$= \frac{(-1)^r}{r!}\int_0^\infty \left\{\sum_{k=r}^n a_{n,k}\frac{k!}{(k-r)!}t^{k-r}\right\}t^r d\alpha(t).$$

Hence

$$P_r(n) = \frac{(-1)^r}{r!}\int_0^\infty t^r\left\{\frac{d^r}{dt^r}f(t)\right\}d\alpha(t).$$

These integral representations have, in our opinion, some advantages over the symbolic representations. They seem to be easier to work with and particularly useful for finding asymptotic formulas for the above probabilities. We hope, in the near future, to derive complete asymptotic expansions for some of the combinatorial numbers treated in the present paper. We also hope that these integral representations might actually settle the Wyman-Moser conjecture [**27**, p. 476] concerning Latin rectangles. In the problems treated in the subsequent sections, the resulting integral representations involve Tchebicheff, Laguerre and Jacobi polynomials. These polynomials satisfy several recurrences; see [**9**; **25**]. Using these recurrences one can derive several recursion relations for the corresponding combinatorial numbers and probabilities.

Recently, Roselle [**22**] discovered the relationship between quasi-symmetric events and labelled digraphs G so that the problems of counting paths and cycles in the complement of G is reduced to calculating the coefficients of the fundamental polynomial associated with these events. This association establishes integral representations for the corresponding graph theoretic problem.

2. Card matching problems. Suppose we permute n objects in such a way that certain positions are prohibited for certain objects. In a card matching problem one is asked to evaluate the probability that these conditions are violated precisely r times and also to compute the number of such ways. Before proceeding to treat specific problems let us introduce some hypergeometric notations.

The shifted factorials $(\sigma)_n$ are defined by

$$(2.1) \qquad (\sigma)_0 = 1 \quad \text{and} \quad (\sigma)_n = \sigma(\sigma+1)\ldots(\sigma+n-1) \quad \text{for } n > 0.$$

A generalized hypergeometric function ${}_pF_q$ is defined as

$$(2.2) \qquad {}_pF_q\left(\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix};x\right) = \sum_{j=0}^\infty \frac{(a_1)_j\ldots(a_p)_j}{(b_1)_j\ldots(b_q)_j}\frac{x^j}{j!}.$$

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The Laguerre polynomials $L_n^{(\alpha)}(x)$ are given by

$$(2.3) \qquad L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!}\,{}_1F_1\left(\begin{matrix}-n\\ \alpha+1\end{matrix};x\right).$$

These polynomials are also called "Rook Polynomials" (see [**21**, pp. 168–174]) because they arise in rectangular chessboard problems involving rooks. We now proceed to treat several card matching problems.

i) *The problème des recontres*, where no object remains in its original place. This problem is also known as the classical derangement problem and Montmort's problem. Kaplansky [**13**; **14**] showed that $\phi_k = (n-k)!/n!$ and $f(E) = (1-E)^n$. These ϕ's have the integral representation

$$\phi_k = \int_0^\infty x^{n-k}\frac{e^{-x}}{n!}\,dx$$

and we get

$$P_0(n) = \int_0^\infty \left\{\sum_{k=0}^n \binom{n}{k}(-1)^k x^{n-k}\right\}\frac{e^{-x}}{n!}\,dx = \int_0^\infty \frac{(x-1)^n}{n!}e^{-x}dx,$$

while the number of ways of the event, N_n, is given by

$$N_n = \int_0^\infty e^{-x}(x-1)^n dx.$$

Integration by parts or writing $(x-1)^n$ as $\sum_{k=0}^n(-1)^{n-k}\binom{n}{k}x^k$ leads to the classical expression

$$N_n = n!\sum_{k=0}^n \frac{(-1)^k}{k!}.$$

Similarly we obtain

$$P_r(n) = \binom{n}{r}\int_0^\infty \frac{(x-1)^{n-r}}{n!}e^{-x}dx = \frac{P_0(n-r)}{r!},$$

as expected.

ii) *Kaplansky's generalization of the derangement problem*. In the present problem (see [**13**]) we have $k_1 + \ldots + k_n$ cards of which k_r are marked "r", $r = 1, 2, \ldots, n$. The cards marked r, or r-cards, are not to appear in any of p_r specified places with no place forbidden simultaneously to both r-cards and s-cards. In [**13**] Kaplansky proved that $\phi_k = k! = \int_0^\infty t^k e^{-t}dt$ and that the number of these arrangements, say $C\begin{pmatrix}k_1, \ldots, k_n\\ p_1, \ldots, p_n\end{pmatrix}$, is $\prod_{j=1}^n H(k_j, p_j, E)\phi_0$, where

$$H(k, p, x) = \frac{x^k}{k!}\,{}_2F_0\left(\begin{matrix}-k, \; -p\\ -\end{matrix};\; -\frac{1}{x}\right).$$

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If $p \leqq k$, $H(k, p, x)$ can be written as

$$H(k, p, x) = \frac{x^k}{k!} \sum_{j=0}^{p} \frac{(-k)_j(-p)_j}{j!} (-x)^{-j}$$
$$= \frac{x^k}{k!} \sum_{j=0}^{p} \frac{(-k)_{p-j}}{(p-j)!} (-p)_{p-j}(-x)^{j-p}$$
$$= \frac{x^{k-p}}{k!} (-k)_p \sum_{j=0}^{p} \frac{(-p)_j}{(k-p+1)_j} \frac{x^j}{j!}$$

that is,

$$(2.3) \qquad H(k, p, x) = x^{k-p}(-1)^p \frac{p!}{k!} L_p^{(k-p)}(x), \quad p \leqq k.$$

Similarly, if $k \leqq p$ we get

$$(2.4) \qquad H(k, p, x) = (-1)^k L_k^{(p-k)}(x), \quad p \geqq k.$$

Therefore

$$(2.5) \qquad C\binom{k_1, \ldots, k_n}{p_1, \ldots, p_n} = \int_0^\infty H(k_1, p_1, x)H(k_2, p_2, x) \ldots H(k_n, p_n, x)e^{-x}dx,$$

where the H's are related to the Laguerre (Rook) polynomials by (2.3) and (2.4). Even and Gillis [**10**] proved (2.5) for $k_j = p_j, j = 1, \ldots, n$ by showing that both sides satisfy the same recurrence. Their argument, however, is both very long and quite complicated. In [**4**], Askey, Ismail and Rashed gave two short proofs of Even and Gillis' representation. The first proof uses MacMahon's Master Theorem [**17**, pp. 93–123] and the second is along the above lines. Furthermore, they obtained the asymptotic formula

$$\lim_{n\to\infty} \frac{(k!)^n}{(kn)!} \underbrace{C\binom{k, k, \ldots, k}{k, k, \ldots, k}}_{n\text{-times}} = e^{-k},$$

or, equivalently,

$$\lim_{n\to\infty} \frac{(k!)^n(-1)^{nk}}{(kn)!} \int_0^\infty e^{-x}\{L_k(x)\}^n dx = e^{-k}.$$

