Combinatorics, Probability and Computing (2022), **31**, pp. 976–1009
doi:10.1017/S0963548322000050

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ARTICLE

The chromatic profile of locally colourable graphs

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(Received 10 February 2021; revised 1 February 2022; accepted 4 April 2022; first published online 10 May 2022)

Abstract
The classical Andrásfai-Erdős-Sós theorem considers the chromatic number of K_{r+1}-free graphs with large minimum degree, and in the case, $r=2$ says that any n-vertex triangle-free graph with minimum degree greater than $2/5 \cdot n$ is bipartite. This began the study of the chromatic profile of triangle-free graphs: for each k, what minimum degree guarantees that a triangle-free graph is k-colourable? The chromatic profile has been extensively studied and was finally determined by Brandt and Thomassé. Triangle-free graphs are exactly those in which each neighbourhood is one-colourable. As a natural variant, Luczak and Thomassé introduced the notion of a locally bipartite graph in which each neighbourhood is 2-colourable. Here we study the chromatic profile of the family of graphs in which every neighbourhood is b-colourable (locally b-partite graphs) as well as the family where the common neighbourhood of every a-clique is b-colourable. Our results include the chromatic thresholds of these families (extending a result of Allen, Böttcher, Griffiths, Kohayakawa and Morris) as well as showing that every n-vertex locally b-partite graph with minimum degree greater than $(1 - 1/(b + 1/7)) \cdot n$ is $(b + 1)$-colourable. Understanding these locally colourable graphs is crucial for extending the Andrásfai-Erdős-Sós theorem to non-complete graphs, which we develop elsewhere.

Keywords: Locally Colourable Graphs; Chromatic Profile

2020 MSC Codes: Primary: 05C15, Secondary: 05C35

1 Introduction

In 1973, Erdős and Simonovits [8] asked the following question: for each graph H and positive integer k, what δ guarantees that every n-vertex H-free graph with minimum degree greater than δn is k-colourable? The values of δ, as k varies, form the chromatic profile of H-free graphs. More generally, for a family of graphs $\mathcal{F}$, the *chromatic profile of* $\mathcal{F}$ is the sequence of values $\delta_\chi(\mathcal{F}, k)$ where

$$\delta_\chi(\mathcal{F}, k) = \inf\left\{d\colon \text{ if } \delta(G) \geqslant d|G| \text{ and } G \in \mathcal{F}, \text{then } \chi(G) \leqslant k\right\}.$$

In the case where $\mathcal{F}$ is the family of H-free graphs, we write $\delta_\chi(H, k)$ for $\delta_\chi(\mathcal{F}, k)$. The question of determining $\delta_\chi(H, k)$, first asked in [8], was re-emphasised by Allen et al. [1]. For general H, very little is known about the chromatic profile and indeed Erdős and Simonovits described it as 'too complicated'.

There has been much greater success with the *chromatic threshold*. The *chromatic threshold of a family* $\mathcal{F}$ is the limit (and so infimum) of the sequence $\delta_\chi(\mathcal{F}, k)$: that is,

* Research supported by an EPSRC grant. Current address: Mathematical Institute, University of Oxford, Woodstock Road, OX2 6GG.

$$\delta_\chi(\mathcal{F}) = \inf_k \delta_\chi(\mathcal{F}, k)$$
$$= \inf\{d \colon \exists C = C(\mathcal{F}, d) \text{ such that if } \delta(G) \geqslant d|G| \text{ and } G \in \mathcal{F}, \text{ then } \chi(G) \leqslant C\}.$$

Allen et al. [1] determined the chromatic threshold of H-free graphs for every graph H. They further obtained the chromatic threshold of *locally bipartite graphs* – the family of graphs in which each neighbourhood is bipartite – confirming the conjecture of Łuczak and Thomassé [16] that this threshold is $1/2$.

Understanding the chromatic profile of locally bipartite graphs is a very natural local to global colouring problem: every neighbourhood is 2-colourable and large, so it seems possible that the whole graph might have small chromatic number. A further motive (that originally brought them to our attention) is to extend the Andrásfai-Erdős-Sós theorem [2] – that theorem gives the first interesting value in the chromatic profile of complete graphs, namely

$$\delta_\chi(K_{r+1}, r) = 1 - \frac{1}{r - 1/3}.$$

This theorem can be seen as a minimum degree analogue of Erdős and Simonovits's stability theorem [6, 7, 19] for the structure of K_{r+1}-free graphs with close to the maximum number of edges. Erdős and Simonovits's result says that any H-free graph with $(1 - 1/r - o(1))\binom{n}{2}$ edges (where $r + 1$ is the chromatic number of H) can be made r-partite by deleting $o(n^2)$ edges. The natural minimum degree analogue asks what minimum degree guarantees that an H-free graph can be made r-partite by deleting $o(n^2)$ edges. The theorem of Andrásfai, Erdős and Sós answers this for cliques. It turns out that, in order to extend the theorem to non-complete H (as in [12]), it is crucial to better understand locally colourable graphs (as defined next), and that is our purpose in this paper.

Definition 1.1. A graph is *a-locally b-partite* if the common neighbourhood of every a-clique is b-colourable. A graph is *locally b-partite* if it is 1-locally b-partite: the neighbourhood of every vertex is b-colourable. We use $\mathcal{F}_{a,b}$ to denote the family of a-locally b-partite graphs.

The family of locally bipartite graphs is $\mathcal{F}_{1,2}$. Note that, in general,

$$\mathcal{F}_{1,\ell} \subset \mathcal{F}_{2,\ell-1} \subset \cdots \subset \mathcal{F}_{\ell,1} = \{G \colon G \text{ is } K_{\ell+2}\text{-free}\}.$$

In particular, $\mathcal{F}_{a,b}$ is a subfamily of $K_{\ell+2}$-free graphs where $\ell = a + b - 1$.

As mentioned, the Andrásfai-Erdős-Sós theorem gives the first interesting value in the chromatic profile of $K_{\ell+2}$-free graphs, that is, of the family $\mathcal{F}_{\ell,1}$. For the basic case of triangle-free graphs, this shows $\delta_\chi(K_3, 2) = 2/5$. Hajnal (see [8]) showed that $\delta_\chi(K_3) \geqslant 1/3$. Häggkvist [10] showed that $\delta_\chi(K_3, 3) \geqslant 10/29$ and Jin [13] proved that equality holds. Finally, building on work of Chen et al. [5], Brandt and Thomassé [4] fully described triangle-free graphs with minimum degree above the chromatic threshold (showing that, in fact, $\delta_\chi(K_3) = \delta_\chi(K_3, 4) = 1/3$). Moreover, Goddard and Lyle [9], and Nikiforov [17] independently showed that the chromatic profile of K_{r+2}-free graphs can be derived straightforwardly from that of triangle-free graphs. The chromatic profile of the $K_{\ell+2}$-free graphs, that is, of $\mathcal{F}_{\ell,1}$, has thus been completely determined.

The chromatic profile of the family $\mathcal{F}_{a,b}$ for $b \geqslant 2$ displays substantially different behaviour to the case $b = 1$, however. In Section 2, we determine the chromatic threshold of the family of a-locally b-partite graphs. This result in the special case $a = 1$ and $b = 2$ was conjectured (with a construction for the lower bound) by Łuczak and Thomassé [16] and proved by Allen et al. [1].

Theorem 1.2. *Let a and b be positive integers with $b \geqslant 2$ and let $\ell = a + b - 1$. Then,*

$$\delta_\chi(\mathcal{F}_{a,b}) = 1 - \frac{1}{\ell},$$

and, in particular, the chromatic threshold of locally b-partite graphs, $\delta_\chi(\mathcal{F}_{1,b})$, is $1 - 1/b$.

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For comparison, the chromatic threshold of the family $\mathcal{F}_{\ell,1}$ of $K_{\ell+2}$-free graphs is $1 - 1/(\ell + 1/2)$.

Now for the chromatic profile. All a-locally b-partite graphs are K_{a+b+1}-free, and furthermore, the n-vertex K_{a+b+1}-free graph with highest minimum degree (and most edges), the Turán graph [20], $T_{a+b}(n)$ is a-locally b-partite and $(a + b)$-chromatic. In particular,

$$\delta_\chi(\mathcal{F}_{a,b}, k) = 1 - \frac{1}{a+b} \text{ for } k = 1, 2, \ldots, \ell := a + b - 1,$$

since there are no K_{a+b+1}-free graphs (and so no a-locally b-partite) with $\delta(G) > (1 - 1/(a + b)) \cdot |G|$. In particular, the first interesting value in the chromatic profile of a-locally b-partite graphs is $\delta_\chi(\mathcal{F}_{a,b}, \ell + 1)$. An upper bound for this is given by Theorems 1.3 ($b \geqslant 3$) and 1.4 ($b = 2$). In Section 4, we prove Theorem 1.3 for locally b-partite graphs (that is, for $a = 1$), and in Section 5, we extend it to all a and prove Theorem 1.4.

Theorem 1.3. *Let a and b be positive integers with $b \geqslant 3$ and let $\ell = a + b - 1$. Then,*

$$\delta_\chi(\mathcal{F}_{a,b}, \ell + 1) \leqslant 1 - \frac{1}{\ell + 1/7},$$

and, in particular, every locally b-partite graph G with $\delta(G) > (1 - 1/(b + 1/7)) \cdot |G|$ is $(b + 1)$-colourable.

Theorem 1.4. *Let a be a positive integer and $\ell = a + 1$. Then,*

$$\delta_\chi(\mathcal{F}_{a,2}, \ell + 1) = 1 - \frac{1}{\ell + 1/3}.$$

Note that the chromatic threshold of the family of $K_{\ell+2}$-free graphs is $1 - 1/(\ell + 1/2)$. In particular, all chromatic profile values for $K_{\ell+2}$-free graphs are greater than the first interesting value for a-locally b-partite ones ($b \geqslant 2$).

To extend the chromatic profile of triangle-free graphs to K_{r+1}-free graphs as mentioned above, Goddard and Lyle [9], and Nikiforov [17] showed that every n-vertex maximal K_{r+1}-free graph with minimum degree greater than $\delta_\chi(K_{r+1}) \cdot n$ consists of an independent set joined to a K_r-free graph. That is, maximal graphs of $\mathcal{F}_{r,1}$ with sufficiently large minimum degree are obtained from those in $\mathcal{F}_{r-1,1}$ by joining an independent set. A simple induction then converts the structure of triangle-free graphs to the structure of K_{r+1}-free graphs. It is natural to ask whether something similar can be done to convert between different families $\mathcal{F}_{a,b}$. Firstly, there does not seem to be an easy way to convert between $\mathcal{F}_{a,b-1}$ and $\mathcal{F}_{a,b}$ (certainly joining on an independent set fails). Although we obtain the upper bound for $\delta_\chi(\mathcal{F}_{1,b}, b + 1)$ in Theorem 1.3 using knowledge of locally bipartite graphs, it is not a straightforward induction.

Joining on an independent set to a graph in $\mathcal{F}_{a-1,b}$ does give a graph in $\mathcal{F}_{a,b}$, but it is not clear that all maximal graphs in $\mathcal{F}_{a,b}$ of large minimum degree are obtained in this way – the lower value of $\delta_\chi(\mathcal{F}_{a,b})$ for $b \geqslant 2$ means the structural lemma of Goddard, Lyle and Nikiforov does not apply. While our arguments extending results from locally b-partite graphs to a-locally b-partite graphs are simpler, they interestingly do require knowledge of locally b'-partite graphs for all $b' \leqslant a + b$. It seems that the crux of understanding locally colourable graphs is understanding locally b-partite graphs, a sentiment we will crystallise in Section 5.

Some of the required knowledge of locally bipartite graphs was obtained by the author in [11], which included results such as

$$\delta_\chi(\mathcal{F}_{1,2}, 3) = 4/7, \quad \text{and} \quad \delta_\chi(\mathcal{F}_{1,2}, 4) \leqslant 6/11.$$

The results in [11] give detailed information about n-vertex locally bipartite graphs with minimum degree greater than $6/11 \cdot n$, including homomorphism properties. However, in order to prove the upper bound for $\delta_\chi(\mathcal{F}_{1,b}, b + 1)$ in Theorem 1.3, we need to extend our structural knowledge

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of locally bipartite graphs down to 8/15, though, fortunately, we do not need any homomorphism properties. We do this in Section 3 and our full knowledge of locally bipartite graphs is summarised there in Theorem 3.1.

1.1 Notation

Let G be a graph and $X \subset V(G)$. We write $\Gamma(X)$ for $\cap_{v\in X}\Gamma(v)$ (the common neighbourhood of the vertices of X) and $d(X)$ for $|\Gamma(X)|$. We often omit the parentheses so $\Gamma(u, v) = \Gamma(u) \cap \Gamma(v)$ and $d(u, v) = |\Gamma(u, v)|$. We write G_X for $G[\Gamma(X)]$ so, for example, $G_{u,v}$ is the induced graph on the common neighbourhood of vertices u and v. Note that G being a-locally b-partite is equivalent to $\chi(G_K) \leqslant b$ for every a-clique K of G. We make frequent use of the fact that for two vertices u and v of G

$$d(u, v) = d(u) + d(v) - |\Gamma(u) \cup \Gamma(v)| \geqslant d(u) + d(v) - |G| \geqslant 2\delta(G) - |G|.$$

Given a set of vertices $X \subset V(G)$, we write $e(X, G)$ for the number of ordered pairs of vertices (x, v) with $x \in X$, $v \in G$ and xv an edge in G. In particular, $e(X, G)$ counts each edge in $G[X]$ twice and each edge from X to $G - X$ once and satisfies

$$e(X, G) = \sum_{x\in X} d(x) = \sum_{v\in G} |\Gamma(v) \cap X|.$$

We generalise this notation to vertex weightings which will appear in many of our arguments. We will take a set of vertices $X \subset V(G)$ and assign weights $\omega\colon X \to \mathbb{Z}_{\geqslant 0}$ to the vertices of X. Then, we define

$$\omega(X, G) = \sum_{x\in X} \omega(x)d(x) = \sum_{v\in G} \text{Total weight of the neighbours of } v \text{ in } X.$$

We will often use the word *circuit* (as opposed to cycle) in our arguments. A circuit is a sequence of (not necessarily distinct) vertices $v_1, v_2, \ldots, v_\ell$ with $\ell > 1$, v_i adjacent to v_{i+1} (for $i = 1, 2, \ldots, \ell - 1$) and v_ℓ adjacent to v_1. Note that in a locally bipartite graph, the neighbourhood of any vertex does not contain an odd circuit (and of course does not contain an odd cycle). We use circuit to avoid considering whether some pairs of vertices are distinct when it is unnecessary to do so.

Given a graph G, a *blow-up* of G is a graph obtained by replacing each vertex v of G by a non-empty independent set I_v and each edge uv by a complete bipartite graph between classes I_u and I_v. We say we have blown-up a vertex v by n if $|I_v| = n$. It is often helpful to think of this as weighting vertex v by n.

A blow-up is *balanced* if the independent sets $(I_v)_{v\in G}$ are as equal in size as possible. We use $G(t)$ to denote the graph obtained by blowing-up each vertex of G by t: $G(t)$ is the balanced blow-up of G on $t|G|$ vertices. Note, for example, that the balanced blow-ups of the clique, K_r, are exactly the Turán graphs, $T_r(n)$. We note in passing that a graph has the same chromatic and clique number as any of its blow-ups. Furthermore, if H is a blow-up of G, then G is a-locally b-partite if and only if H is a-locally b-partite.

Given two graphs G and H, the *join* of G and H, denoted $G + H$, is the graph obtained by taking disjoint copies of G and H and joining each vertex of the copy of G to each vertex of the copy of H. Note that the chromatic and clique numbers of $G + H$ are the sum of the chromatic and clique numbers of G and H.