In fact they proved the more general result

$$\lim_{n\to\infty} \frac{(k!)^n(-1)^{nk}}{\Gamma(nk+\alpha+1)} \int_0^\infty e^{-x}x^\alpha\{L_k^\alpha(x)\}^n dx = e^{-k-\alpha}.$$

The numbers

$$G(\alpha; k_1, \ldots, k_n) = \frac{(-1)^{k_1+\ldots+k_n}}{\Gamma(\alpha+1)} \int_0^\infty e^{-x}x^\alpha L_{k_1}^\alpha(x) \ldots L_{k_n}^\alpha(x)dx,$$
$$\alpha = 0, 1, 2, \ldots,$$

also have a combinatorial interpretation. They have the generating function

$$(2.6)\quad \begin{aligned}\Gamma(\alpha+1)&\sum_{k_1,\ldots,k_n=0}^{\infty} G(\alpha;k_1,\ldots,k_n)r_1^{k_1}\ldots r_n^{k_n}\\ &= \int_0^\infty e^{-x}x^\alpha \prod_{j=1}^{n}\left\{(1+r_j)^{-\alpha-1}\exp\frac{xr_j}{1+r_j}\right\}dx,\end{aligned}$$

since the generalized Laguerre polynomials $\{L_n^\alpha(x)\}_{n=0}^\infty$ have the generating function (see [**25**, (5.1.9)])

$$(2.7)\quad \sum_{n=0}^{\infty} L_n^\alpha(x)t^n = (1-t)^{-\alpha-1}\exp\left(\frac{-xt}{1-t}\right).$$

The evaluation of the integral on the right hand side of (2.6) shows that

$$(2.8)\quad \begin{cases}\displaystyle\sum_{k_1,\ldots,k_n=0}^{\infty} G(\alpha;k_1,\ldots,k_n)r_1^{k_1}\ldots r_n^{k_n}\\ \displaystyle\qquad = \left\{1-\sum_{l=0}^{n}\frac{r_l}{1+r_l}\right\}^{-\alpha-1}\prod_{j=1}^{n}(1+r_j)^{-\alpha-1}\\ \qquad = \{1-\sigma_2-2\sigma_3-\ldots-(n-1)\sigma_n\}^{-\alpha-1},\end{cases}$$

where $\sigma_j = \sigma_j(r_1,\ldots,r_n)$ is the elementary symmetric function of degree j, $j = 1,\ldots,n$ in the variables $r_1,\ldots,r_n$. Explicitly,

$$(2.9)\quad \sigma_j = \sum_{i_1<\ldots<i_j} r_{i_1}\ldots r_{i_j}.$$

Note that

$$G(0,k_1,\ldots,k_n) = C\binom{k_1,\ldots,k_n}{k_1,\ldots,k_n}.$$

The generating function (2.7) and

$$\sum_{k_1,\ldots,k_n=0}^{\infty} G(0;k_1,\ldots,k_n)r_1^{k_1}\ldots r_n^{k_n} = \{1-\sigma_2-2\sigma_3-\ldots-(n-1)\sigma_n\}^{-1}$$

show that $G(\alpha;k_1,\ldots,k_n)$ is the α-fold convolution of

$$C\binom{k_1,\ldots,k_n}{k_1,\ldots,k_n},$$

and leads to the following combinatorial interpretation. Assume that we have n boxes and the jth box contains k_j distinguishable objects, $j = 1,2,\ldots,n$. The number of ways of redistributing these objects such that at the end the jth box still has k_j objects and no object remains in its original box is

$$C\binom{k_1,\ldots,k_n}{k_1,\ldots,k_n}.$$

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Instead of doing that we randomly divide the objects in each box to $\alpha + 1$ sets or bundles, (some might be empty), then take a bundle from each box and redistribute the objects in these bundles in such a way that each bundle ends up with the same number of objects it started with and no object remains in its original bundle. The number of ways of doing so is simply $G(\alpha; k_1, \ldots, k_n)$ because it is the α-fold convolution of

$$C\binom{k_1, \ldots, k_n}{k_1, \ldots, k_n}.$$

iii) *The "problème des ménages"*. This problem asks for the number of permutations of $1, 2, \ldots, n$ so that 1 is not in position 1 or 2, 2 is not in position 2 or $3, \ldots, n$ is not in position n or 1. Let u_n be the number of ways of doing so. There is another related problem which is also called the problème des ménages, namely: find the number of ways of seating at a circular table n married couples, husbands and wives alternating, so that no husband is next to his own wife. The number of such ways is $2(n!)u_n$ (see Kaplansky and Riordan [**15**]). It is clear that the probability U_n of the conditions fulfilled is the same in both problems. In [**14**] Kaplansky proved that $\phi_k = (n-k)!/n!$ and $U_n = f(E)\phi_0$ with

$$(2.10)\quad f(E) = \sum_{k=0}^{n} \frac{2n}{2n-k}\binom{2n-k}{k}(-E)^k.$$

The substitution of

$$E^k\phi_0 = \phi_k = \int_0^\infty \frac{t^{n-k}}{n!}e^{-t}dt$$

in (2.10) proves the following integral representation

$$\begin{aligned} U_n &= \sum_{k=0}^{n} \frac{2n}{2n-k}\binom{2n-k}{k}(-1)^k \int_0^\infty \frac{t^{n-k}}{n!}e^{-t}dt \\ &= \frac{(-1)^n}{n!}\int_0^\infty e^{-t}\left\{\sum_{k=0}^{n} \frac{2n}{n+k}\binom{n+k}{n-k}(-t)^k\right\}dt \\ &= 2(-1)^n\int_0^\infty e^{-t}\left\{\sum_{k=0}^{n} \frac{(n)_k(-t)^k}{(2k)!(n-k)!}\right\}dt. \end{aligned}$$

Using the duplication formula $(2k)! = 2^{2k}(k!)(\frac{1}{2})_k$ we get $U_0 = 1$ and

$$(2.11)\quad U_n = \frac{2(-1)^n}{n!}\int_0^\infty e^{-t}{}_2F_1\left(\begin{matrix}-n, n\\ \frac{1}{2}\end{matrix}; \frac{t}{4}\right)dt, \quad n = 1, 2, \ldots.$$

The polynomial ${}_2F_1\left(\begin{matrix}-n, n\\ \frac{1}{2}\end{matrix}; \frac{t}{4}\right)$ is a multiple of the Tchebycheff polynomial of the first kind, see Szegö [**25**]. Furthermore the menages numbers u_n have the

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integral representation

$$u_n = 2(-1)^n \int_0^\infty e^{-t} {}_2F_1\left(\begin{matrix}-n, n\\ \frac{1}{2}\end{matrix}; \frac{t}{4}\right) dt, \quad n = 1, 2, \ldots.$$

Let $u_{n,r}$ be the number of permutations with r elements in forbidden positions. These numbers have the integral representation

$$\begin{aligned} u_{n,r} &= \sum_{k=0}^{n} \frac{2n}{2n-k} \binom{2n-k}{k} \binom{k}{r} (-1)^{k+r} \int_0^\infty e^{-t} t^{n-k} dt \\ &= 2n(-1)^{r+n} \int_0^\infty \left\{ \sum_{k=0}^{n-r} \binom{n+k}{n-k} \binom{n-k}{r} \frac{(-t)^k}{n+k} \right\} e^{-t} dt \\ &= \frac{2(n!)(-1)^{r+n}}{r!} \int_0^\infty \left\{ \sum_{k=0}^{n-r} \frac{(n)_k (-t)^k}{(2k)!(n-r-k)!} \right\} e^{-t} dt \\ &= 2(-1)^{n+r} \binom{n}{r} \int_0^\infty {}_2F_1\left(\begin{matrix}-n+r, n\\ \frac{1}{2}\end{matrix}; \frac{t}{4}\right) e^{-t} dt, \end{aligned}$$

that is $u_{n,r} = 0$ if $n < r$ and

$$(2.12) \quad u_{n+r,r} = 2(-1)^n \frac{(r+1)_n}{n!} \int_0^\infty {}_2F_1\left(\begin{matrix}-n, n+r\\ \frac{1}{2}\end{matrix}; \frac{t}{4}\right) e^{-t} dt,$$

$$n = 0, 1, \ldots, \quad r > 0.$$

The polynomials ${}_2F_1\left(\begin{matrix}-n, n+r\\ \frac{1}{2}\end{matrix}; \frac{t}{4}\right)$ are multiples of the Jacobi polynomials $P_n^{(-\frac{1}{2}, r-\frac{1}{2})}\left(1 - \frac{t}{2}\right)$, since

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!} {}_2F_1\left(\begin{matrix}-n, n+\alpha+\beta+1\\ \alpha+1\end{matrix}; \frac{1-x}{2}\right).$$

For properties of Jacobi polynomials see Szegö [**25**].

Touchard [**26**] obtained the following generating function for the ménage numbers u_n

$$(2.13) \quad \sum_{n=0}^{\infty} u_n I_n(2t) = e^{-2t}(1-t)^{-1},$$

where $I_\nu(z)$ is the modified Bessel function

$$I_\nu(z) = \sum_{m=0}^{\infty} \frac{(z/2)^{\nu+2m}}{m!\Gamma(\nu+m+1)} = \frac{(z/2)^\nu}{\Gamma(\nu+1)} {}_0F_1\left(\begin{matrix}-\\ \nu+1\end{matrix}; \frac{z^2}{4}\right).$$

The generating function (2.13) is a Neumann expansion of the function $e^{-2t}(1-t)^{-1}$ in terms of this modified Bessel function. We shall obtain a similar expansion for $u_{n,r}$.

Srivastava [**23**] proved

$$(z/2)^{\delta}{}_pF_q\begin{pmatrix}a_1, \ldots, a_p; & zx\\ b_1, \ldots, b_q & \end{pmatrix}$$

$$(2.15)\qquad = \frac{\Gamma(\delta)}{\Gamma(2\delta)} e^{z/2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (\delta + n)\Gamma(2\delta + n) I_{n+\delta}(z/2)$$

$$\cdot {}_{p+2}F_{q+1}\begin{pmatrix}-n, n + 2\delta, & a_1, \ldots, a_p; & x\\ \delta + \frac{1}{2}, & b_1, \ldots, b_q & \end{pmatrix}.$$

In (2.14) we let $x = u/4$, $z = 4t$, $p = q = 1$, $a_1 = \delta + \frac{1}{2}$, $b_1 = \frac{1}{2}$, $\delta = r/2$ to get

$$(2t)^{r/2}e^{-2t}{}_1F_1\begin{pmatrix}(r + 1)/2; & tu\\ 1/2 & \end{pmatrix}$$

$$(2.15)\qquad = \frac{\Gamma(r/2)}{\Gamma(r)} \sum_{n=0}^{\infty} \cdot \frac{(-1)^n(r/2 + n)\Gamma(r + n)}{n!} I_{n+r/2}(2t)$$

$$\cdot {}_2F_1\begin{pmatrix}-n, n + r; & \frac{u}{4}\\ \frac{1}{2} & \end{pmatrix}.$$

Multiply (2.15) by e^{-u} and integrate over $u \in (0, \infty)$. Using (2.12) and the difference formula $\Gamma(z + 1) = z\Gamma(z)$ establishes the following generating function for $\{u_{n+r,r}\}_{n=0}^{\infty}$:

$$(2t)^{r/2}e^{-2t}{}_2F_1\begin{pmatrix}(r + 1)/2, 1; & t\\ 1/2 & \end{pmatrix}$$

$$(2.16)\qquad = \frac{\Gamma\left(\frac{r}{2} + 1\right)}{\Gamma(r + 1)} \sum_{n=0}^{\infty} \frac{\frac{r}{2} + n}{r + n} I_{n+r/2}(2t)u_{n+r,r}.$$

Touchard's generating function (2.13) is the limiting case $r \to 0$ of (2.16). In evaluating this limit one has to be careful with the term $n = 0$ since $u_{0,0} = \frac{1}{2}\lim_{r\to 0} u_{r,r}$, where $u_{r,r}$ is given by (2.12) for $r > 0$ (not necessarily an integer). We now apply the transformation

$${}_2F_1\begin{pmatrix}a, b; & z\\ c & \end{pmatrix} = (1 - z)^{-b}{}_2F_1\begin{pmatrix}c - a, b; & \frac{z}{z - 1}\\ c & \end{pmatrix}$$

to the left hand side of (2.16) to derive the equivalent generating function

$$(2t)^{r/2}(1 - t)^{-1}e^{-2t}{}_2F_1\begin{pmatrix}-r/2, 1; & \frac{t}{t - 1}\\ 1/2 & \end{pmatrix}$$

$$(2.17)\qquad = \frac{\Gamma\left(\frac{r}{2} + 1\right)}{\Gamma(r + 1)} \sum_{n=0}^{\infty} \frac{\frac{r}{2} + n}{r + n} I_{n+r/2}(2t)u_{n+r,r}.$$

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The above formula (2.17) is a Neumann expansion for

$$(2t)^{r/2}(1-t)^{-1}{}_2F_1\left(\begin{matrix}-r/2,\ 1\\ 1/2\end{matrix};\frac{t}{t-1}\right)$$

in terms of the modified Bessel functions and is a proper extension of Touchard's generating function (2.13). If r is even, then ${}_2F_1\left(\begin{matrix}-r/2,\ 1\\ 1/2\end{matrix};\ z\right)$ is a polynomial of degree $r/2$ in z and is very easy to compute, especially for small r.