A graph G is *homomorphic* to a graph H, written $G \to H$, if there is a map $\varphi\colon V(G) \to V(H)$ such that for any edge uv of G, $\varphi(u)\varphi(v)$ is an edge of H. A graph G is homomorphic to a graph H if and only if G is a subgraph of some blow-up of H. In particular, if $G \to H$, then $\chi(G) \leqslant \chi(H)$ and, moreover, if H is a-locally b-partite, then G is also.

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1.2 Importance to extending the Andrásfai-Erdős-Sós theorem

As previously mentioned, our initial motivation for studying locally colourable graphs came from trying to extend the Andrásfai-Erdős-Sós theorem [2] to non-complete graphs and obtain a minimum degree analogue of Erdős and Simonovits's classical stability theorem [6, 7, 19]. In particular, we wish to determine, for each $(r+1)$-chromatic H, the value of

$$\delta_H = \inf\{c\colon \text{if } |G| = n,\ \delta(G) \geqslant cn,\ \text{and } G \text{ is } H\text{-free}, \\ \text{then } G \text{ can be made } r\text{-partite by deleting } o(n^2) \text{ edges}\}.$$

Here we sketch the link between this and locally colourable graphs deferring a thorough treatment to [12]. When $r = 2$ the situation is particularly clean.

Theorem 1.5. *Let H be a 3-chromatic graph. There is a smallest positive integer g such that H is not homomorphic to C_{2g+1}. Then*

$$\delta_H = \frac{2}{2g+1}.$$

The graph that shows $\delta_H \geqslant 2/(2g+1)$ is a balanced blow-up of the cycle C_{2g+1}. The intuitive explanation for the theorem is that the main obstacle for being close to (that is, within $o(n^2)$ edges of) bipartite is containing some blow-up of an odd cycle. That odd cycle must be consistent with being H-free (in particular, the blow-up of the odd cycle must be H-free), and hence, it is the first odd cycle to which H is not homomorphic that determines δ_H.

Understanding locally colourable graphs becomes crucial when extending this theorem to $r \geqslant 3$. For concreteness, consider $r = 3$: we are interested in which graphs' blow-ups are the main obstacles for being close to tripartite. Given the importance of odd cycles when $r = 2$, it seems natural that odd wheels (the join of a single vertex and an odd cycle) would be obstacles here and indeed they are. However, there are further obstacles that do not contain odd wheels and so are locally bipartite. These observations suggest we should pay attention to 4-chromatic locally bipartite graphs as these may be obstacles for being close to tripartite. Many of the graphs appearing in our theorems here also play a leading role for minimum degree stability.

2 Chromatic thresholds

The $(r-1)$-locally 1-partite graphs are exactly the K_{r+1}-free graphs, and their chromatic threshold was determined by Goddard and Lyle [9], and Nikiforov [17].

$$\delta_\chi(\mathcal{F}_{r-1,1}) = \delta_\chi(K_{r+1}) = 1 - \frac{1}{r - 1/2}.$$

In this section, we prove Theorem 1.2, showing that $\delta_\chi(\mathcal{F}_{a,b}) = 1 - 1/(a+b-1)$ for all $a \geqslant 1$ and $b \geqslant 2$. The upper bound follows from the work of Allen et al. [1].

Proof of upper bound in Theorem 1.2. Fix a and b positive integers with $b \geqslant 2$ and let $\ell = a + b - 1$. Let $d > 1 - 1/\ell$ and let $G \in \mathcal{F}_{a,b}$ with $\delta(G) \geqslant d|G|$. Now $K_{\ell-1} + C_5$ has an a-clique whose common neighbourhoods contains $K_{b-2} + C_5$ so is not b-colourable. In particular, G is $(K_{\ell-1} + C_5)$-free.

Now, in the language of [1], $K_{\ell-1} + C_5$ is $(\ell+2)$-near-acyclic, so the chromatic threshold of the family of $(K_{\ell-1} + C_5)$-free graphs is $1 - 1/\ell$. In particular, there is a constant C depending only upon d and ℓ (and not on G) such that $\chi(G) \leqslant C$. Hence,

$$\delta_\chi(\mathcal{F}_{a,b}) \leqslant 1 - \frac{1}{\ell} = 1 - \frac{1}{a+b-1}. \qquad \square$$

For the lower bound, it suffices to give examples of graphs $G \in \mathcal{F}_{a,b}$ with $\delta(G) \geqslant (1 - 1/(a+b-1) - o(1)) \cdot |G|$ which have arbitrarily large chromatic number. Allen et al. [1] used graphs of

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large girth and high chromatic number as well as Borsuk-Hajnal graphs for their lower bounds. However, these depend upon the forbidden subgraph H and do not seem to be applicable here where the collection of forbidden subgraphs is infinite. Łuczak and Thomassé [16] modified a Borsuk graph to give the lower bound of 1/2 for the chromatic threshold of locally bipartite graphs. We will give a somewhat simpler example which gives the lower bound for all $\mathcal{F}_{a,b}$ $(a \geqslant 1, b \geqslant 2)$.

Our example is based upon the classical *Schrijver graph* [18]. The *Kneser graph*, KG (n, k), is the graph whose vertex set is all k-subsets of $\{1, 2, \ldots, n\}$ with two vertices adjacent if the corresponding k-sets are disjoint. In 1955, Kneser [14] conjectured that the chromatic number of KG (n, k) is $n - 2k + 2$. This conjecture remained open for two decades and was first proved by Lovász [15] using homotopy theory (see also Bárány [3] for a very short proof).

The *Schrijver graph*, SG (n, k), is the graph whose vertex set is all k-subsets of $\{1, 2, \ldots, n\}$ which do not contain both i and $i + 1$ for any $i = 1, 2, \ldots, n - 1$ and do not contain both n and 1. Again, vertices are adjacent if the corresponding k-sets are disjoint. Put another way, SG (n, k) is the induced subgraph of KG (n, k) obtained by deleting all vertices whose corresponding sets are supersets of any of $\{1, 2\}, \{2, 3\}, \{3, 4\}, \ldots, \{n - 1, n\}, \{n, 1\}$. Schrijver [18] showed that SG (n, k) is vertex-critical with chromatic number $\chi(\text{SG}(n, k)) = \chi(\text{KG}(n, k)) = n - 2k + 2$.

Proof of lower bound in Theorem 1.2. Fix a and b positive integers with $b \geqslant 2$ and let $\ell = a + b - 1$. Fix k and let $n = 2k + f(k)$ where $f(k)$ is a non-negative integer less than k, and both $f(k) \to \infty$, $f(k)/k \to 0$ as $k \to \infty$. Since $k > n/3$, the graph SG (n, k) is triangle-free. We will eventually consider a blow-up of the graph shown in Figure 1.

- Each rectangle is an independent set of n vertices: $v_{1,2}, v_{2,3}, \ldots, v_{n-1,n}, v_{n,1}$ – we always consider indices modulo n.
- There are $\ell - 1$ rectangles and the vertices in rectangles form a complete $(\ell - 1)$-partite graph with n vertices in each part.
- The vertex v is adjacent to all of the $v_{i,i+1}$ but has no neighbours in the copy of SG (n, k) – in particular, the rectangles together with v form a complete ℓ-partite graph.
- Finally, $A \in \text{SG}(n, k)$ is adjacent to $v_{i,i+1}$ if either i or $i + 1$ is in A (note, by the definition of the Schrijver graph, that it is impossible for both i and $i + 1$ to be in A).

We first check that the graph, G, shown in Figure 1 is locally ℓ-partite. Fix a vertex u of G. If $u = v$, then G_u consists of the $\ell - 1$ rectangles and so is $(\ell - 1)$-colourable. Next suppose that u is a vertex in the copy of SG (n, k), so $v \notin \Gamma(u)$. As SG (n, k) is triangle-free, $\Gamma(u)$ consists of an independent set in SG (n, k) together with some vertices from the $\ell - 1$ rectangles. In particular, G_u is ℓ-colourable. Finally, suppose that u lies in the $\ell - 1$ rectangles. By symmetry, we may take u to be a $v_{1,2}$. Then, $\Gamma(u)$ consists of two independent sets in SG (n, k) (sets containing 1 and sets containing 2), $\ell - 2$ rectangles and v. There are no edges from v to the copy of SG (n, k), so G_u is ℓ-colourable.

Now we consider a suitable blow-up of G. Let s be a multiple of n that is much larger than $|\text{SG}(n, k)|$. We will blow-up each $v_{i,i+1}$ by s/n so that all rectangles contain s vertices, and we will blow up v by s also. Note that v together with the rectangles form a complete ℓ-partite graph with s vertices in each part. Keep the copy of SG (n, k) as it is. Call the resulting graph G'.

- G' has $\ell s + |\text{SG}(n, k)|$ vertices.
- Any $A \in \text{SG}(n, k)$ is adjacent to $2k/n$ proportion of the $v_{i,i+1}$ so has degree at least $(\ell - 1)s(2k)/n = (\ell - 1)s/(1 + f(k)/(2k))$.
- Any other vertex has degree at least $(\ell - 1)s$.
- $\chi(G') \geqslant \chi(\text{SG}(n, k)) = n - 2k + 2 = f(k) + 2$.

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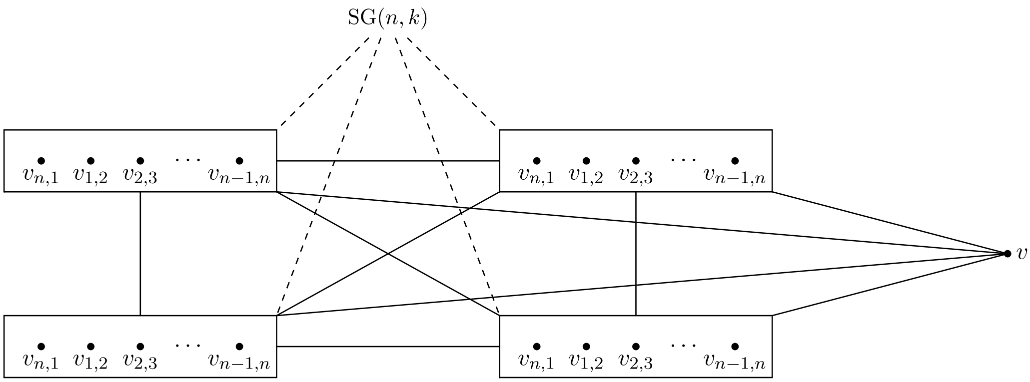

Figure 1. In this diagram, $\ell = 5$.

Given any $C, \varepsilon > 0$, we may choose k large enough so that $f(k) \geqslant C$ and $(\ell - 1)/(1 + f(k)/(2k)) \geqslant \ell - 1 - \varepsilon$ and then choose s large enough so that $\ell s + |\mathrm{SG}(n, k)| \leqslant (\ell + \varepsilon)s$. The resulting graph G' has chromatic number at least $f(k) + 2 \geqslant C$ and

$$\frac{\delta(G')}{|G'|} \geqslant \frac{\ell - 1 - \varepsilon}{\ell + \varepsilon} \geqslant 1 - \frac{1}{\ell} - \varepsilon.$$

Being a-locally b-partite is preserved when taking blow-ups. Now G is locally ℓ-partite and so G' is too. As $\mathcal{F}_{1,\ell} \subset \mathcal{F}_{a,b}$, G' is a-locally b-partite. Thus,

$$\delta_\chi(\mathcal{F}_{a,b}) \geqslant 1 - \frac{1}{\ell} - \varepsilon = 1 - \frac{1}{a + b - 1} - \varepsilon,$$

but $\varepsilon > 0$ was arbitrary and so we have the required result. □

3 Structure of locally bipartite graphs down to 8/15

Our understanding of the structure of locally bipartite graphs can be summarised as follows. The graphs $\overline{C}_7$, H_2^+, H_2, etc. can be seen in Figure 2 where they are discussed more thoroughly.

Theorem 3.1. *(locally bipartite graphs) Let G be a locally bipartite graph.*

a. *If $\delta(G) > 4/7 \cdot |G|$, then G is 3-colourable.*
b. *If $\delta(G) > 5/9 \cdot |G|$, then G is homomorphic to $\overline{C}_7$. Also, G is either 3-colourable or contains $\overline{C}_7$.*
c. *There is an absolute constant $\varepsilon > 0$ such that if $\delta(G) > (5/9 - \varepsilon) \cdot |G|$, then G is homomorphic to either $\overline{C}_7$ or H_2^+.*
d. *If $\delta(G) > 6/11 \cdot |G|$, then G is 4-colourable. Also, G is either 3-colourable or contains $\overline{C}_7$ or H_2^+. In the first two cases, G is homomorphic to $\overline{C}_7$.*
e. *If $\delta(G) > 7/13 \cdot |G|$, then G is either 3-colourable or contains H_2.*
f. *If $\delta(G) > 8/15 \cdot |G|$, then G is either 3-colourable or contains H_2 or T_0.*

In [11], we considered the structure of locally bipartite graphs with minimum degree down to 6/11, and, indeed, the first four parts of Theorem 3.1 are proved there. The purpose of this section is to prove the final two parts. In Section 3.1, we motivate this proof and also show that many of the constants in the theorem are tight.

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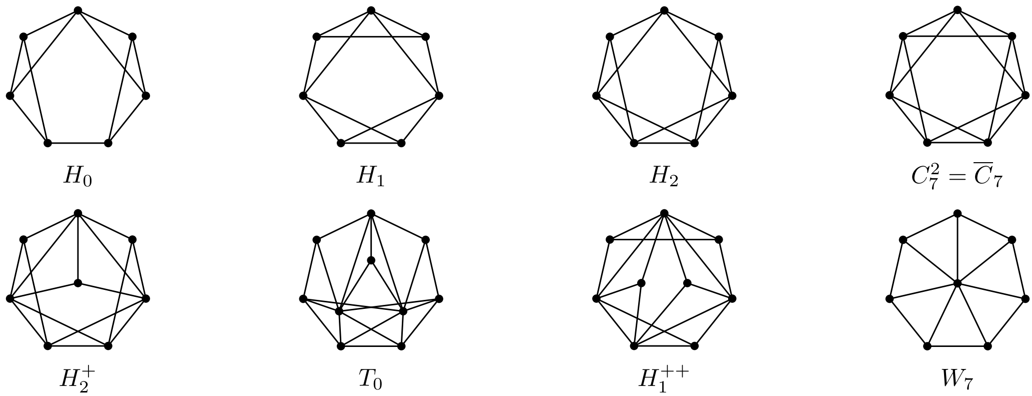

Figure 2. The graphs of Theorem 3.1.

Figure 2 displays the graphs of Theorem 3.1 as well as some that appear in the proof.

- All graphs shown are 4-chromatic, and all bar W_7 are locally bipartite.
- The graph H_0 is isomorphic to the *Moser Spindle* – the smallest 4-chromatic unit distance graph. H_0 is also the smallest 4-chromatic locally bipartite graph, and so it is natural that it should play such an integral part in many of our results. The graph $\overline{C}_7$ is the complement (and also the square) of the 7-cycle.
- Adding a single edge to H_0 while maintaining local bipartiteness can give rise to two non-isomorphic graphs, one of which is H_1. The other will appear fleetingly in the proof of Claim 3.9. Adding a single edge to H_1 while maintaining local bipartiteness gives rise to a unique (up to isomorphism) graph – H_2. There is only one way to add a single edge to H_2 and maintain local bipartiteness – this gives $\overline{C}_7$. H_2^+ is H_2 with a degree 3 vertex added. H_1^{++} is H_1 with two degree 3 vertices added.
- $\overline{C}_7$ and H_2^+ are both edge-maximal locally bipartite graphs.
- T_0 is a 7-cycle (the outer cycle) together with two vertices each joined to six of the seven vertices in the outer cycle (with the 'seventh' vertices two apart) and finally a vertex of degree three is added.
- W_7, is called the *7-wheel*. More generally, a single vertex joined to all the vertices of a k-cycle is called a *k-wheel* and is denoted by W_k. We term any edge from the central vertex to the cycle a *spoke* of the wheel and any edge of the cycle a *rim* of the wheel. Note that a graph is locally bipartite exactly if it does not contain any odd wheel (there is no such nice characterisation for a graph being locally tripartite, locally 4-partite, . . .).