iv) *Some related problems.* The analysis of the previous sub-section also gives an integral representation for the number of permutations discordant with two given ones, as follows. Consider one of the given permutations as a permutation of the other one and write it as a product of distinct cycles of length $s_1, s_2, \ldots, s_m$. The corresponding fundamental polynomial is then $U_{s1}, U_{s2} \ldots U_{s_m}$, where U_n is as in subsection (iii) and $U_1 = 1 - E$ (see Kaplansky [**14**]), and $\phi_k = (n-k)!/n!$. Thus when $s_j > 1, j = 1, \ldots, m$ we get

$$\begin{aligned}U_{s1}\ldots U_{sm}\phi_0 &= \frac{1}{n!}\int_0^\infty t^n e^{-t}\prod_{j=1}^m\left\{2(-1)^{s_j}\sum_{k=0}^{s_j}\frac{(s_j)_k(-s_j)_k}{k!(\frac{1}{2})_k}\frac{t^{k-s_j}}{4^k}\right\}dt\\ &= \frac{2^m(-1)^{s1+\ldots+s_m}}{n!}\int_0^\infty t^{n-s1-\ldots-s_m}e^{-t}\\ &\qquad\cdot\prod_{j=1}^m {}_2F_1\left(\begin{matrix}s_j,\ -s_j\\ \frac{1}{2}\end{matrix};\ \frac{t}{4}\right)dt.\end{aligned}$$

If some s's are unity a similar formula holds. The problème des ménages is the case when the two given permutations are $(1, 2, \ldots, n)$ and $(2, 3, \ldots, n, 1)$. Similarly one can find integral representations for the three line Latin rectangles, see Riordan [**20**].

Another related problem is the following. Divide the integers $1, 2, \ldots, n$ into subsets of m each, $m|n$. We permute these integers so that every integer is forbidden to appear in any of the places originally occupied by another of its subset, with its original position permitted. The corresponding fundamental polynomial is (see Kaplansky [**14**])

$$\begin{aligned}f(E) &= \left\{\sum_{j=0}^m\binom{m}{j}E^j\sum_{k=0}^{m-j}\binom{m-j}{k}\binom{m-j}{k}k!(-E)^k\right\}^{n/m}\\ &= \left\{\sum_{j=0}^m\binom{m}{j}E^j\sum_{k=0}^{m-j}\binom{m-j}{k}\binom{m-j}{k}\right.\\ &\qquad\left.\cdot(m-j-k)!(-E)^{m-j-k}\right\}^{n/m},\end{aligned}$$

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so that

$$(2.18)\quad f(E) = \left\{\sum_{k=0}^{m} C_{m,k}E^{m-k}\right\}^{n/m},$$

with

$$C_{m,k} = \sum_{j=0}^{m-k} \binom{m}{j}\binom{m-j}{k}\binom{m-j}{k}(m-j-k)!(-1)^{m-j-k}.$$

Expand the right hand side of (2.18) by the multinomial theorem. A typical term in this expansion is the multinomial coefficient multiplied by $\prod_{l=0}^{n/m}\{C_{m,l}E^{m-l}\}^{p_l}$, where $\sum_{l=0}^{n/m}p_l = n/m$, that is

$$\left\{\prod_{l=0}^{n/m} C_{m,l}^{p_l}\right\}E^{n-\sum_{l=0}^{n/m} lp_l}.$$

The effect of that term, apart from the multinomial coefficient, on ϕ_0 is

$$\left\{\sum_{l=0}^{n/m} C_{m,l}^{p_l}\right\}\left(\sum_{l=0}^{n/m} lp_l\right)!/n!,$$

that is

$$\int_0^\infty \frac{e^{-t}}{n!}t^{\sum_{l=0}^{n/m} lp_l}\left\{\prod_{l=0}^{n/m} C_{m,l}^{p_l}\right\}dt.$$

This proves that

$$\begin{aligned} f(E)\phi_0 &= \int_0^\infty \frac{e^{-t}}{n!}\left\{\sum_{k=0}^{m} C_{m,k}t^k\right\}^{n/m} dt \\ &= \int_0^\infty \frac{e^{-t}}{n!}\left\{\sum_{j=0}^{m} (-m)_j L_j(t)\right\}^{n/m} dt, \end{aligned}$$

where

$$(2.19)\quad L_j(x) = L_j^{0}(x) = {}_1F_1\left(\begin{matrix}-j\\1\end{matrix};\ x\right).$$

3. A discrete analogue of Even and Gillis' integral representation. The Laguerre polynomials $L_n^\alpha(x)$ are orthogonal on $[0, \infty)$ with respect to the weight function $x^\alpha e^{-x}$. The discrete analogues of these polynomials are the Meixner polynomials $M_n(x, \alpha, c)$ (see [**1**])

$$M_n(x, \alpha, c) = {}_2F_1\left(\begin{matrix}-n,\ -x\\ \alpha\end{matrix};\ 1 - c^{-1}\right),\quad 0 < c < 1,\quad \alpha > 0.$$

These polynomials are orthogonal on the discrete set $\{0, 1, 2, \ldots\}$ with respect to the weight function $w(x) = \dfrac{{}^{(\alpha)}x}{x!}\, c^x$, $x = 0, 1, 2, \ldots$. Furthermore they have the generating function (see [**1**])

$$(3.1)\quad \sum_{n=0}^{\infty} \frac{{}^{(\alpha)}n}{n!} M_n(x, \alpha, c)t^n = (1 - t)^{-\alpha}\left(\frac{1 - t/c}{1 - t}\right)^x.$$

In the present section we give a combinatorial interpretation for the numbers

$$(3.2)\qquad A(k_1,\ldots,k_n) = (1-c)(-1)^{k_1+\ldots+k_n} \cdot \sum_{x=0}^{\infty} M_{k_1}(x,1,c)M_{k_2}(x,1,c)\ldots M_{k_n}(x,1,c)c^x.$$

As $c \to 1$ these numbers tend to

$$C\binom{k_1,\ldots,k_n}{k_1,\ldots,k_n}$$

of Section 2 (see (2.5)).

Let us first state the Master Theorem of MacMahon [**17**, pp. 93–98].

THE MASTER THEOREM. *Set*

$$V_n = (-1)^n x_1\ldots x_n \begin{vmatrix} a_{11}-x_1^{-1} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22}-x_2^{-1} & \ldots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \ldots & a_{nn}-x_n^{-1} \end{vmatrix}.$$

Then the coefficient of $x_1^{k_1}x_2^{k_2}\ldots x_n^{k_n}$ in the expansion of $1/V_n$ is the same as the coefficient of the same term in $(a_{11}x_1+\ldots+a_{1n}x_n)^{k_1}\ldots(a_{n1}x_1+\ldots+a_{nn}x_n)^{k_n}$.

In order to find a combinatorial interpretation for $A(k_1,\ldots,k_n)$ of (3.2) we will use the Master Theorem; we start by computing a generating function for these A's. Using (3.2) we obtain the generating function

$$\begin{aligned}&\sum_{k_1,\ldots,k_n=0}^{\infty} A(k_1,\ldots,k_n)r_1^{k_1}\ldots r_n^{k_n}\\ &= (1-c)\sum_{x=0}^{\infty} c^x \prod_{j=1}^{n}\left\{\sum_{k_j=0}^{\infty}(-r_j)^{k_j}M_{k_j}(x,1,c)\right\}\\ &= (1-c)\prod_{l=1}^{n}(1+r_l)^{-1}\sum_{x=0}^{\infty} c^x\prod_{j=1}^{n}\left(\frac{1+r_jc^{-1}}{1+r_j}\right)^x, \text{ by (3.1).}\end{aligned}$$

Simplifying the above generating function we get

$$\begin{aligned}&\sum_{k_1,\ldots,k_n=0}^{\infty} A(k_1,\ldots,k_n)r_1^{k_1}\ldots r_n^{k_n}\\ &= (1-c)\left\{\prod_{l=1}^{n}(1+r_l)^{-1}\right\}\left\{1-c\prod_{j=1}^{k}\left(\frac{1+r_jc^{-1}}{1+r_j}\right)\right\}^{-1}\\ &= (1-c)\left\{1+\sigma_1+\ldots+\sigma_k - c\left(1+\frac{\sigma_1}{c}+\ldots+\frac{\sigma_n}{c^n}\right)\right\}^{-1},\end{aligned}$$

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where $\sigma_j = \sigma_j(r_1, \ldots, r_n)$ is the elementary symmetric function of degree $j, j = 1, \ldots, n$ in the variables $r_1, \ldots, r_n$, as defined by (2.9). This establishes the generating function

$$\text{(3.3)} \quad \sum_{k_1,\ldots,k_n=0}^{\infty} A(k_1, \ldots, k_n) r_1^{k_1} \ldots r_n^{k_n}$$
$$= (1-c)\left\{(1-c) - \left(\frac{1-c}{c}\right)\sigma_2 - \left(\frac{1-c^2}{c^2}\right)\sigma_3 - \ldots - \frac{(1-c^{n-1})}{c^{n-1}}\sigma_n\right\}^{-1}.$$

Now consider the $n \times n$ determinant Γ_n,

$$\text{(3.4)} \quad \Gamma_n(d) = \begin{vmatrix} -y_1 & d & \ldots & d & d \\ 1 & -y_2 & \ldots & d & d \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 1 & 1 & \ldots & -y_{n-1} & d \\ 1 & 1 & \ldots & 1 & -y_n \end{vmatrix},$$

that is the (j, j) entry is $-y_j$, the entries above the main diagonal are all equal to d while the entries below the main diagonal are all equal to unity. Let $\sigma_1', \ldots, \sigma_n'$ be the elementary symmetric functions of the n symbols $y_1, \ldots, y_n$. It is clear that Γ_n is a symmetric function of $y_1, \ldots, y_n$ of degree n. Thus Γ_n can be expressed in terms of the elementary symmetric function $\sigma_1', \ldots, \sigma_n'$. Obviously the coefficient of σ_n' in $\Gamma_n(d)$ is $(-1)^n$. Let

$$\text{(3.5)} \quad \Gamma_n(d) = (-1)^n\sigma_n' + \sum_{j=0}^{n-1} (-1)^j \sigma_j' B_{n-j}(d).$$

From the definition (3.4) of Γ_n it is quite obvious that $B_j(d)$ is the $j \times j$ determinant with zeros along the main diagonal, d in all the entries above the main diagonal and 1 in all the places below the main diagonal, that is,

$$B_j(d) = \begin{vmatrix} 0 & d & \ldots & d & d \\ 1 & 0 & \ldots & d & d \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 1 & 1 & \ldots & 0 & d \\ 1 & 1 & \ldots & 1 & 0 \end{vmatrix}.$$

The determinant $B_j(d)$ is the special case $c = 0$, $a = 1$, $b = d$ of Exercise 8, page 441 in Muir [**19**]. Therefore

$$B_j(d) = (-1)^{j+1}(d + d^2 + \ldots + d^{j-1}) \quad j > 1.$$

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It is clear that $B_1(d) = 0$. Therefore (3.5) reduces to

$$(3.6) \qquad \Gamma_n(d) = (-1)^n\left\{\sigma_n' - \sum_{j=2}^{n}\left(\frac{d^j - d}{d-1}\right)\sigma'_{n-j}\right\}.$$

Setting $y_j = 1/r_j$, $j = 1, \ldots, n$ and $d = 1/c$ in (3.11) we have

$$(3.7) \qquad (-1)^n r_1 \ldots r_n \Gamma_n(1/c) = 1 - \sum_{j=2}^{n}\left(\frac{1-c^{j-1}}{1-c}\right)\frac{\sigma_j}{c^{j-1}}.$$

From the Master Theorem, (3.3), and (3.7), we see that $A(k_1, \ldots, k_n)$ is the coefficient of $r_1^{k_1}r_2^{k_2}\ldots r_n^{k_n}$ in $R_1^{k_1}R_2^{k_2}\ldots R_n^{k_n}$, where

$$\begin{bmatrix} R_1 \\ R_2 \\ \cdot \\ \cdot \\ \cdot \\ R_n \end{bmatrix} = \begin{bmatrix} 0 & c^{-1} & c^{-1} & \ldots & c^{-1} & c^{-1} \\ 1 & 0 & c^{-1} & \ldots & c^{-1} & c^{-1} \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ 1 & 1 & 1 & \ldots & 0 & c^{-1} \\ 1 & 1 & 1 & \ldots & 1 & 0 \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \\ \cdot \\ \cdot \\ \cdot \\ r_r \end{bmatrix}.$$

Consider the following combinatorial problem. We have n boxes, where the jth box contains k_j objects of type j, $j = 1, \ldots, n$. We redistribute these objects in such a way that each box ends up with the same number of objects it originally contained and no object remains in its original container. Moreover we assign weights to the above derangements. A derangement has the weight $c^{-\theta}$ where θ is the number of objects that ended up in a box of lower index than the box it originally occupied; that is, θ is the number of objects that "retreated." It is clear there is a one to one correspondence between these weighted derangements and the contribution of different factors of $R_1^{k_1}R_2^{k_2}\ldots R_n^{k_n}$ to the coefficient of $r_1^{k_1}\ldots r_n^{k_n}$ in $R_1^{k_1}R_2^{k_2}\ldots R_n^{k_n}$. This proves the following theorem.