The following observation gives a useful link between local bipartiteness and some of these graphs. We will use it frequently when copies of H_0, H_1, H_2 or $\overline{C}_7$ appear.

Remark 3.2. Any five vertices of H_0 contain a triangle or a 5-cycle. In particular, if G is a locally bipartite graph, then any vertex can have at most four neighbours in any copy of H_0 appearing in G.

Containment and homomorphisms between the first seven graphs of Figure 2 are summarised in Figure 3. Note that a full arrow pointing from H to G signifies that H is a subgraph of G and a dashed arrow from H to G signifies that H is homomorphic to G. Furthermore, for two graphs H and G in the diagram, H is homomorphic to G exactly if there is a sequence of arrows starting at H and ending at G. We verify Figure 3 in the appendix.

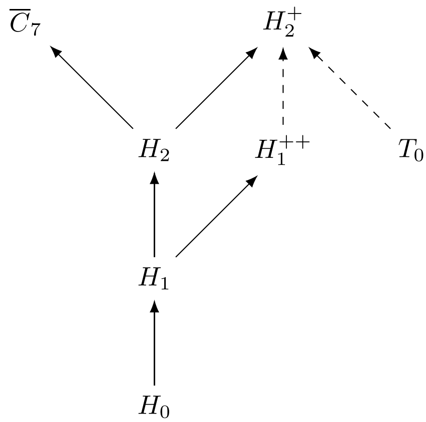

Figure 3. Full arrows denote containment, dashed arrows denote homomorphisms.

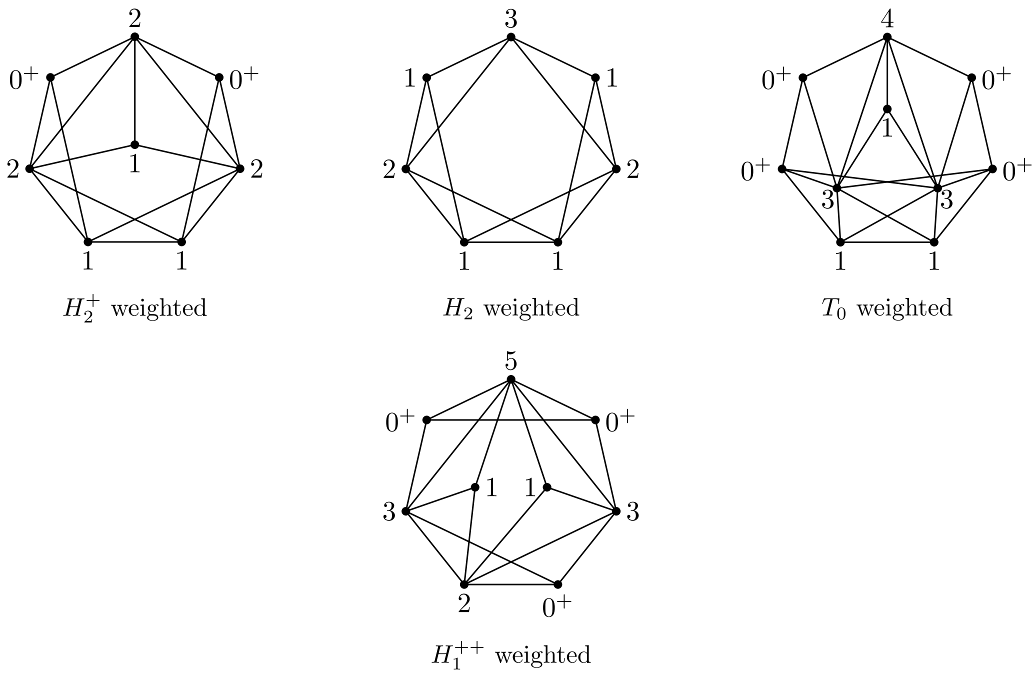

Figure 4. Weightings of H_2^+, H_2, T_0 and H_1^{++}.

3.1 The components of Theorem 3.1 and initial observations

We will start by noting which parts of Theorem 3.1 are tight. We will need suitable blow-ups of some of the graphs in Figure 2. In Figure 4 we give these blow-ups which have been chosen so that the ratio between the minimum degree and order of the graph is as large as possible. When we 'weight a vertex by 0^+', we are actually giving it some tiny positive weight (we have not deleted the vertices entirely just given them a small weight relative to the rest).

Now we address one-by-one the tightness of the constants in Theorem 3.1. First note that $\overline{C}_7$ is locally bipartite, 4-regular with 7 vertices and chromatic number 4. Hence, balanced blow-ups of $\overline{C}_7$ show that 4/7 is tight in part *a*. Figure 4 shows that there are *n*-vertex blow-ups of H_2^+

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with minimum degree at least $5/9 \cdot n - \mathcal{O}(1)$. These blow-ups of H_2^+ are 4-chromatic (as H_2^+ is), do not contain $\overline{C}_7$ nor are homomorphic to $\overline{C}_7$ (as neither of $\overline{C}_7$ nor H_2^+ is homomorphic to the other). This gives the tightness of 5/9 in part *b*. Figure 4 shows that there are *n*-vertex blow-ups of H_2 with minimum degree $\lfloor 6/11 \cdot n \rfloor$. These blow-ups are 4-chromatic and contain neither $\overline{C}_7$ nor H_2^+, since neither of these is homomorphic to H_2. Hence, in part *d*, 6/11 is tight for the conclusion that G is either 3-colourable or contains $\overline{C}_7$ or H_2^+. Tightness is not known for 4-colourability, and in fact, it seems likely that $\delta_\chi(\mathcal{F}_{1,2}, 4) < 6/11$. Similar arguments show that the weightings of T_0 and H_1^{++} in Figure 4 give the tightness of 7/13 and 8/15 in parts *e* and *f* respectively.

Our second aim in this section is to motivate the proof of parts *e* and *f* of Theorem 3.1. We will deduce them from the following two theorems.

Theorem 3.3. *Let G be a locally bipartite graph. If $\delta(G) > 8/15 \cdot |G|$, then G is either 3-colourable or contains H_0 or T_0. If $\delta(G) > 7/13 \cdot |G|$, then G is either 3-colourable or contains H_0.*

Theorem 3.4. *Let G be a locally bipartite graph that contains H_0. If $\delta(G) > 8/15 \cdot |G|$, then G contains H_2.*

Proof of parts e and f. Let G be a locally bipartite graph. If $\delta(G) > 7/13 \cdot |G|$, then, by Theorem 3.3, G is either 3-colourable or contains H_0. If G contains H_0, then Theorem 3.4 shows it actually contains H_2. This gives part *e*. Suppose instead that $\delta(G) > 8/15 \cdot |G|$. By Theorem 3.3, G is either 3-colourable, contains T_0 or contains H_0. In the last case Theorem 3.4 shows that G contains H_2 and so we have part *f*. □

We prove Theorem 3.4 in Section 3.2 by counting edges between the copy of H_0 and G. Theorem 3.3 is more involved. As motivation, consider a locally bipartite H_2-free G with $\delta(G) > 7/13 \cdot |G|$ (we will ignore the 8/15 conclusion here). By Theorem 3.4, G is H_0-free and our aim is to show that G is 3-colourable. We may as well assume that G is edge-maximal: the addition of any edge will give a copy of H_0 or a vertex with non-bipartite neighbourhood. Thus, any non-edge of G is either the missing edge of a K_4, a missing rim of an odd wheel, a missing spoke of an odd wheel or a missing edge of an H_0. This motivates a key definition that also appeared in [11]. Recall that $G_{u,v}$ denotes the subgraph of G induced by the common neighbourhood of vertices u, v.

Definition 3.5. (dense and sparse) A pair of non-adjacent, distinct vertices u, v in a graph G is *dense* if $G_{u,v}$ contains an edge and *sparse* if $G_{u,v}$ does not contain an edge.

First note that every pair of distinct vertices in any graph is exactly one of 'adjacent', 'dense' or 'sparse'. Another way to view being dense is as being the missing edge of a K_4. Locally bipartite graphs are K_4-free so any pair of distinct vertices with an edge in their common neighbourhood must be non-adjacent and so dense. Our initial observations above show that each sparse pair of vertices in G is either the missing rim or spoke of an odd wheel or a missing edge of an H_0. In Section 3.3, we will rule out sparse pairs of vertices being the missing spoke of an odd wheel, and in Section 3.4, we will finish the proof by showing that G is 3-colourable.

The distinction between dense and sparse pairs turns out to be crucial. We borrow from [11] the following three simple but useful lemmas. The final one hints at the importance of H_0.

Lemma 3.6. *Let G be a graph with $\delta(G) > 1/2 \cdot |G|$ and let I be any largest independent set in G. Then, for every distinct $u, v \in I$, the pair u, v is dense.*

Lemma 3.7. *Let G be a graph with $\delta(G) > 1/2 \cdot |G|$ and suppose C is an induced 4-cycle in G. Then at least one of the non-edges of C is a dense pair.*

Lemma 3.8. *Let G be a locally bipartite graph which does not contain H_0. For any vertex v of G,*

$$D_v := \{u \colon \textit{the pair } u, v \textit{ is dense}\}$$

is an independent set of vertices.

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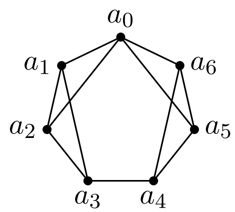
3.2 From H_0 to H_2 – proof of Theorem 3.4

In this subsection, we prove Theorem 3.4 in two steps. The strategy is to start with a copy of H_0 and consider edges between it and the rest of G. Using the high minimum degree, we are able to find a vertex with the correct neighbours in the copy of H_0 so that a copy of H_1 is present. We then play the same game to get a copy of H_2. To aid the reader, we will use '$G[v_1, v_2, \ldots, v_7]$ is a copy of H_1' to mean that G has a copy of H_1 in which v_1 is the top vertex (as displayed in Figure 2) and $v_1, v_2, \ldots, v_7$ precede anticlockwise round the figure. We do similarly for copies of H_2.

Claim 3.9. Let G be a locally bipartite graph containing H_0. If $\delta(G) > 1/2 \cdot |G|$, then G contains H_1.

Proof. We label a copy of H_0 in G as below and let $X = \{a_0, a_1, \ldots, a_6\}$.



Let U_4 be the set of vertices with exactly four neighbours in X. Remark 3.2 shows that no vertex has five neighbours in a copy of H_0, so every non-U_4 vertex has at most three neighbours in X. Hence,

$$7/2 \cdot |G| < 7\delta(G) \leqslant e(X, G) \leqslant 4|U_4| + 3(|G| - |U_4|) = 3|G| + |U_4|,$$

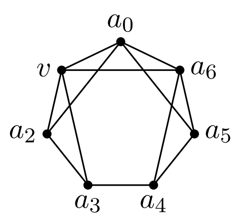
and so

$$|U_4| > 1/2 \cdot |G|.$$

Now $|U_4| + d(a_0) > |G|$ and so some vertex v is adjacent to a_0 and has exactly four neighbours in X. Note that v cannot be adjacent to both a_1, a_2 as otherwise $va_0a_1a_2$ is a K_4, so by symmetry we may assume that v is not adjacent to a_1. Similarly, we may assume that v is not adjacent to a_5. But v has four neighbours in X so must be adjacent to at least one of a_2, a_6 – by symmetry, we may assume v is adjacent to a_2.

There are two possibilities: v is adjacent to a_0, a_2, a_3, a_4, or v is adjacent to a_0, a_2, a_6 and one of a_3, a_4. In the latter case, we may assume by symmetry that v is adjacent to a_3. Hence, there are two possibilities for $\Gamma(v) \cap X$: $\{a_0, a_2, a_3, a_4\}$ and $\{a_0, a_2, a_3, a_6\}$. In both cases, v cannot be any a_i except for possibly a_1.

If $\Gamma(v) \cap X = \{a_0, a_2, a_3, a_4\}$, then $G[v, a_3, a_4, a_5, a_6, a_0, a_2]$ is a copy of H_1. If $\Gamma(v) \cap X = \{a_0, a_2, a_3, a_6\}$, then G contains the following graph where, in particular, v is not adjacent to a_4.

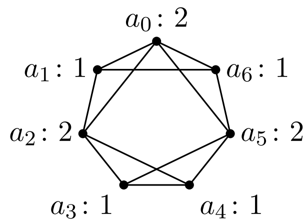


Vertex a_6 is not adjacent to a_3 else G_{a_6} contains the 5-cycle $a_0va_3a_4a_5$. Hence, $va_6a_4a_3$ is an induced 4-cycle in G. By Lemma 3.7, at least one of the pairs v, a_4 and a_3, a_6 is dense. By symmetry, we may assume that v, a_4 is dense: let $a_2'a_3'$ be an edge in G_{v,a_4}. Vertex v is not adjacent to a_5, so a_5

is neither a_2' nor a_3'. Note that a_0 is not adjacent to a_4 (else $a_0a_4a_5a_6$ is a K_4) and so a_0 is neither a_2' nor a_3'. If $a_6 = a_2'$, then G_{a_6} contains the 5-cycle $a_3'va_0a_5a_4$, which is impossible. Similarly, $a_6 \neq a_3'$. Hence, a_2', a_3' are distinct from a_4, a_5, a_6, a_0, v and so $G[a_6, a_0, v, a_2', a_3', a_4, a_5]$ is a copy of H_1. □

Claim 3.10. Let G be a locally bipartite graph containing H_1. If $\delta(G) > 8/15 \cdot |G|$, then G contains H_2.

Proof. Consider a copy of H_1 with vertices $X = \{a_0, a_1, \ldots, a_6\}$. We assign a weighting $\omega\colon X \to \mathbb{Z}_{\geqslant 0}$ as shown in the diagram below, so, for example $\omega(a_0) = 2$ and $\omega(a_1) = 1$ (recall this notation from Section 1.1). We will often use diagrams to give weightings in this way. For each vertex $v \in G$, let $f(v)$ be the total weight of the neighbours of v in X.



We will assume that G does not contain H_2. We first show that any vertex v has $f(v) \leqslant 6$ and further that if $f(v) = 6$, then $\Gamma(v) \cap X = \{a_1, a_2, a_5, a_6\}$.

Let v be a vertex with $f(v) \geqslant 6$. No vertex has five neighbours in a copy of H_1 (as noted in Remark 3.2), so v is adjacent to at most four of the a_i. Thus, v is adjacent to at least two of the vertices of weight two, that is, to at least two of a_0, a_2, a_5. If v is adjacent to all of a_0, a_2, a_5, then G_{a_0} contains the odd circuit $va_5a_6a_1a_2$. Thus, v is adjacent to exactly two of a_0, a_2 and a_5.

Suppose v is adjacent to a_0. By symmetry, we may assume that v is adjacent to a_2 but not to a_5. Then,v is not adjacent to a_1, else $va_0a_1a_2$ is a K_4. Similarly, v cannot be adjacent to both a_3 and a_4. Hence,v is adjacent to a_0, a_2, a_6 and one of a_3, a_4. By symmetry, we may assume v is adjacent to a_3. Now v cannot be a_4 nor a_5 as it would then have five neighbours in X. Hence, $G[a_5, a_6, a_0, v, a_2, a_3, a_4]$ is a copy of H_2.