THEOREM 3.1. *The numbers* $A(k_1, \ldots, k_n)$ *of* (3.2) *are the sum of the above mentioned weighted derangements.*

4. Further integrals of products of Laguerre polynomials. MacMahon's Master Theorem can be combined with some work on rational functions with positive power series coefficients to obtain some results which are far from obvious. One example is the following.

The coefficient of $r_1^{k_1}r_2^{k_2}r_3^{k_3}$ in the expansion of

$$(4.1) \qquad (r_1 - r_2 - r_3)^{k_1}(-r_1 + r_2 - r_3)^{k_2}(-r_1 - r_2 + r_3)^{k_3}$$

is a positive integer.

The first problem which leads to a result of this type is an old conjecture of

Friedrichs and Lewy. They conjectured that

$$\text{(4.2)}\quad \frac{1}{(1-r)(1-s)+(1-r)(1-t)+(1-s)(1-t)} = \sum_{k,m,n=0}^{\infty} A_{k,m,n} r^k s^m t^n$$

has positive power series coefficients. Szegö [**24**] proved this and also gave a couple of extensions. Other proofs were given by Kaluza [**12**], and Askey and Gasper [**2**]. Askey and Gasper [**3**] proved the stronger result that

$$\text{(4.3)}\quad \frac{1}{(1-r)(1-s)(2+t)+(1-r)(2+s)(1-t)+(2+r)(1-s)(1-t)} = \sum_{k,m,n=0}^{\infty} B_{k,m,n} r^k s^m t^n,$$

has positive power series coefficients. Stated in this form it is impossible (at least at present) to see that (4.3) implies (4.2). The clue to the "right" way to consider (4.2) was given by Szegö [**24**] when he stated that

$$\text{(4.4)}\quad A_{k,m,n} = \int_0^\infty L_k(x)L_m(x)L_n(x)e^{-3x}dx.$$

Similarly

$$\text{(4.5)}\quad B_{k,m,n} = \frac{1}{3}\int_0^\infty L_k(x)L_m(x)L_n(x)e^{-2x}dx.$$

Since we have earlier seen that

$$\text{(4.6)}\quad (-1)^{k+m+n}\int_0^\infty L_k(x)L_m(x)L_n(x)e^{-x}dx \geqq 0,$$

this naturally raises the question of what is the sign behavior of

$$\text{(4.7)}\quad \int_0^\infty L_{n_1}(x)\ldots L_{n_k}(x)e^{-\mu x}dx.$$

When $k = 1$ or 2 it is possible to answer this question completely, but the answer is not totally typical of the more general case.

THEOREM 4.1. *The following inequalities hold.*

$$\text{(4.8)}\quad \int_0^\infty L_n(x)e^{-\mu x}dx \geqq 0, \quad \mu \geqq 1$$

$$\text{(4.9)}\quad (-1)^n\int_0^\infty L_n(x)e^{-\mu x}dx \geqq 0, \quad 0 < \mu \leqq 1$$

$$\text{(4.10)}\quad \int_0^\infty L_m(x)L_n(x)e^{-\mu x}dx \geqq 0, \quad \mu \geqq 1$$

$$\text{(4.11)}\quad (-1)^{m+n}\int_0^\infty L_m(x)L_n(x)e^{-\mu x}dx \geqq 0, \quad 0 < \mu \leqq 1.$$

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Since $L_0(x) = 1$, it will be sufficient to prove (4.10) and (4.11). The second of these holds in general, and so will be given below. The other one is an instance of an important result of Karlin and McGregor [**16**]. If $\{p_n(x)\}_0^\infty$ is a set of polynomials orthogonal on $[0, \infty)$ with respect to a positive measure $d\alpha(x)$, if the polynomials are normalized by $p_n(0) = 1$, and if $\epsilon > 0$ then

$$(4.12) \quad \int_0^\infty p_m(x)p_n(x)e^{-\epsilon x}d\alpha(x) > 0, \quad m, n = 0, 1, \ldots .$$

If $p_n(x) = L_n(x)$, $d\alpha(x) = e^{-x}dx$ and $\epsilon = \mu - 1 > 0$ then (4.12) reduces to (4.10), and when $\mu = 1$ (4.10) is just the orthogonality relation

$$\int_0^\infty L_m(x)L_n(x)e^{-x}dx = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}.$$

The case $k = 3$ is quite interesting and complete results are unknown. The state of present knowledge is summarized in

THEOREM 4.2. *Define* $A(k, m, n; \mu)$ *by*

$$(4.13) \quad A(k, m, n; \mu) = \int_0^\infty L_k(x)L_m(x)L_n(x)e^{-\mu x}dx, \quad \mu > 0.$$

Then

$$(4.14) \quad \sum_{k,m,n=0}^\infty A(k, m, n; \mu)r^k s^m t^n$$
$$= [\mu + (1 - \mu)(r + s + t) + (\mu - 2)(rs + st + tr) + (3 - \mu)rst]^{-1}.$$

The sign behavior of these integrals is given by

$$(4.15) \quad A(k, m, n; \mu) > 0 \qquad \mu \geqq 2, \qquad k, m, n = 0, 1, \ldots,$$

$$(4.16) \quad (-1)^{k+m+n}A(k, m, n; \mu) \geqq 0 \quad 0 < \mu \leqq 1, \quad k, m, n = 0, 1, \ldots,$$

$$(4.17) \quad (-1)^k A(k, n, n; \tfrac{3}{2}) \geqq 0, \qquad k = 0, 1, \ldots, n.$$

The generating function follows easily from (2.7). Inequality (4.16) for $\mu = 1$ is quite old, having been given by Erdélyi [**8**]. Other proofs and references are given in [**6**].

The remaining results in (4.16) follow from

$$(4.18) \quad L_n(\mu x) = \sum_{k=0}^n \binom{n}{k}\mu^{n-k}(1 - \mu)^k L_{n-k}(x) \quad [\mathbf{25}, \text{problem } 67].$$

This was pointed out by Gillis [**11**]. The inequality (4.15) for $\mu = 2$ was proved by Askey and Gasper [**3**], and this inequality for $\mu > 2$ also follows from (4.18) and the case $\mu = 2$. The details are given in Askey and Gasper [**3**]. The case $\mu = 3$ is equivalent to the Friedrichs and Lewy conjecture. Finally the inequality (4.17) was proved by Debbi and Gillis [**7**].

Next we see what MacMahon's Master Theorem implies when used in connection with Theorem 4.2.

The generating function (4.14) is

$$\frac{1}{1-(1-1/\mu)(r+s+t)+(1-2/\mu)(rs+st+tr)-(1-3/\mu)rst}$$
$$=\sum_{k,m,n=0}^{\infty} \mu A(k,m,n;\mu) r^k s^m t^n.$$

A simple calculation shows that

$$1-(1-1/\mu)(r+s+t)+(1-2/\mu)(rs+st+tr)-(1-3/\mu)rst$$

$$(4.19)\qquad = -rst\begin{vmatrix} -1/r+(1-1/\mu) & a & -ab\mu \\ 1/\mu^2 a & -1/s+(1-1/\mu) & b \\ -1/\mu^3 ab & 1/\mu^2 b & -1/t+(1-1/\mu) \end{vmatrix}$$

$$= \begin{vmatrix} 1-(1-1/\mu)r & -as & ab\mu t \\ -r/\mu^2 a & 1-(1-1/\mu)s & -bt \\ r/\mu^3 ab & -s/\mu^2 b & 1-(1-1/\mu)t \end{vmatrix}$$

MacMahon's Master Theorem says that if

$$(4.20)\qquad [(1-1/\mu)r+as-ab\mu t]^k[(1/\mu^2 a)r+(1-1/\mu)s+bt]^m[(-1/\mu^3 ab)r + (1/\mu^2 b)s+(1-1/\mu)t]^n = \sum_{j_1,j_2,j_3} c(j_1,j_2,j_3) r^{j_1}s^{j_2}t^{j_3}$$

then

$$c(k,m,n) = \mu A(k,m,n;\mu).$$

Thus if $\mu \geqq 2$ the coefficient of $r^k s^m t^n$ in the expansion (4.20) is positive. The special case $\mu = 2$, $a = b = -\frac{1}{2}$ is equivalent to the first result mentioned at the beginning of this section (see (4.1)). The special case $\mu = 3$, $a = b = -\frac{1}{3}$ is equivalent to

$$(4.21)\qquad (2r-s-t)^k(-r+2s-t)^m(r+s-2t)^n = \sum_{j_1,j_2,j_3} c(j_1,j_2,j_3) r^{j_1}s^{j_2}t^{j_3}$$

with $c(k,m,n) > 0$. These two results are far from obvious. If $\mu = 2$, $a = b = \frac{1}{2}$ then the result is equivalent to the positivity of the coefficient of $r^k s^m t^n$ in the expansion of

$$(4.22)\qquad (r+s-t)^k(r+s+t)^m(-r+s+t)^n.$$

This seems much more reasonable, but it is still far from obvious. Another fact which is not obvious is that $c(k,m,n)$ in (4.20) is independent of a and b.

When $0 < \mu \leqq 1$ then

$$(-1)^{k+m+n} c(k, m, n) \geqq 0.$$

For $\mu = 1$ this is equivalent to the nonnegativity of the coefficient of $r^k s^m t^n$ in the expansion of

$$(-as + abt)^k (-r/a - bt)^m (r/ab - s/b)^n. \tag{4.23}$$

This is obvious when $a = b = -1$, but is not obvious when $a = b = 1$.

Finally (4.17) implies the nonnegativity of the coefficient of $r^k (st)^n$ in

$$(-r - as + (ab/2)t)^k (4r/a + s + bt)^n ((-8/ab)r + (4/b)s + t)^n, \quad k = 0, 1, \ldots, n. \tag{4.24}$$

Again this does not seem to be obvious for any choice of a and b.

Similar results can be obtained for k variables. One consequence of (4.15) and (4.16) in Theorem 4.2 is the following.

THEOREM 4.3. *If* $0 < \mu \leqq 1$ *then*

$$(-1)^{n_1 + \ldots + n_k} \int_0^\infty L_{n_1}(x) \ldots L_{n_k}(x) e^{-\mu x} dx \geqq 0. \tag{4.25}$$

If $\mu \geqq k - 1$ *then*

$$\int_0^\infty L_{n_1}(x) \ldots L_{n_k}(x) e^{-\mu x} dx \geqq 0. \tag{4.26}$$

The combinatorial interpretation of (4.25) when $\mu = 1$ is the result which started our thinking on these problems, and it has been adequately treated above in [**4**] and in [**10**]. Another interesting consequence follows from (4.26) when $\mu = k - 1$ and $\mu = k$. We will treat only one of many possible interpretations of it and leave further exploration to the reader or the future. A simple argument using the generating function (2.7) gives

$$\begin{aligned} &\mu \sum_{n_1, \ldots, n_k = 0}^{\infty} r_1^{n_1} \ldots r_k^{n_k} \int_0^\infty L_{n_1}(x) \ldots L_{n_k}(x) e^{-\mu x} dx \\ &= \left[1 + \sum_{j=1}^{k} (-1)^j (1 - j/\mu) \sigma_j \right]^{-1}. \end{aligned} \tag{4.27}$$

To be able to give an interpretation of (4.26) we need to find a_{ij} so that

$$\begin{vmatrix} 1 - a_{11} r_1 & -a_{12} r_2 & \ldots & -a_{1k} r_k \\ -a_{21} r_1 & 1 - a_{22} r_2 & \ldots & -a_{2k} r_k \\ \ldots & \ldots & \ldots & \ldots \\ -a_{k1} r_1 & -a_{k2} r_2 & \ldots & 1 - a_{kk} r_k \end{vmatrix} = 1 + \sum_{j=1}^{k} (-1)^j (1 - j/\mu) \sigma_j.$$

We have not tried to find the general solution to this set of equations but one particular solution is both reasonably easy to find and interesting.

LEMMA 4.4.

$$D_k = \det|I - A_k| = 1 + \sum_{j=1}^{k} (-1)^j (1 - \mu/j)\sigma_j,$$

where I is the identity matrix and A_k is given by

$$a_{ii} = (1 - 1/\mu) r_i, \quad i = 1, \ldots, k$$
$$a_{ij} = (-1/\mu) r_j, \quad i \neq j, \quad 1 \leqq i, \quad j \leqq k.$$

Proof. Factoring out $(-r_1) \ldots (-r_k)$ and subtracting the last row from every other row gives

$$D_k = (-1)^k r_1 \ldots r_k \begin{vmatrix} 1 - 1/\mu - 1/r_1 & -1/\mu & \ldots & -1/\mu \\ -1/\mu & 1 - 1/\mu - 1/r_2 & \ldots & -1/\mu \\ \ldots & \ldots & \ldots & \ldots \\ -1/\mu & -1/\mu & \ldots & 1 - 1/\mu - 1/r_k \end{vmatrix}$$

If we factor $-1/\mu$ out of each row then

$$D_k = r_1 \ldots r_k/\mu^k \begin{vmatrix} 1 - \mu + \mu/r_1 & 1 \ldots 1 \\ 1 & 1 - \mu + \mu/r_2 \ldots 1 \\ , & \\ , & \\ , & \\ 1 & 1 \ldots 1 - \mu + \mu/r_k \end{vmatrix}$$

and this determinant is A_{11} in Exercise 2, p. 389 of Muir [**19**] with $a_i = \mu(-1 + 1/r_i) = \mu(r_i - 1)/r_i$.

$$D_k = (1 - r_1) \ldots (1 - r_{k-1})\left(1 - \frac{\mu - 1}{\mu} r_k\right) - \frac{1}{\mu} \sum_{i=1}^{k-1} r_i \prod_{\substack{j=1 \\ j \neq i}}^{k} (1 - r_j)$$
$$= (1 - r_1) \ldots (1 - r_k) - \frac{1}{\mu} \sum_{i=1}^{k} r_i \prod_{\substack{j=1 \\ j \neq i}} (1 - r_j).$$

The above sum can be simplified to give

$$D_k = 1 + \sum_{j=1}^{k} (-1)^j (1 - j/\mu)\sigma_j.$$

Applying MacMahon's Master Theorem we have the following result.