Thus, v is not adjacent to a_0 and so is adjacent to both a_2 and a_5. Note v cannot be adjacent to both a_3, a_4 else $va_2a_3a_4$ is K_4, so we may assume that v is adjacent to a_1. If v were adjacent to one of a_3, a_4, then we may assume by symmetry that v is adjacent to a_4 and not a_3. Now v cannot be a_0 nor a_6 as it would then have five neighbours in X. But then, $G[a_5, a_6, a_0, a_1, a_2, v, a_4]$ is a copy of H_2. Therefore, v is adjacent to neither a_3 nor a_4. But $f(v) \geqslant 6$, so v is adjacent to a_1, a_6 and thus $\Gamma(v) \cap X = \{a_1, a_2, a_5, a_6\}$.

So every vertex v has $f(v) \leqslant 6$, and furthermore, all $v \in \Gamma(a_0) \cup \Gamma(a_3) \cup \Gamma(a_4)$ have $f(v) \leqslant 5$. We first claim that there is $i \in \{3, 4\}$ such that $\Gamma(a_0, a_2, a_i) = \varnothing$. If not, there is $a_1' \in \Gamma(a_0, a_2, a_3)$ and $a_1'' \in \Gamma(a_0, a_2, a_4)$. But then, G_{a_2} contains the odd circuit $a_1'a_3a_4a_1''a_0$. Similarly there is $j \in \{3, 4\}$ such that $\Gamma(a_0, a_5, a_j) = \varnothing$.

Next we claim that there is $i \in \{3, 4\}$ with $\Gamma(a_0, a_2, a_i) = \Gamma(a_0, a_5, a_i) = \varnothing$. If not, then without loss of generality there is $a_1' \in \Gamma(a_0, a_2, a_3)$ and $a_6' \in \Gamma(a_0, a_5, a_4)$. Certainly, $\Gamma(a_1') \cap X \neq \{a_1, a_2, a_5, a_6\}$, so $f(a_1') \leqslant 5$. But $\omega(a_0) + \omega(a_2) + \omega(a_3) = 5$, so $\Gamma(a_1') \cap X = \{a_0, a_2, a_3\}$. Similarly $\Gamma(a_6') \cap X = \{a_0, a_4, a_5\}$. In particular, all of $a_0, \ldots, a_6, a_1', a_6'$ are distinct. But then, $G[a_0, a_1', a_2, a_3, a_4, a_5, a_6']$ is a copy of H_2. Thus, without loss of generality, $\Gamma(a_0, a_2, a_3) = \Gamma(a_0, a_5, a_3) = \varnothing$. Then, $\Gamma(a_3)$, $\Gamma(a_0, a_2)$, $\Gamma(a_0, a_5)$ are pairwise disjoint (we already showed that $\Gamma(a_0, a_2, a_5) = \varnothing$). Hence, the set $Y = \Gamma(a_3) \cup \Gamma(a_0, a_2) \cup \Gamma(a_0, a_5)$ has size

$$|Y| = d(a_3) + d(a_0, a_2) + d(a_0, a_5) \geqslant \delta(G) + 2(2\delta(G) - |G|) = 5\delta(G) - 2|G|.$$

Now all $v \in Y$ have $f(v) \leqslant 5$. We bound $\omega(X, G)$ from both directions (recalling this notation from Section 1.1) to get

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$$10\delta(G) \leqslant \sum_{x\in X} \omega(x)d(x) = \sum_{v\in G} f(v) \leqslant 5|Y| + 6(|G| - |Y|) = 6|G| - |Y| \leqslant 8|G| - 5\delta(G),$$

which contradicts $\delta(G) > 8/15 \cdot |G|$. □

These two claims combine to give Theorem 3.4.

3.3 Ruling out sparse pairs being spokes of odd wheels

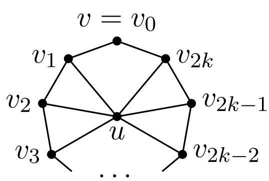
In this subsection, we make a start on the proof of Theorem 3.3, by ruling out the possibility that G contains a sparse pair of vertices which is the spoke of an odd wheel (Corollary 3.18). Our next three lemmas show that any sparse pair which is the missing spoke of an odd wheel is, in fact, the missing spoke of a 7-wheel. The following straightforward lemma appeared in [11].

Lemma 3.11. *Let G be a locally bipartite graph with $\delta(G) > 1/2 \cdot |G|$ and which does not contain H_0. Then,G does not contain a sparse pair u, v with uv being the missing spoke of a 5-wheel.*

We will use the following technical lemma on various occasions.

Lemma 3.12. *Let G be a locally bipartite graph and let u, v be a sparse pair of vertices in G. Suppose that C is the shortest odd cycle which both passes through v and satisfies $C \setminus \{v\} \subset \Gamma(u)$ (i.e. $G[C \cup \{u\}]$ contains an odd wheel missing the spoke uv). Then, every neighbour of u has at most two neighbours in C and if two, then they are two apart on C. In particular, C is an induced cycle.*

Proof. Label the configuration as follows and write v_0 for v (we consider indices modulo $2k+1$). Note $C = \{v, v_1, \ldots, v_{2k}\}$.



Consider a vertex x which is adjacent to u. Suppose that x is adjacent to two vertices in C which are not two apart: x is adjacent to v_i and v_{i+r} where $r \in \{1, 3, 4, \ldots, k\}$. Firstly, if $r = 1$, then either G contains the K_4 uxv_iv_{i+1} or $G_{u,v}$ contains an edge (if one of v_i or v_{i+1} is v) which contradicts the sparsity of u, v. Secondly, if $r > 1$ is odd, then $C' = xv_iv_{i+1} \cdots v_{i+r}$ is an odd circuit which is shorter than C. But then, C' contains an odd cycle C'' which is shorter than C. Either C'' is in G_u (if $v \notin C''$) contradicting the local bipartiteness of G or we have found a shorter odd cycle than C which satisfies the properties of C (if $v \in C''$). Finally, if $r > 2$ is even, then $C' = xv_{i+r}v_{i+r+1} \cdots v_{i-1}v_i$ is an odd circuit which is shorter than C. Again C' must contain an odd cycle C'' which is shorter than C. We either obtain an odd cycle in G_u or contradict the minimality of C. Hence, every neighbour of u has at most two neighbours in C, and if two, then they are v_i, v_{i+2} for some i.

All of $v_1, \ldots, v_{2k}$ are neighbours of u so have two neighbours in C. Hence, C is induced. □

Lemma 3.13. *Let G be a locally bipartite graph with $\delta(G) > 8/15 \cdot |G|$ which does not contain H_0. Any sparse pair in G that is the missing spoke of an odd wheel is the missing spoke of a 7-wheel.*

Proof. Consider a sparse pair u, v that is the missing spoke of a $(2k+1)$-wheel. Choose the odd wheel so that k is minimal. Without loss of generality, u is the central vertex and v is in the outer $(2k+1)$-cycle which we call C. Lemma 3.11 shows that $k > 2$. We are done if we can show that $k = 3$.

By Lemma 3.12, every neighbour of u has at most two neighbours in C. All vertices have at most $2k$ neighbours in C as otherwise G contains a $(2k+1)$-wheel and so is not locally bipartite. Hence,

$$(2k+1)\delta(G) \leqslant e(G, C) \leqslant 2d(u) + 2k(|G| - d(u))$$
$$= 2k|G| - (2k-2)d(u) \leqslant 2k|G| - (2k-2)\delta(G),$$

so

$$\frac{8}{15} < \frac{\delta(G)}{|G|} \leqslant \frac{2k}{4k-1},$$

which implies that $k < 4$. □

Thus, to rule out sparse pairs being a missing spoke of an odd wheel we only need to rule out there being a sparse pair which is the missing spoke of a 7-wheel. This is where the graph T_0 becomes relevant. We will prove the following.

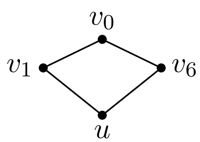
Proposition 3.14. *Let G be a locally bipartite graph which does not contain H_0. If either $\delta(G) > 7/13 \cdot |G|$, or $\delta(G) > 8/15 \cdot |G|$ and G does not contain T_0, then G does not contain a sparse pair that is the missing spoke of a 7-wheel.*

Proof. Let G be a locally bipartite graph which does not contain H_0 and either satisfies $\delta(G) > 7/13 \cdot |G|$, or $\delta(G) > 8/15 \cdot |G|$ and G does not contain T_0. In a slight abuse of notation, we will say that a vertex u is *sparse to a cycle C* if u is adjacent to $|C| - 1$ vertices of C and is in a sparse pair with the final vertex. We are required to show that there is no vertex u and no 7-cycle C with u sparse to C. □

By Lemma 3.11, there is no 5-cycle C and vertex u with u sparse to C. Hence, if a vertex u is sparse to a 7-cycle C, then, by Lemma 3.12, C is an induced 7-cycle and any neighbour of u has at most two neighbours in C.

Claim 3.15. If a vertex u is sparse to a 7-cycle $C = v_0 v_1 \dots v_6$ with the pair u, v_0 sparse, then there is some vertex which has six neighbours in C and is adjacent to all of v_6, v_0, v_1.

Proof of Claim. First consider the induced 4-cycle $u v_6 v_0 v_1$. The pair u, v_0 is sparse, so $\Gamma(u, v_6, v_0) = \Gamma(v_0, v_1, u) = \varnothing$. Also $\Gamma(v_1, u, v_6) = \varnothing$ as otherwise G_u contains an odd circuit. Hence, all $z \notin \Gamma(v_6, v_0, v_1)$ have at most two neighbours in $\{u, v_6, v_0, v_1\}$.



Thus,

$$4\delta(G) \leqslant e(\{v_0, v_1, u, v_6\}, G) \leqslant 3|\Gamma(v_6, v_0, v_1)| + 2(|G| - |\Gamma(v_6, v_0, v_1)|),$$

so $|\Gamma(v_6, v_0, v_1)| \geqslant 4\delta(G) - 2|G|$.

If the claim is false, then all vertices in $\Gamma(v_6, v_0, v_1)$ have at most five neighbours in C. Also note that $\Gamma(u)$ and $\Gamma(v_6, v_0, v_1)$ are disjoint and that all vertices have at most six neighbours in C (otherwise G contains a 7-wheel) so,

$$7\delta(G) \leqslant e(C, G) \leqslant 2d(u) + 5|\Gamma(v_6, v_0, v_1)| + 6(|G| - d(u) - |\Gamma(v_6, v_0, v_1)|)$$
$$= 6|G| - 4d(u) - |\Gamma(v_6, v_0, v_1)| \leqslant 6|G| - 4\delta(G) - 4\delta(G) + 2|G|,$$

which contradicts $\delta(G) > 8/15 \cdot |G|$. □

Claim 3.16. Let C be a 7-cycle such that there is some vertex which is sparse to C. Then, every vertex with at least six neighbours in C is sparse to C.

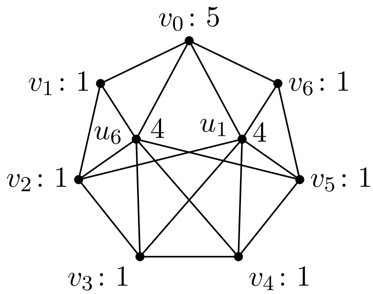
Proof of Claim. Let vertex u be sparse to the 7-cycle C and let x be a vertex with at least six neighbours in C. Since G is locally bipartite, x has exactly six neighbours in C. Let y be the vertex of C to which x is not adjacent. There is a vertex u' which has six neighbours in C and is adjacent to y. Indeed, if u is adjacent to y then take $u' = u$ and if u is not adjacent to y, then Claim 3.15 gives the desired u'.

Now $\Gamma(u', x)$ contains two consecutive vertices of C, so the pair u', x is dense. But u' is adjacent to y so, by Lemma 3.8, the pair y, x cannot be dense. In particular, x, y is sparse and so x is sparse to C. □

Now fix a 7-cycle $C = v_0 v_1 \ldots v_6$ such that there is some vertex which is sparse to C. Say vertex v_i is *lonely* if there is some vertex u which is adjacent to all of $C \setminus \{v_i\}$ – by the previous claim u is sparse to C and the pair u, v_i is sparse.

Claim 3.17. For all i, v_i and v_{i+2} are not both lonely.

Proof of Claim. Suppose for contradiction that v_1 and v_6 are both lonely: let u_1 and u_6 be sparse to C with both the pairs u_1, v_1 and u_6, v_6 sparse. Lemma 3.12 shows that any neighbour of u_1 (or u_6) has at most two neighbours in C. Let $X = \{u_1, u_6, v_0, \ldots, v_6\}$ and give a weighting ω to the vertices in X as shown below.



For each vertex $v \in G$, let $f(v)$ be the total weight of the neighbours of v in X. We shall show that if $v \notin \Gamma(u_6, u_1, v_0)$, then $f(v) \leqslant 10$ and if $v \in \Gamma(u_6, u_1, v_0)$, then $f(v) = 13$. Let v be a vertex with $f(v) \geqslant 11$. It suffices to show that v is adjacent to u_6, u_1, v_0 and none of $v_1, \ldots, v_6$.

If v is adjacent to neither u_1 nor u_6, then $f(v) \leqslant 11$ with equality only if v is adjacent to all of C which would give a 7-wheel so, in fact, $f(v) \leqslant 10$. Thus, we may assume v is adjacent to u_1. But then, v must have at most two neighbours in C. If neither of these is v_0, then $f(v) \leqslant 4 + 4 + 1 + 1 = 10$. Hence, we may assume v is adjacent to both v_0 and u_1.

Since $f(v) \geqslant 11$ and v has at most two neighbours in C, v must be adjacent to u_6 as well. Now, v is adjacent to both u_1 and v_0 so, by Lemma 3.12, the only other possible neighbour of v in C is one of v_2 and v_5. However, if v is adjacent to v_2, then G_{u_1} contains the odd circuit $v_0 v v_2 v_3 \ldots v_6$ while if v is adjacent to v_5, then G_{u_6} contains the odd circuit $v_0 v_1 \ldots v_5 v$. In conclusion, v is adjacent to u_1, u_6, v_0, and no other vertices of C.

Thus,

$$19\delta(G) \leqslant \omega(X, G) = \sum_{v \in G} f(v) \leqslant 13|\Gamma(v_0, u_1, u_6)| + 10(|G| - |\Gamma(v_0, u_1, u_6)|),$$

so

$$3|\Gamma(v_0, u_1, u_6)| \geqslant 19\delta(G) - 10|G|. \tag{1}$$

Any $v \in \Gamma(v_0, u_1, u_6)$ satisfies $f(v) = 13$ so has only one neighbour in C. Also any neighbour of u_1 has at most two neighbours in C. Hence,

$$\begin{aligned} 7\delta(G) \leqslant e(C, G) &\leqslant |\Gamma(v_0, u_1, u_6)| + 2(d(u_1) - |\Gamma(v_0, u_1, u_6)|) + 6(|G| - d(u_1)) \\ &= 6|G| - 4d(u_1) - |\Gamma(v_0, u_1, u_6)|, \end{aligned}$$

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so

$$|\Gamma(v_0, u_1, u_6)| \leqslant 6|G| - 11\delta(G). \tag{2}$$

Combining inequalities (1) and (2) gives

$$19\delta(G) - 10|G| \leqslant 3|\Gamma(v_0, u_1, u_6)| \leqslant 18|G| - 33\delta(G),$$

so $\delta(G) \leqslant 7/13 \cdot |G|$ and so G is T_0-free.

Inequality (1) and $\delta(G) > 8/15 \cdot |G|$ show that $\Gamma(v_0, u_1, u_6)$ is non-empty. Let v be a common neighbour of v_0, u_1, u_6. As G is T_0-free, v must be one of the v_i. But $C = v_0v_1 \ldots v_6$ is induced, so v must be one of v_1, v_6. This means one of the edges u_1v_1, u_6v_6 is present. However, these are both sparse pairs giving the required contradiction. □

We now finish the proof of Proposition 3.14. By the choice of C, some v_i is lonely. Without loss of generality, v_0 is lonely. By Claim 3.17, neither v_2 nor v_5 is lonely. By Claims 3.15 and 3.16, at least one of v_3 and v_4 is lonely. By Claim 3.17, we may assume v_3 is lonely. By lonely, v_1 is not lonely and at most one of v_4, v_6 is. Again, by symmetry, we may assume v_6 is not lonely. In conclusion, v_1, v_2, v_5, v_6 are all not lonely, v_0 and v_3 are lonely and v_4 may or may not be.

Let U_6 be the set of vertices with six neighbours in C. By Claim 3.16, any vertex in U_6 is sparse to C so cannot be adjacent to all of v_0, v_3, v_4 (else some other v_i is lonely). In particular,

$$U_6 \subset \overline{\Gamma(v_0)} \cup \overline{\Gamma(v_3)} \cup \overline{\Gamma(v_4)}.$$

As v_0 is lonely, there is a vertex u that is sparse to C with u, v_0 sparse. No two of v_0, v_3, v_4 are two apart on C and so, by Lemma 3.12, any neighbour of u is in at least two of $\overline{\Gamma(v_0)}, \overline{\Gamma(v_3)}, \overline{\Gamma(v_4)}$ and is not in U_6. Hence

$$|U_6| + 2d(u) \leqslant |\overline{\Gamma(v_0)}| + |\overline{\Gamma(v_3)}| + |\overline{\Gamma(v_4)}| \leqslant 3|G| - 3\delta(G),$$

and so $|U_6| \leqslant 3|G| - 5\delta(G)$. Since every neighbour of u has at most two neighbours in C,

$$\begin{aligned} 7\delta(G) &\leqslant e(C, G) \leqslant 6|U_6| + 2d(u) + 5(|G| - |U_6| - d(u)) = 5|G| + |U_6| - 3d(u) \\ &\leqslant 5|G| + 3|G| - 5\delta(G) - 3\delta(G) = 8|G| - 8\delta(G), \end{aligned}$$

which contradicts $\delta(G) > 8/15 \cdot |G|$. □

Lemmas 3.11 and 3.13 and Proposition 3.14 together give the result we need.

Corollary 3.18. *Let G be a locally bipartite graph which does not contain H_0. If either $\delta(G) > 7/13 \cdot |G|$, or $\delta(G) > 8/15 \cdot |G|$ and G does not contain T_0, then G does not contain a sparse pair which is the missing spoke of an odd wheel.*

3.4 The proof of Theorem 3.3

Here we will prove Theorem 3.3 which we restate for convenience.

Theorem 3.3. *Let G be a locally bipartite graph. If $\delta(G) > 8/15 \cdot |G|$, then G is either 3-colourable or contains H_0 or T_0. If $\delta(G) > 7/13 \cdot |G|$, then G is either 3-colourable or contains H_0.*

We start with a locally bipartite graph G which does not contain H_0 and satisfies either $\delta(G) > 7/13 \cdot |G|$, or $\delta(G) > 8/15 \cdot |G|$ and G does not contain T_0. We are required to show that G is 3-colourable. We may assume that G is edge-maximal: for any sparse pair u, v of G, the addition of uv to G introduces an odd wheel, a copy of H_0 or a copy of T_0. By Theorem 3.4, the addition of uv to G introduces an odd wheel, a copy of H_2 (since G itself does not contain H_2) or a copy of T_0.

Firstly, if the addition of uv introduces an odd wheel, then, by Corollary 3.18, uv must be a rim of that wheel – this case is depicted in Figure 5(a) below. Secondly, if the addition of uv introduces a copy of H_2, then that copy of H_2 less the edge uv must not contain H_0 – this case is depicted in

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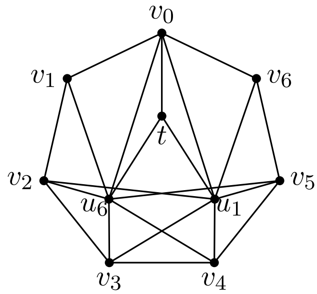
Figure 5(b) to (f) below. Finally, suppose the addition of uv introduces a copy of T_0 (but not an odd wheel nor a copy of H_0). Label this copy of T_0 in $G + uv$ as follows.



Note that $G + uv$ is locally bipartite and does not contain H_0 so, by Lemma 3.8, for any vertex x, $D_x = \{y\colon \text{the pair } x, y \text{ is dense}\}$ is an independent set. In $G + uv$, $t \in D_{v_1}$ and t is adjacent to u_1 so the pair u_1, v_1 is not dense. Also,u_1v_1 is not an edge in $G + uv$ (else $G + uv$ contains a 7-wheel centred at u_1), so the pair u_1, v_1 is sparse in $G + uv$. Therefore, u_1, v_1 is a sparse pair in G. Now, by Corollary 3.18, G does not contain an odd wheel missing a sparse spoke so uv must either be one of the edges v_iv_{i+1} or one of the edges u_1v_i. Similarly, uv must either be one of the edges v_iv_{i+1} or one of the edges u_6v_i. Thus, in fact, uv must be one of the edges $v_iv_{i+1,}$ and by symmetry, we may take $i = 0, 1, 2, 3$ – this case is depicted in Figure 5(g) to (j) below.

Thus, in G, any sparse pair u, v must appear in one of the configurations shown in Figure 5 (with the labels of u and v possibly swapped).

We will now consider a largest independent set in G: an independent set I of size $\alpha(G)$. We will shortly show that all vertices are either in I or adjacent to all of I.

By Lemma 3.6, for every $u \in I$, $I \subset D_u \cup \{u\}$. We now show there is set equality.

Proposition 3.19. *For every $u \in I$, $I = D_u \cup \{u\}$*

Proof. By Lemma 3.8 and the definition of dense, $D_u \cup \{u\}$ is an independent set. However, it contains the maximal independent set I, so must equal it. □

The following definition will be helpful.

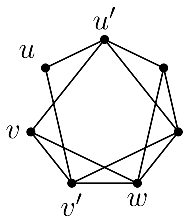
Definition 3.20. (quasidense) A pair of vertices u, v is *quasidense* if there is a sequence of vertices $u = d_1, d_2, \ldots, d_k, d_{k+1} = v$ such that all pairs d_i, d_{i+1} are dense ($i = 1, 2, \ldots, k$).

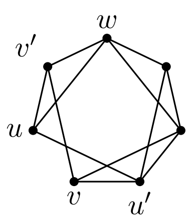
Proposition 3.19 immediately implies that if the pair u, v is quasidense and $u \in I$, then $v \in I$ also.

Proposition 3.21. *Every vertex of G is either in I or adjacent to all of I.*

Proof. Fix a vertex $u \in I$ and let v be any other vertex which is not adjacent to u. It suffices to show that $v \in I$. If the pair u, v is (quasi)dense, then $v \in I$, so we may assume that u, v is sparse (and not quasidense). Thus, u, v appears in one of the configurations given in Figure 5 (with labels u and v possibly swapped). However, in each of Figure 5(a), (b), (e) and (g) to (j) the pair u, v is quasidense. Hence, we may assume that u, v appear in one of Figure 5(c), (d) and (f).

We consider Figure 5(c) and (d) together. For ease, we label some more of the vertices as follows.





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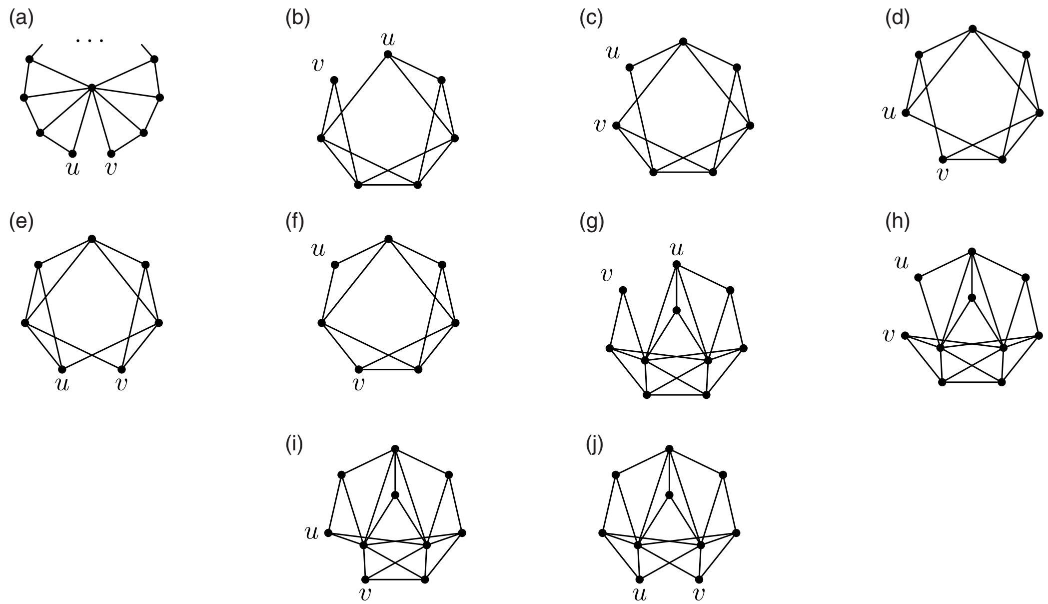
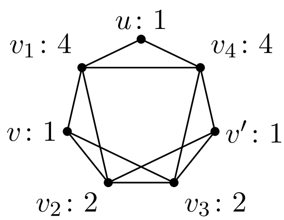


Figure 5. Configurations in which a sparse pair u, v may appear (labels u and v possibly swapped).

In both cases, the pair u', w is dense, and so, by Lemma 3.8, the pair u', v' is not dense. However, $u'v'$ is not an edge, as the pair u, v is sparse, and so u', v' is a sparse pair. But then, $uu'vv'$ is an induced 4-cycle in which both non-edges are sparse which contradicts Lemma 3.7.

Finally, we consider Figure 5(f) which we label as follows. Let $X = \{u, v, v', v_1, v_2, v_3, v_4\}$ and give a weighting ω to the vertices of X as shown.



The pair v, v' is dense, so, if u, v' is quasidense, then u, v is quasidense, a contradiction. Also, as G is H_0-free, uv' is not an edge. Hence, the pair u, v' is sparse and not quasidense. Now v, v' is dense and so, by Lemma 3.8, the pair vv_4 is not. But vv_4 is not an edge (else u, v is dense), so v, v_4 is a sparse pair. Similarly, v', v_1 is a sparse pair. To summarise, the pairs u, v and u, v' are sparse and not quasidense and the pairs v, v_4 and v', v_1 are sparse. It follows that $G[X]$ contains no more edges than shown.

If a vertex x has five neighbours in X, then it is adjacent to three consecutive vertices round the 7-cycle, so x is either adjacent to a triangle or to all of u, v_1, v or to all of v', v_4, u. The first gives a K_4 while the latter two contradict the pairs u, v and u, v' being sparse. Hence, all vertices have at most four neighbours in X.

For each vertex x in G, let $f(x)$ be the total weight of the neighbours of x in X. Now

$$\sum_{x\in G} f(x) = \omega(X, G) \geqslant 15\delta(G) > 8|G|,$$

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so some vertex x has $f(x) \geqslant 9$. All vertices of X have f value at most eight, so $x \notin X$. As x has at most four neighbours in X, either x is adjacent to both v_1 and v_4 or x is adjacent to exactly one of v_1, v_4 and both of v_2, v_3.

First suppose that x is adjacent to both v_1 and v_4. As v, v_4 is sparse, x is not adjacent to v. Similarly x is not adjacent to v'. If x is adjacent to v_2, then x, v is dense (the edge v_1v_2). But also u, x is dense (the edge v_1v_4), so u, v is quasidense, a contradiction. Hence, x is not adjacent to v_2. Similarly x is not adjacent to v_3, and so $f(x) = 8$, a contradiction.

In the second case, we may assume, by symmetry, that x is adjacent to v_1, v_2, v_3 but not to v_4. Then, x, v and x, v' are both dense pairs (the edge v_2v_3) so x is adjacent to neither v nor v'. Finally, if x is adjacent to u, then x, v_4 is dense (the edge uv_1). But then, the edge v_4v' is in D_x, contradicting Lemma 3.8. Hence, $f(x) = 8$, a contradiction. □

Proof of Theorem 3.3. Let $u \in I$. Proposition 3.21 gives $G[V(G) \setminus I] = G_u$, so $G[V(G) \setminus I]$ is 2-colourable. Using a third colour for the independent set I gives a 3-colouring of G. □

4 Locally *b*-partite graphs

In this section, we prove Theorem 1.3 in the case $a = 1$, showing that, for $b \geqslant 3$, any locally b-partite graph G with minimum degree greater than $(1 - 1/(b + 1/7)) \cdot |G|$ is $(b + 1)$-colourable.

The proof of this will be an induction upon b, with some ideas from the proof of Theorem 3.3 persisting. In this introduction to the section, we generalise some of our previous ideas to the locally b-partite case, and then, we give a sketch of the proof. We first generalise dense and sparse pairs.

Definition 4.1. (b-dense and b-sparse) A pair of non-adjacent, distinct vertices u, v in a graph G is *b-dense* if $G_{u,v}$ contains a b-clique and is *b-sparse* if $G_{u,v}$ does not contain a b-clique.

This extends the notion of dense and sparse given in Definition 3.5 – the definitions given there are identical to those of 2-dense and 2-sparse. Note that any pair of distinct vertices is exactly one of 'adjacent', 'b-dense' or 'b-sparse'. Another way to view being b-dense is being the missing edge of a K_{b+2}. Locally b-partite graphs are K_{b+2}-free so any pair of distinct vertices with a b-clique in their common neighbourhood must be non-adjacent and so b-dense. The following lemma will be helpful for lifting results from the locally bipartite case.

Lemma 4.2. (lifting) *Let b, s be positive integers and γ any real with $b + \gamma > s$. Let G be a graph with $\delta(G) > (1 - 1/(b + \gamma)) \cdot |G|$. For any s-set $X \subset V(G)$, we have*

$$|G_X| \geqslant s\delta(G) - (s-1)|G| > \left(1 - \tfrac{s}{b+\gamma}\right) \cdot |G|, \text{ and}$$

$$\delta(G_X) > \left(1 - \tfrac{1}{b-s+\gamma}\right) \cdot |G_X|.$$

Proof. Let $X = \{x_1, \ldots, x_s\}$. Note that for each $v \in V(G)$

$$\mathbb{1}(v \in G_X) \geqslant \mathbb{1}(vx_1 \in E(G)) + \cdots + \mathbb{1}(vx_s \in E(G)) - (s-1),$$

and summing over $v \in V(G)$ gives

$$|G_X| \geqslant s\delta(G) - (s-1)|G| > s\left(1 - \tfrac{1}{b+\gamma}\right)|G| - (s-1)|G| = \left(1 - \tfrac{s}{b+\gamma}\right) \cdot |G|.$$

Note that $\delta(G_X) \geqslant \delta(G) - (|G| - |G_X|) = |G_X| - (|G| - \delta(G))$ so

$$\frac{\delta(G_X)}{|G_X|} \geqslant 1 - \frac{|G| - \delta(G)}{|G|} \cdot \frac{|G|}{|G_X|} > 1 - \left(1 - \frac{\delta(G)}{|G|}\right) \cdot \frac{1}{1 - s/(b+\gamma)}$$
$$> 1 - \frac{1}{b+\gamma} \cdot \frac{1}{1 - s/(b+\gamma)} = 1 - \frac{1}{b+\gamma-s}.$$

□

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Remark 4.3. If, in Lemma 4.2, X is an s-clique and G is locally b-partite, then G_X is $(b-s+1)$-colourable and for any $u, v \in G_X$, if the pair u, v is b-sparse in G, then u, v is $(b-s)$-sparse in G_X. (The former can be seen by taking a vertex $x \in X$ and noting that G_x is b-colourable and contains the $(s-1)$-clique $X - \{x\}$ joined to G_X; the latter by noting that if vertices $u, v \in G_X$ form a $(b-s)$-dense pair in G_X, then they form a b-dense pair in G.)