COROLLARY 4.5. *The coefficient of $r_1^{n_1} \ldots r_k^{n_k}$ in the expansion of*

$$[(k - c)r_1 - r_2 - \ldots r_k]^{n_1}[-r_1 + (k - c)r_2 - r_3 - \ldots - r_k]^{n_2} \ldots [-r_1 - r_2 - \ldots - r_{k-1} + (k - c)r_k]^{n_k}$$

is positive when $c \leqq 2$.

When $k = 3$ and $c = 1$, Corollary 4.5 is a consequence of the Friedrichs-Lewy conjecture and the case $c = 2$ is the result mentioned at the beginning of this section.

The following combinatorial interpretation of Corollary 4.5 can be given. Consider first the case $k = 3$, $c = 2$. Take three boxes with n_1, n_2, n_3 distinguishable objects in them. Rearrange these objects so that n_1 end up in the first box, n_2 in the second, n_3 in the third. In each box the objects are distinguishable, but the order in a given box after the rearrangement does not matter. Then Corollary 4.5 says that the number of such rearrangements in which an even number of objects have moved from their original box to a different box is larger than the number of arrangements in which an odd number of objects have moved.

For example, consider the case of two objects a_1, a_2 in the first box, and one each in the second and third, say b and c respectively. The different rearrangements are

Box 1	Box 2	Box 3
a_1a_2	b	c
a_1a_2	c	b
a_1b	a_2	c
a_1b	c	a_2
a_1c	a_2	b
a_1c	b	a_2
a_2b	a_1	c
a_2b	c	a_1
a_2c	a_1	b
a_2c	b	a_1
bc	a_1	a_2
bc	a_2	a_1

and eight of these have an even number of objects which have moved to a different box while only four have an odd number of objects which have moved. Actually even more can be obtained, since the difference of these two numbers is the coefficient of $r^{n_1}s^{n_2}t^{n_3}$ in the expansion of $(r - s - t)^{n_1}(-r + s - t)^{n_2}(-r - s + t)^{n_3}$, and so by the Master Theorem it is the coefficient of $r^{n_1}s^{n_2}t^{n_3}$ in a certain power series expansion of a generating function; and then it is an appropriate integral of the product of three Laguerre polynomials times e^{-2x}. Explicitly it is

$$C_{n_1,n_2,n_3} = 2^{n_1+n_2+n_3+1} \int_0^\infty L_{n_1}(x)L_{n_2}(x)L_{n_3}(x)e^{-2x}dx,$$

which in the case $n_1 = 2$, $n_2 = 1$, $n_3 = 1$ can be computed to give 4.

The Laguerre polynomials can be expanded using (2.19) to give

$$C_{n_1,n_2,n_3} = 2^{n_1+n_2+n_3} \sum_{k_1,k_2,k_3 \geqq 0} \binom{n_1}{k_1}\binom{n_2}{k_2}\binom{n_3}{k_3} \times \frac{(k_1+k_2+k_3)!}{k_1!k_2!k_3!}\left(-\tfrac{1}{2}\right)^{k_1+k_2+k_3}$$

Again the positivity of this sum is not obvious. There is another sum which was given in [**3**] and repeated in [**1**, pp. 51–54]. A simple calculation from the results given there gives

$$C_{n_1,n_2,n_3} = \frac{2^{2n_3}(n_1+n_2-n_3)!}{(n_2-n_3)!n_1!} \sum_{j=0}^{n_3} \frac{(\frac{1}{2})_{n_3-j}}{(1)_{n_3-j}} \frac{(n_2-n_1-n_3)_{2j}}{(n_2-n_3+1)_j j! 2^{2j}}$$

when $n_2 \geqq n_3$, and this is obviously positive. When n_3 is a small integer this is a useful formula. In particular when $n_3 = 1$ it is

$$C_{n_1,n_2,1} = \frac{(n_1+n_2-1)!}{n_1!n_2!} [(n_1-n_2)^2 + n_1 + n_2], \quad n_1, n_2 \geqq 1.$$

Also when $n_2 = n_1 + n_3$ this formula simplifies. In this case it is

$$C_{n_1,n_1+n_3,n_3} = \binom{2n_1}{n_1}\binom{2n_3}{n_3}.$$

The older result of Szegö on the Friedrichs-Lewy conjecture has a combinatorial meaning of the same sort. Now the rearrangements are listed in two columns, those with an even number of objects which have moved and those with an odd number of objects which have moved. Instead of just counting the number of elements in each column, they are counted with weights. A rearrangement is counted 2^i times when exactly i objects remain in their original boxes. Then Corollary 4.5 with $c = 1$ says that the weighted sum of those rearrangements with an even number of objects that move is greater than the weighted sum of those rearrangements with an odd number of objects that moved. The weighting factor can be a real number $a \geqq 1$, since $c = 3 - a$ and Corollary 4.5 holds when $k = 3$, $c \leqq 2$.

When considering more than three boxes it seems to be essential to consider weighted sums of the rearrangements. For when $k = 5, n_1 = n_2 = n_3 = n_4 = n_5 = 1$ there are 120 different rearrangements, 64 of which have an odd number of elements that move to a different box and only 56 have an even number of elements that move to a different box. However with weights there is a correct theorem.

THEOREM 4.6. *Let k boxes with n_i objects in the ith box be given and assume that all the objects are distinguishable but the order of the objects in a given box does not matter. Rearrange these $n_1 + \ldots + n_k$ objects so that box i still has n_i objects. Form the sum of $(k-2+\mu)^a(-1)^{n_1+\cdots+n_k-a}$ over all such rearrangements, where a is the number of objects in a given rearrangement that stay in the same box. Then this number is positive if $\mu \geqq 0$.*

This is just a reinterpretation of Corollary 4.5. In the case $k = 5$, $n_i = 1$ and $c = 3 + \mu$ the sum is

$$c^5 + 10c^3 - 20c^2 + 45c - 44,$$

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which is -8 (as it should be) when $c = 1$ and is positive when $c \geqq 2$. Theorem 4.6 only gives the positivity for $c \geqq 3$.

Because of these combinatorial interpretations Theorem 4.3 takes on more significance, and the problem of finding further values of μ for which it holds becomes more interesting. Also the question of finding a direct combinatorial proof of the results in this section is clearly of interest.

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