The next lemma extends Lemma 3.8 and partly explains why the situation for locally b-partite is simpler for $b \geqslant 3$ than for $b = 2$. Lemma 3.8 said that in any locally bipartite H_0-free graph, the set of vertices which form a dense pair with a fixed vertex is independent. Here, in place of H_0-free, we give a minimum degree condition which guarantees this – for $b \geqslant 3$ this minimum degree condition falls below the chromatic threshold so is, for our purposes, automatic. When $b = 2$, this minimum degree condition is 4/7, which corresponds to $\overline{C}_7$.

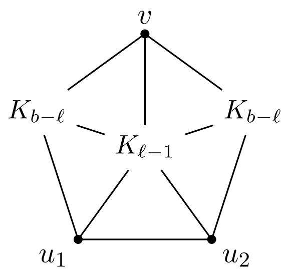
Lemma 4.4. *Let $b \geqslant 2$ be an integer and G be a locally b-partite graph with $\delta(G) > 2b/(2b+3) \cdot |G|$. For each vertex v of G,*

$$D_v := \{u\colon \textit{the pair } u, v \textit{ is } b\textit{-dense}\}$$

is an independent set of vertices.

Proof. Suppose that in fact there are vertices v, u_1 and u_2 with both pairs v, u_1 and v, u_2 b-dense as well as u_1 adjacent to u_2. Let Q_1 and Q_2 be b-cliques in G_{v,u_1} and $G_{v,u_2,}$ respectively. Choose Q_1 and Q_2 so that $\ell = |V(Q_1) \cap V(Q_2)|$ is maximal.

Firstly if $\ell \geqslant 1$, then fix $y \in V(Q_1) \cap V(Q_2)$. Now G_y contains



The pair v, u_1 is $(b-1)$-dense in G_y so in any b-colouring of G_y, v and u_1 are the same colour. Similarly in any b-colouring of G_y, v and u_2 are the same colour. In particular, G_y is not b-colourable which contradicts the local b-colourability of G.

Hence, $\ell = 0$. Let $X = V(Q_1) \cup V(Q_2) \cup \{v, u_1, u_2\}$ which is a set of $2b+3$ vertices. As $\delta(G) > 2b/(2b+3) \cdot |G|$, some vertex has at least $2b+1$ neighbours in X – call this vertex x. As G_x is K_{b+1}-free, x has a non-neighbour $x_1 \in V(Q_1) \cup \{u_1\}$ and a non-neighbour $x_2 \in V(Q_2) \cup \{u_2\}$. These must be the only non-neighbours of x in X. In particular, x is adjacent to v and so, as G_x is K_{b+1}-free, x_1 must be in $V(Q_1)$ and x_2 must be in $V(Q_2)$. In particular, $Q_1' = Q_1 - \{x_1\} + \{x\}$ is a b-clique in G_{v,u_1} and $Q_2' = Q_2 - \{x_2\} + \{x\}$ is a b-clique in G_{v,u_2}. But $\left|V(Q_1') \cap V(Q_2')\right| = 1$ which contradicts the maximality of ℓ. □

Note that

$$1 - \tfrac{1}{b} \geqslant \tfrac{2b}{2b+3},$$

for all $b \geqslant 3$. We are only interested in locally b-partite graphs with $\delta(G) > (1 - 1/b) \cdot |G|$ (the chromatic threshold) and the conclusion of Lemma 4.4 holds for all such graphs.

We are now ready to give a sketch of Theorem 1.3 for locally b-partite graphs. Let G be a locally b-partite graph ($b \geqslant 3$) with $\delta(G) > (1 - 1/(b + 1/7))|G|$. We aim to show that G is $(b+1)$-colourable. We may assume that G is edge-maximal: for any missing edge uv, $G' = G + uv$

is not locally b-partite. Using induction on b and the lifting lemma, we will show there is some $(b-2)$-clique K with G'_K not 3-colourable.

We will rule out the configurations where at least one of u, v is in K and so $u, v \notin K$. Since G is locally b-partite, G_K is 3-colourable and also, using the lifting lemma, we will obtain $\delta(G_K) > 8/15 \cdot |G_K|$. As G_K is 3-colourable while G'_K is not, we must have $G'_K = G_K + uv$. Further, by Theorem 3.3, the addition of uv to G_K introduces an odd wheel, a copy of H_2 or a copy of T_0. We are then in a position to use Lemma 4.4 and finish in similar (although more involved) way to Section 3.4.

The ruling out of configurations is done in Section 4.1 while the rest of the proof is presented in Section 4.2.

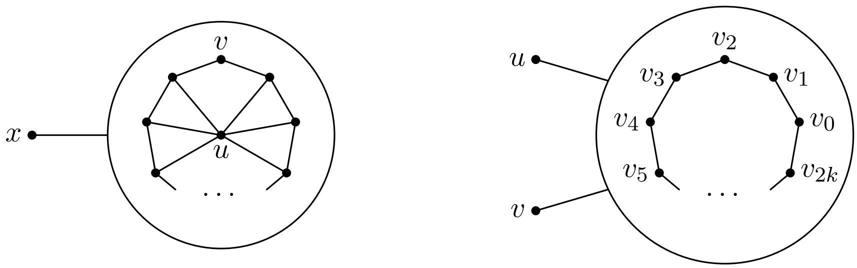
4.1 Ruling out configurations

In this section, we will rule out various configurations from locally b-partite graphs in a similar vein to Section 3.3. From now on, we will use C_{odd} to denote any odd cycle.

Proposition 4.5. *Fix an integer $b \geqslant 3$, let G be a locally b-partite graph with $\delta(G) > (1 - 1/(b + 1/7)) \cdot |G|$ and let u, v be a b-sparse pair in G. Then $G + uv$ does not contain a $K_{b-1} + C_{\text{odd}}$ where at least one of u, v is in the K_{b-1}.*

Proof. We will prove this for $b = 3$ and then use the lifting lemma for larger b.

For $b = 3$, G is a locally tripartite graph with $\delta(G) > 15/22 \cdot |G| > 2/3 \cdot |G|$ and u, v is a 3-sparse pair in G. Suppose the conclusion does not hold. Each neighbourhood of G is 3-colourable so does not contain an odd wheel. In particular, G does not contain $K_2 + C_{\text{odd}}$, and so G contains one of the following two configurations (corresponding to whether only one of u, v is in the clique or they both are). In the left figure, x is adjacent to all vertices inside the ring, and in the right figure, u and v are adjacent to all vertices inside the ring. In the future, we will use rings in this way.



We deal with the left-hand configuration first. Odd wheels, H_0 and T_0 are not 3-colourable and so G_x is locally bipartite, H_0-free and T_0-free. Applying Lemma 4.2 with $X = \{x\}$ and $\gamma = 1/7$ gives $\delta(G_x) > (1 - 1/(2 + 1/7)) \cdot |G_x| = 8/15 \cdot |G_x|$ so, by Corollary 3.18, G_x does not contain a sparse pair which is the missing spoke of an odd wheel. In particular, the pair u, v is dense in G_x so must be 3-dense in G, a contradiction.

Now consider the right-hand configuration. We may assume that k is minimal. As the pair u, v is 3-sparse, $k > 1$. Let $X = \{u, v, v_0, v_1, \ldots, v_{2k}\}$. We consider indices modulo $2k + 1$.

First consider a vertex x adjacent to both u and v. We claim that x is adjacent to at most two of the v_i so has at most four neighbours in X. For $r \notin \{0, \pm 2\}$, x cannot be adjacent to both v_i and v_{i+r}. Indeed if $r = \pm 1$, then xv_iv_{i+r} is a triangle in $G_{u,v}$ while other r give a shorter odd cycle in $G_{u,v}$.

Now consider any other vertex x: this is not adjacent to at least one of u or v. If this is the only non-neighbour of x in X, then G_x contains an odd-wheel, which is not 3-colourable. Thus, all vertices have at most $2k + 1$ neighbours in X. Hence,

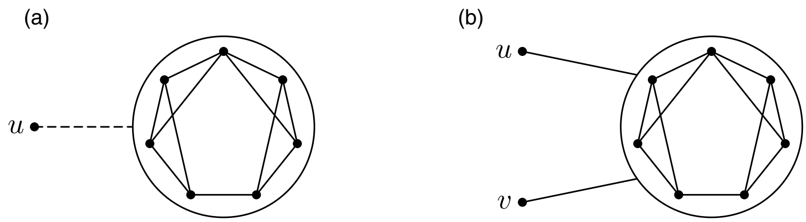


Figure 6. Configurations in Proposition 4.6.

$$(2k+3)\delta(G) \leqslant e(G,X) \leqslant 4\left|G_{u,v}\right| + (2k+1)(|G| - \left|G_{u,v}\right|)$$
$$\leqslant (2k+1)|G| - (2k-3)(2\delta(G) - |G|) = (4k-2)|G| - (4k-6)\delta(G),$$

which contradicts $\delta(G) > 2/3 \cdot |G|$.

Suppose now that $b \geqslant 4$ and that $G + uv$ does contain a $K_{b-1} + C_{\text{odd}}$ where at least one of u, v is in the K_{b-1}. The $(b-1)$-clique contains a $(b-3)$-clique, L, with $u, v \notin L$. Thus L is a $(b-3)$-clique in G.

Now G_L is a 4-colourable (so locally tripartite) graph in which u, v is 3-sparse (by Remark 4.3). Applying Lemma 4.2 with $X = L$, $\gamma = 1/7$ gives $\delta(G_L) > (1 - 1/(b - (b-3) + 1/7)) \cdot |G_L| = 15/22 \cdot |G_L|$. Finally, $G_L + uv$ contains a $K_2 + C_{\text{odd}}$ where at least one of u, v is in the 2-clique. This contradicts the result for $b = 3$. □

Proposition 4.6. *Fix an integer $b \geqslant 3$, let G be a locally b-partite graph with $\delta(G) > (1 - 1/(b + 1/7)) \cdot |G|$ and let u, v be a b-sparse pair in G. Then, $G + uv$ does not contain a $K_{b-2} + H_0$ where at least one of u, v is in the K_{b-2}.*

Proof. We split into two cases depending upon whether only one of u, v is in the $(b-2)$-clique or both are. In each case, we will prove the result for small b and then use the lifting lemma for larger b.

Suppose only one of u, v is in the $(b-2)$-clique. We first prove the result for $b = 3$: G is locally tripartite, the pair u, v is 3-sparse, and $\delta(G) > 15/22 \cdot |G| > 2/3 \cdot |G|$. Suppose the conclusion does not hold. Since G does not contain $K_1 + H_0$, G contains the configuration shown in Figure 6(a) where u is adjacent to all of the copy of H_0 except for one vertex (v) to which is it 3-sparse.

Let X be the vertex set of the H_0. First, consider a vertex x adjacent to u: x cannot be adjacent to a triangle or 5-cycle in $G[X]$ otherwise $G + uv$ contains a 2-clique ux joined to an odd cycle which contradicts Proposition 4.5. In particular, any neighbour of u has at most four neighbours in X.

Consider any vertex x: $\chi(G[X]) = 4$ so x has at most six neighbours in X. Hence,

$$7\delta(G) \leqslant e(G,X) \leqslant 4d(u) + 6(|G| - d(u)) \leqslant 6|G| - 2\delta(G),$$

which contradicts $\delta(G) > 15/22 \cdot |G| > 2/3 \cdot |G|$.

Now let $b \geqslant 4$ and suppose $G + uv$ does contain a $K_{b-2} + H_0$ where exactly one of u, v (say u) is in the $(b-2)$-clique. Graph G is locally b-partite so does not contain $K_{b-2} + H_0$, so v is in the copy of H_0. Let L be the $(b-2)$-clique without u: L is a $(b-3)$-clique in G and so G_L is 4-colourable and so locally tripartite. Also, by Lemma 4.2, $\delta(G_L) > 15/22 \cdot |G_L|$ and, by Remark 4.3, u, v is a 3-sparse pair in G_L. Finally, $G_L + uv$ contains $u + H_0$, which contradicts the result we just proved for $b = 3$.

Now consider the second case, where both u and v are in the $(b-2)$-clique: this means that $b \geqslant 4$. We first prove the result for $b = 4$: G is locally 4-partite, the pair u, v is 4-sparse, and $\delta(G) > 22/29 \cdot |G| > 3/4 \cdot |G|$. If the result is false, then G contains the configuration shown in Figure 6(b).

Let $X = V(H_0) \cup \{u\}$ – a set of eight vertices. First, consider a vertex x adjacent to both u and v: x cannot be adjacent to a triangle or 5-cycle in $V(H_0)$ as this would contradict Proposition 4.5. Hence, x has at most four neighbours in $V(H_0)$ and so at most five in X.

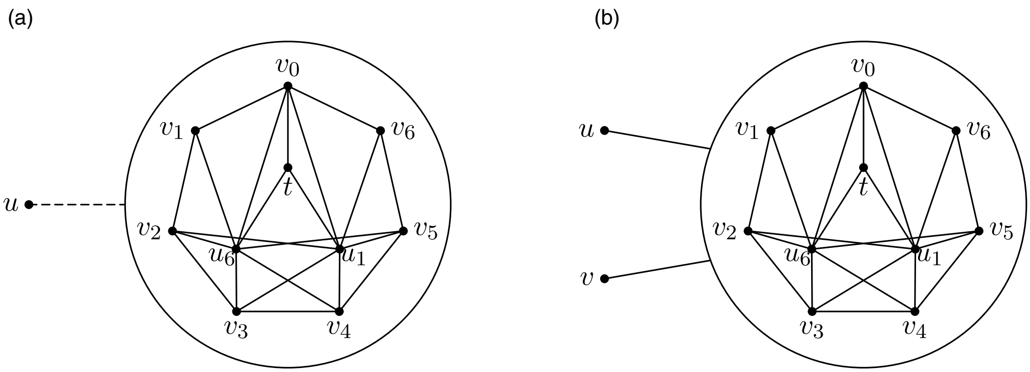


Figure 7. Configurations in Proposition 4.7.

Since $G[X]$ is not 4-colourable, all vertices have at most seven neighbours in X. Hence,

$$8\delta(G) \leqslant 5\left|G_{u,v}\right| + 7(|G| - \left|G_{u,v}\right|)$$
$$\leqslant 7|G| - 2(2\delta(G) - |G|) = 9|G| - 4\delta(G),$$

which contradicts $\delta(G) > 22/29 \cdot |G| > 3/4 \cdot |G|$.

Finally, suppose $b \geqslant 5$ and $G + uv$ does contain a $K_{b-2} + H_0$ where both of u, v are in the $(b-2)$-clique. Let L be the $(b-2)$-clique without u, v: L is a $(b-4)$-clique in G so, by Remark 4.3, G_L is 5-colourable and so locally 4-partite. Also, by Lemma 4.2, $\delta(G_L) > 22/29 \cdot |G_L|$ and u, v is a 4-sparse pair in G_L. Finally, $G_L + uv$ contains $uv + H_0$, which contradicts the result we obtained for $b = 4$. □

Proposition 4.7. *Fix an integer $b \geqslant 3$, let G be a locally b-partite graph with $\delta(G) > (1 - 1/(b + 1/7)) \cdot |G|$ and let u,v be a b-sparse pair in G. Then $G + uv$ does not contain a $K_{b-2} + T_0$ where at least one of u, v is in the $(b-2)$-clique.*

Proof. Again we split into two cases depending upon whether only one of u, v is in the $(b-2)$-clique or both are, and in each case, we will prove the result for small b and then use the lifting lemma for larger b.

Suppose only one of u, v is in the $(b-2)$-clique. We first prove the result for $b = 3$: G is locally tripartite, the pair u, v is 3-sparse, and $\delta(G) > 15/22 \cdot |G| > 2/3 \cdot |G|$. Suppose the conclusion does not hold. Since G does not contain $K_1 + T_0$, G contains the configuration shown in Figure 7(a) (labels have been added for convenience) where u is adjacent to all of the copy of T_0 except for one vertex (v) to which it is 3-sparse.

We first show that at least one of the pairs u_1, v_1 and u_6, v_6 is 3-sparse. Neither of these is an edge as otherwise $G + uv$ contains u joined to a 7-wheel which contradicts Proposition 4.5. If $v \notin \{u_1, u_6, v_3, v_4\}$, then u_1, u_6 is 3-dense (triangle uv_3v_4) and so u_1, v_1 is 3-sparse, by Lemma 4.4. On the other hand, suppose $v \in \{u_1, u_6, v_3, v_4\}$ – by symmetry we may assume $v \neq u_6$. Then, v_1, t is 3-dense (triangle uv_0u_6) and so u_1, v_1 is 3-sparse, by Lemma 4.4. From now on, we will assume that the pair u_1, v_1 is 3-sparse in G.

Let $G' = G + uv$ – we work in G'. From Proposition 4.5, G'_u contains no odd wheel, i.e. is locally bipartite, and from Proposition 4.6, G'_u is H_0-free. Now u_1v_1 is not an edge (else there is a 7-wheel in G'_u) and the pair v_1, t is 2-dense in G'_u so u_1, v_1 is 2-sparse in G'_u, by Lemma 3.8. Note that $\delta(G') > 2/3 \cdot \left|G'\right|$ and so applying Lemma 4.2 with $b = 3$, $\gamma = 0$, $s = 1$ and $X = \{u\}$ gives

$$\delta(G'_u) > \left(1 - \tfrac{1}{3-1}\right) \cdot |G'_u| = 1/2 \cdot |G'_u|.$$

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Hence, we may apply Lemma 3.11 to show that u_1v_1 is not a missing spoke of a 5-wheel in G'_u. Hence, by Lemma 3.12, any neighbour of u_1 in G'_u is adjacent to at most two of the v_i. In particular, any common neighbour of u and u_1 in G' is adjacent to at most two of the v_i.

Let $X = \{u, u_1, v_0, v_1, \ldots, v_6\}$. What we have just shown is that any common neighbour of u and u_1 in G' has at most four neighbours in X. Consider a vertex x which is not adjacent to both u and u_1: x cannot be adjacent to all of $X \setminus \{u_1\}$ otherwise $G'_{u,x}$ contains a 7-cycle which contradicts Proposition 4.5. Also, x cannot be adjacent to all of $X \setminus \{u\}$ as otherwise $G'_x = G_x$ contains a 7-wheel missing the spoke u_1v_1 which is 3-sparse in G. This again contradicts Proposition 4.5. Hence, any vertex has at most seven neighbours in X. Thus,

$$\begin{aligned}9\delta(G) &\leqslant 9\delta(G') \leqslant 4|G'_{u,u_1}| + 7(\left|G'\right| - |G'_{u,u_1}|) = 7|G| - 3|G'_{u,u_1}| \\ &\leqslant 7|G| - 3\left|G_{u,u_1}\right| \leqslant 7|G| - 3(2\delta(G) - |G|) = 10|G| - 6\delta(G),\end{aligned}$$

which contradicts $\delta(G) > 2/3 \cdot |G|$.

Now let $b \geqslant 4$ and suppose $G + uv$ does contain a $K_{b-2} + T_0$ where exactly one of u, v (say u) is in the $(b-2)$-clique. Graph G is locally b-partite so does not contain $K_{b-2} + T_0$ so v is in the copy of T_0. Let L be the $(b-2)$-clique without u. We make use of Remark 4.3: L is a $(b-3)$-clique in G and so G_L is 4-colourable and so locally tripartite. Also, by Lemma 4.2, $\delta(G_L) > 15/22 \cdot |G_L|$ and u, v is a 3-sparse pair in G_L. Finally, $G_L + uv$ contains $u + T_0$, which contradicts the result we just proved for $b = 3$.

Now consider the second case, where both u and v are in the $(b-2)$-clique: this means that $b \geqslant 4$. We first prove the result for $b = 4$: G is locally 4-partite, the pair u, v is 4-sparse, and $\delta(G) > 22/29 \cdot |G| > 3/4 \cdot |G|$. If the result is false, then G contains the configuration shown in Figure 7(b) (labels have been added for convenience).

By Propositions 4.5 and 4.6, $G_{u,v}$ is locally bipartite and H_0-free. By Lemma 3.2, $\delta(G_{u,v}) > 1/2 \cdot \left|G_{u,v}\right|$. Since $G_{u,v}$ does not contain an odd wheel, u_1v_1 is not an edge. Also, v_1, t is a 2-dense pair, tu_1 is an edge and $G_{u,v}$ is H_0-free, so, by Lemma 3.8, u_1, v_1 is a 2-sparse pair in $G_{u,v}$. Similarly, u_6, v_6 is 2-sparse in $G_{u,v}$.

Now let $X = \{u, u_1, u_6, v_0, \ldots, v_6\}$ (note that this does not contain t or v). Within $G_{u,v}$, u_1 together with the v_i form a 7-wheel missing the spoke u_1v_1 which is a sparse pair. Since $G_{u,v}$ is locally bipartite with $\delta(G_{u,v}) > 1/2 \cdot \left|G_{u,v}\right|$, Lemma 3.11 implies that $G_{u,v}$ does not contain any 5-wheels missing a sparse spoke. Hence, by Lemma 3.12, any neighbour of u_1 in $G_{u,v}$ is adjacent to at most two of the v_i. Thus, any neighbour of u, v, u_1 has at most five neighbours in X (two amongst v_i together with possibly u_1, u_6, u). Similarly, any neighbour of u, v, u_6 has at most five neighbours in X. Next, consider a vertex x adjacent to both u, v but to neither u_1 nor u_6. As $G_{u,v}$ is locally bipartite, x is adjacent to at most six of the v_i so x has at most seven neighbours in X. Finally, $\chi(G[X]) = 5$ so all vertices have at most nine neighbours in X. Hence,

$$\begin{aligned}10\delta(G) &\leqslant 5|\Gamma(u,v,u_1) \cup \Gamma(u,v,u_6)| + 7(\left|G_{u,v}\right| - |\Gamma(u,v,u_1) \cup \Gamma(u,v,u_6)|) + 9(|G| - \left|G_{u,v}\right|) \\ &= 9|G| - 2\left|G_{u,v}\right| - 2|\Gamma(u,v,u_1) \cup \Gamma(u,v,u_6)| \leqslant 9|G| - 2\left|G_{u,v}\right| - 2\left|G_{u,v,u_1}\right| \\ &\leqslant 9|G| - 2(2\delta(G) - |G|) - 2(3\delta(G) - 2|G|) = 15|G| - 10\delta(G),\end{aligned}$$

which contradicts $\delta(G) > 3/4 \cdot |G|$.

Now let $b \geqslant 5$ and suppose $G + uv$ does contain a $K_{b-2} + T_0$ where both of u, v are in the $(b-2)$-clique. Let L be the $(b-2)$-clique without u, v: L is a $(b-4)$-clique in G and so G_L is 5-colourable and so locally 4-partite. Also, by Lemma 4.2, $\delta(G_L) > 22/29 \cdot |G_L|$ and, by Remark 4.3, the pair u, v is 4-sparse in G_L. Finally, $G_L + uv$ contains $uv + T_0$, which contradicts the result just proved for $b = 4$. □

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4.2 Finishing the proof

Here we will prove Theorem 1.3 for locally b-partite graphs.

Proof. Take an edge-maximal locally b-partite graph G with $\delta(G) > (1 - 1/(b + 1/7)) \cdot |G|$. We need to show that G is $(b+1)$-colourable. We may assume by induction that the theorem holds for all b' with $3 \leqslant b' < b$ (if there are any).

We first show that for any b-sparse pair u, v of G, $G' = G + uv$ contains a $(b-2)$-clique K with G'_K not 3-colourable. Indeed, for $b = 3$, G' is not locally tripartite (by edge-maximality) so there is a vertex in G' whose neighbourhood is not 3-colourable. Take K to be this vertex. For $b > 3$, G' is not locally b-partite so contains a vertex w_1 with G'_{w_1} not b-colourable. Applying Lemma 4.2 with $X = \{w_1\}$ and $\gamma = 1/7$ gives

$$\delta(G'_{w_1}) > \left(1 - \frac{1}{b - 1 + 1/7}\right) \cdot |G'_{w_1}|.$$

By the induction hypothesis, if G'_{w_1} was locally $(b-1)$-partite, then it would be b-colourable. In particular, G'_{w_1} is not locally $(b-1)$-partite and so there is a vertex w_2 in G'_{w_1} with G'_{w_1,w_2} not $(b-1)$-colourable. Repeating this argument gives a $(b-2)$-clique K with G'_K not 3-colourable.

Now, applying Lemma 4.2 with $X = K$ and $\gamma = 1/7$ gives

$$\delta(G'_K) > \left(1 - \frac{1}{b - (b-2) + 1/7}\right) \cdot |G'_K| = 8/15 \cdot |G'_K|.$$

By Theorems 3.3 and 3.4, G'_K contains either an odd wheel, a copy of H_2, or a copy of T_0. Hence, G' contains either $K_{b-2} + W_{\text{odd}} = K_{b-1} + C_{\text{odd}}$, $K_{b-2} + H_2$ or $K_{b-2} + T_0$. Note that G cannot contain any of these so uv is a missing edge from one of these configurations. Propositions 4.5 to 4.7 mean that both u and v lie in the C_{odd}, the H_2 or the T_0.

In particular, $u, v \notin K$ so K is a $(b-2)$-clique in G and $V(G_K) = V(G'_K)$. We have the following facts.

- By Remark 4.3, G_K is 3-colourable and so locally bipartite.
- By Remark 4.3, u, v is a 2-sparse pair in G_K.
- Applying Lemma 4.2 with $X = K$ and $\gamma = 1/7$ gives $\delta(G_K) > 8/15 \cdot |G_K|$.
- The graph G_K contains no odd wheel, H_0 or T_0 (G_K is 3-colourable) but the addition of uv introduces an odd wheel, a copy of H_2, or a copy of T_0.

Using the argument at the start of Section 3.4, we deduce that, within G_K, there must be one of the configurations appearing in Figure 5 (with labels u and v possibly swapped). Note in that proof we only used edge-maximality to show that uv was the missing edge of an odd wheel, a copy of H_2 or a copy of T_0 (and so here we do not need G_K to be an edge-maximal locally bipartite graph).

We now mimic the remainder of the proof of Theorem 3.3. Let I be a largest independent set in G: $|I| = \alpha(G)$.

Proposition 4.8. *For all distinct $u, v \in I$, the pair u, v is b-dense and furthermore every $u \in I$ has $I = D_u \cup \{u\}$.*

Proof. Fix distinct $u, v \in I$. We will first show that $G_{u,v}$ is not $(b-1)$-colourable. Note that $\Gamma(u), \Gamma(v) \subset V(G) \setminus I$ so $|\Gamma(u) \cup \Gamma(v)| \leqslant |G| - |I|$. Also $I \subset V(G) \setminus \Gamma(u)$, so $|I| \leqslant |G| - d(u) \leqslant |G| - \delta(G)$. Hence,

$$\begin{aligned}
|\Gamma(u, v)| &= d(u) + d(v) - |\Gamma(u) \cup \Gamma(v)| \geqslant 2\delta(G) + |I| - |G| \\
&= b\delta(G) - (b-1)|G| + (b-2)(|G| - \delta(G)) + |I| \\
&\geqslant b\delta(G) - (b-1)|G| + (b-1)|I| > (b-1)|I|,
\end{aligned}$$

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where we used $\delta(G) > (1-1/b)\cdot|G|$ in the final inequality. But I was a largest independent set in G so $G_{u,v}$ is not $(b-1)$-colourable.

Now, we will show that u, v is b-dense. Suppose not and so they form a b-sparse pair. If $b=3$, then $G_{u,v}$ is not bipartite and so contains an odd cycle. This contradicts Proposition 4.5. For $b \geqslant 4$, we will find a $(b-4)$-clique K in $G_{u,v}$ with $G_{u,v,K}$ not 3-colourable. For $b=4$, we take $K=\varnothing$ and this suffices. For larger b, we note that, by Lemma 4.2, $\delta(G_{u,v}) > (1-1/(b-2+1/7))\cdot\left|G_{u,v}\right|$. By the induction hypothesis, if $G_{u,v}$ were locally $(b-2)$-partite, then $G_{u,v}$ would be $(b-1)$-colourable, which it is not. Thus, there is $w_1 \in G_{u,v}$ with G_{u,v,w_1} not $(b-2)$-colourable. Repeating this argument we obtain a $(b-4)$-clique K with $G_{u,v,K}$ not 3-colourable. Applying Lemma 4.2 with $X=\{u,v\}\cup K$ and $\gamma=1/7$ gives

$$\delta(G_{u,v,K}) > (1-1/(b-(b-2)+1/7))\cdot\left|G_{u,v,K}\right| = 8/15\cdot\left|G_{u,v,K}\right|.$$

Then, Theorem 3.3 gives that $G_{u,v,K}$ either contains an odd wheel, a copy of H_0 or a copy of T_0. These contradict Propositions 4.5 to 4.7 (applied to G). We have shown that u, v must be b-dense. Thus, $I\subset D_u\cup\{u\}$.

On the other hand, by the definition of density and Lemma 4.4, $D_u\cup\{u\}$ is an independent set. It contains the maximal independent set I so must equal it. □

Definition 4.9. (b-quasidense) A pair of vertices u, v is b-quasidense if there is a sequence of vertices $u=d_1, d_2, \ldots, d_k, d_{k+1}=v$ such that all pairs d_i, d_{i+1} are b-dense ($i=1,2,\ldots,k$).

Proposition 4.8 immediately implies that if u, v is b-quasidense and $u\in I$, then $v\in I$ as well. Now we can finish the proof. It suffices to show that every vertex is either in I or is adjacent to all of I. Indeed, we may then fix $u\in I$ and note that $G[V(G)\setminus I]=G_u$ so $G[V(G)\setminus I]$ is b-colourable. Using a further colour for the independent set I gives a $(b+1)$-colouring of G.

Suppose instead there is $u\in I$ and $v\notin I$ with u not adjacent to v. In particular, the pair u, v cannot be b-quasidense and so is b-sparse. Thus, from our remarks just preceding Proposition 4.8, there is a $(b-2)$-clique K in G such that G_K contains one of the configurations appearing in Figure 5 (with labels u and v possibly swapped) and the pair u, v is 2-sparse in G_K.

Focus on G_K – this graph is 3-colourable so locally bipartite, H_0-free and T_0-free and the pair u, v is 2-sparse in G_K. Also, by Lemma 4.2, $\delta(G_K) > (1-1/(b-(b-2)+1/7))\cdot|G_K| = 8/15\cdot|G_K|$. In the proof of Proposition 3.21, we used these facts alone to show that u, v is quasidense in every configuration appearing in Figure 5. Hence, the pair u, v is quasidense in G_K so is b-quasidense in G. This is our required contradiction. □

5 *a*-locally *b*-partite graphs

In this section, we relate the chromatic profile of a-locally b-partite graphs to the chromatic profile of locally b-partite graphs, making precise our comment in the introduction that to understand a-locally b-partite graphs it seems to be enough to understand locally b-partite graphs. This is elucidated at the end of Section 5.1 and just before Theorem 5.6. Along the way, we will prove Theorems 1.3 and 1.4.

5.1 The first threshold – proving Theorems 1.3 and 1.4

As noted in the introduction, the first interesting threshold is $\delta_\chi(\mathcal{F}_{a,b}, a+b)$ – what values of c guarantee that any a-locally b-partite graph with $\delta(G)\geqslant c|G|$ is $(a+b)$-colourable? We already know $\delta_\chi(\mathcal{F}_{1,2},3)=4/7$ and $\delta_\chi(\mathcal{F}_{1,b},b+1)\leqslant 1-1/(b+1/7)$ and will extend these to all values of a. To simplify the statements of our results and make comparisons between different values of a and b, it is helpful to write

$$\delta_\chi(\mathcal{F}_{a,b}, a+b) = 1-\frac{1}{a+b-1+\gamma_{a,b}},$$

and to focus our attention on the $\gamma_{a,b}$. As $\delta_\chi(\mathcal{F}_{a,b}, a+b) \geqslant \delta_\chi(\mathcal{F}_{a,b})$ we have, from Theorem 1.2,

$$\gamma_{a,b} \geqslant 0.$$

We collect some other basic properties of the $\gamma_{a,b}$.

Lemma 5.1. *For all positive integers a and b the following hold.*

- $\delta_\chi(\mathcal{F}_{a,b+1}, a+b) \leqslant \delta_\chi(\mathcal{F}_{a+1,b}, a+b)$ *and so*
$$\gamma_{a,b+1} \leqslant \gamma_{a+1,b}. \tag{3}$$
- $1/(2-\delta_\chi(\mathcal{F}_{a,b}, a+b)) \leqslant \delta_\chi(\mathcal{F}_{a+1,b}, a+b+1)$ *and so*
$$\gamma_{a,b} \leqslant \gamma_{a+1,b}. \tag{4}$$

Also $\gamma_{1,2} = 1/3$ and $\gamma_{1,b} \leqslant 1/7$ for all $b \geqslant 3$.

Proof. In [11], it was shown that $\delta_\chi(\mathcal{F}_{1,2}, 3) = 4/7$ giving $\gamma_{1,2} = 1/3$ while, for $b \geqslant 3$, Section 4 showed $\delta_\chi(\mathcal{F}_{1,b}, b+1) \leqslant 1 - 1/(b+1/7)$ so $\gamma_{1,b} \leqslant 1/7$.

Now $\mathcal{F}_{a,b+1} \subset \mathcal{F}_{a+1,b}$ from which $\delta_\chi(\mathcal{F}_{a,b+1}, a+b) \leqslant \delta_\chi(\mathcal{F}_{a+1,b}, a+b)$ immediately follows. This gives inequality (3).

Finally, let $d < \delta_\chi(\mathcal{F}_{a,b}, a+b)$: there is an a-locally b-partite graph G with $\delta(G) \geqslant d|G|$ and $\chi(G) > a+b$. Let G' be G joined to an independent set of size $|G| - \delta(G)$, that is,

$$G' = K_1(|G| - \delta(G)) + G.$$

Since G is a-locally b-partite, it is also $(a+1)$-locally $(b-1)$-partite. From both of these, it follows that G' is $(a+1)$-locally b-partite. Also, $\chi(G') = \chi(G) + 1 > a+b+1$, and

$$\frac{\delta(G')}{|G'|} = \frac{|G|}{2|G| - \delta(G)} = \frac{1}{2 - \delta(G)|G|^{-1}} \geqslant \frac{1}{2-d},$$

so $\delta_\chi(\mathcal{F}_{a+1,b}, a+b+1) \geqslant 1/(2-d)$. This holds for all $d < \delta_\chi(\mathcal{F}_{a,b}, a+b)$, so $\delta_\chi(\mathcal{F}_{a+1,b}, a+b+1) \geqslant 1/(2-\delta_\chi(\mathcal{F}_{a,b}, a+b))$. Thus,

$$1 - \frac{1}{a+b+\gamma_{a+1,b}} \geqslant \frac{1}{1 + \frac{1}{a+b-1+\gamma_{a,b}}} = \frac{a+b-1+\gamma_{a,b}}{a+b+\gamma_{a,b}} = 1 - \frac{1}{a+b+\gamma_{a,b}},$$

and so $\gamma_{a+1,b} \geqslant \gamma_{a,b}$, as required. □

Inequality (4) gives a lower bound for $\gamma_{a+1,b}$ in terms of $\gamma_{a,b}$. The next lemma, which lies at the heart of our analysis, gives an upper bound.

Lemma 5.2. *For all positive integers a and b,*

$$\gamma_{a,b} \leqslant \gamma_{a+1,b} \leqslant \max\{\gamma_{a,b}, \gamma_{1,a+b}\}. \tag{5}$$

Proof. The left-hand inequality is just inequality (4). Let $\gamma = \max\{\gamma_{a,b}, \gamma_{1,a+b}\}$. Let G be an $(a+1)$-locally b-partite graph with

$$\delta(G) > \left(1 - \frac{1}{a+b+\gamma}\right) \cdot |G|.$$

It suffices to show that $\chi(G) \leqslant a+b+1$ as then

$$1 - \frac{1}{a+b+\gamma} \geqslant \delta_\chi(\mathcal{F}_{a+1,b}, a+b+1) = 1 - \frac{1}{a+b+\gamma_{a+1,b}}.$$

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Fix any $u \in V(G)$ and consider G_u: G_u is an a-locally b-partite graph with

$$\delta(G_u) > \left(1 - \frac{1}{a+b-1+\gamma}\right) \cdot |G_u|,$$

by the lifting lemma, Lemma 4.2. But $\gamma \geqslant \gamma_{a,b}$ so

$$\delta(G_u) > \delta_\chi(\mathcal{F}_{a,b}, a+b) \cdot |G_u|,$$

and hence G_u is $(a+b)$-colourable. Thus, the graph G is locally $(a+b)$-partite. Also $\gamma \geqslant \gamma_{1,a+b}$, so

$$\delta(G) > \delta_\chi(\mathcal{F}_{1,a+b}, a+b+1) \cdot |G|.$$

Thus, G is $(a+b+1)$-colourable. □

From this, one can immediately deduce Theorems 1.3 and 1.4.

Corollary 5.3. *(Theorems 1.3 and 1.4) For all positive integers a and for all* $b \geqslant 3$*,*

$$\gamma_{a,2} = 1/3, \quad \gamma_{a,b} \leqslant 1/7,$$

and so

$$\delta_\chi(\mathcal{F}_{a,2}, a+2) = 1 - \frac{1}{a+1+1/3}, \qquad \delta_\chi(\mathcal{F}_{a,b}, a+b) \leqslant 1 - \frac{1}{a+b-1+1/7}.$$

Proof. From Lemma 5.1, $\gamma_{1,2} = 1/3$ and $\gamma_{1,b} \leqslant 1/7$ for any $b \geqslant 3$. By Lemma 5.2, for any a and any $b \geqslant 2$,

$$\gamma_{a,b} \leqslant \gamma_{a+1,b} \leqslant \max\{\gamma_{a,b}, \gamma_{1,a+b}\} \leqslant \max\{\gamma_{a,b}, 1/7\}.$$

An easy induction gives $\gamma_{a,2} = 1/3$ for all a and $\gamma_{a,b} \leqslant 1/7$ for any $b \geqslant 3$. □

Note that inequalities (3) and (4) give

$$\gamma_{a,b} \geqslant \gamma_{1,b'},$$

for all $b \leqslant b' \leqslant a+b-1$. In line with our inductive arguments in Section 4, we believe that in fact $\gamma_{1,b} \geqslant \gamma_{1,b+1}$ for all b. If this were true, then $\gamma_{a,b} \geqslant \gamma_{1,a+b}$, and so Lemma 5.2 would give $\gamma_{a,b} = \gamma_{a+1,b}$ for all a, b. Of course, this implies $\gamma_{a,b} = \gamma_{1,b}$ and so $\delta_\chi(\mathcal{F}_{a,b}, a+b)$ would be determined by $\delta_\chi(\mathcal{F}_{1,b}, b+1)$ – a particular manifestation of our aforementioned belief that to understand a-locally b-partite graphs, we should focus on locally b-partite graphs. It also highlights the following question.

Question 5.4. Is the sequence $\gamma_{1,b}$ non-increasing in b?

5.2 a-locally bipartite graphs

One could replicate the elementary approach of the previous section to try to evaluate $\delta_\chi(\mathcal{F}_{a,b}, k)$ for $k > a+b$. Indeed, one might define $\gamma_{a,b,m}$ by

$$\delta_\chi(\mathcal{F}_{a,b}, a+b+m) = 1 - \frac{1}{a+b-1+\gamma_{a,b,m}},$$

so that $\gamma_{a,b,0} = \gamma_{a,b}$. Many of the properties of the $\gamma_{a,b}$ pass over: the $\gamma_{a,b,m}$ are non-negative (and, in fact, $\lim_{m\to\infty} \gamma_{a,b,m} = 0$) and both inequalities (3) and (4) extend easily ($\gamma_{a,b+1,m} \leqslant \gamma_{a+1,b,m}$ and $\gamma_{a,b,m} \leqslant \gamma_{a+1,b,m}$). However, there seems to be no argument to produce inequality (5) or anything similar. A more involved approach would be required.

The next threshold to consider is $\delta_\chi(\mathcal{F}_{a,b}, a+b+1)$. For locally bipartite graphs, we showed $\delta_\chi(\mathcal{F}_{1,2}, 4) \leqslant 6/11$ and had many structural results (some of which we will extend). For $b \geqslant 3$, we know very little about $\delta_\chi(\mathcal{F}_{1,b}, b+2)$ beyond it being at least $\delta_\chi(\mathcal{F}_{1,b}) = 1 - 1/b$ and at most $\delta_\chi(\mathcal{F}_{1,b}, b+1) \leqslant 1 - 1/(b+1/7)$. The question for $b = 3$ is of particular interest. Tantalisingly,

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the complement of the 9-cycle is locally tripartite, 5-chromatic and has minimum degree $6 = 2/3 \cdot 9$.

Question 5.5. Is there a locally tripartite graph G with minimum degree greater than $2/3 \cdot |G|$ which is not 4-colourable?

We now focus on a-locally bipartite graphs. The following theorem, which will be essential for extending the Andrásfai-Erdős-Sós theorem [12], should be compared to Theorem 3.1 – again we see that the key to understanding a-locally bipartite graphs is to understand locally bipartite ones. The proof is an induction combining our results for locally bipartite (Theorem 3.1) and locally b-partite (Theorem 1.3) graphs.

Theorem 5.6. (a-locally bipartite graphs) *Let G be an a-locally bipartite graph.*

- a. *If $\delta(G) > (1 - 1/(a + 4/3)) \cdot |G|$, then G is $(a + 2)$-colourable. Suitable blow-ups of $K_{a-1} + \overline{C}_7$ show that this is tight.*
- b. *If $\delta(G) > (1 - 1/(a + 5/4)) \cdot |G|$, then G is either $(a + 2)$-colourable or contains $K_{a-1} + \overline{C}_7$.*
- c. *If $\delta(G) > (1 - 1/(a + 6/5)) \cdot |G|$, then G is either $(a + 2)$-colourable or contains $K_{a-1} + \overline{C}_7$ or $K_{a-1} + H_2^+$.*
- d. *If $\delta(G) > (1 - 1/(a + 7/6)) \cdot |G|$, then G is either $(a + 2)$-colourable or contains $K_{a-1} + H_2$.*
- e. *If $\delta(G) > (1 - 1/(a + 8/7)) \cdot |G|$, then G is either $(a + 2)$-colourable or contains $K_{a-1} + H_2$ or $K_{a-1} + T_0$.*

Proof. The graph $K_{a-1} + \overline{C}_7$ is a-locally bipartite with chromatic number $a + 3$. Hence, $K_{a-1}(3) + \overline{C}_7$ also has these properties and is, furthermore, $(3a + 1)$-regular with $3a + 4$ vertices. Thus, balanced blow-ups of $K_{a-1}(3) + \overline{C}_7$ give the tightness of the first bullet point. Proving everything else is a simple induction on a (with Theorem 3.1 covering the base case). Indeed, we will just demonstrate it for part e. Let G be an a-locally bipartite graph with $\delta(G) > (1 - 1/(a + 8/7)) \cdot |G|$. Fix any vertex u of G and consider G_u: this is $(a - 1)$-locally bipartite, and by Lemma 4.2, $\delta(G_u) > (1 - 1/(a - 1 + 8/7)) \cdot |G_u|$. By induction, either G_u contains one of $K_{a-2} + H_2$, $K_{a-2} + T_0$ or is $(a + 1)$-colourable. If there is some vertex u with G_u not $(a + 1)$-colourable, then G contains one of $K_{a-1} + H_2$, $K_{a-1} + T_0$. Otherwise, G is locally $(a + 1)$-partite. But, by Theorem 1.3,

$$\delta_\chi(\mathcal{F}_{1,a+1}, a + 2) \leqslant 1 - \frac{1}{a + 8/7},$$

so G is $(a + 2)$-colourable, as required. □

Acknowledgement

It is a pleasure to thank Andrew Thomason for many helpful discussions. I am grateful to the anonymous referees for their careful reading and excellent suggestions for improving the presentation.

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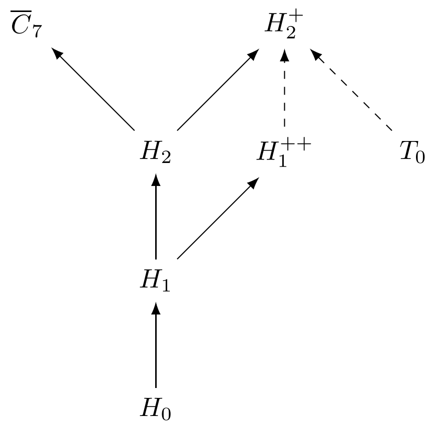
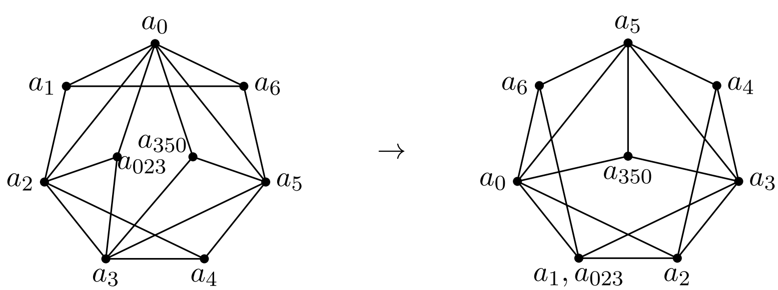
A Verifying Figure 3

In this appendix, we verify Figure 3 which, for convenience, we display again here.



The reader will recall that full arrows represent containment and dashed arrows represent homomorphisms. Furthermore, H is homomorphic to G in the diagram if there is a sequence of arrows starting at H and ending at G.

All the containments are clear. The following figure gives a homomorphism from H_1^{++} to H_2^+: the left diagram is a labelling of the vertices of H_1^{++} and the right diagram shows the images of those vertices under the map.



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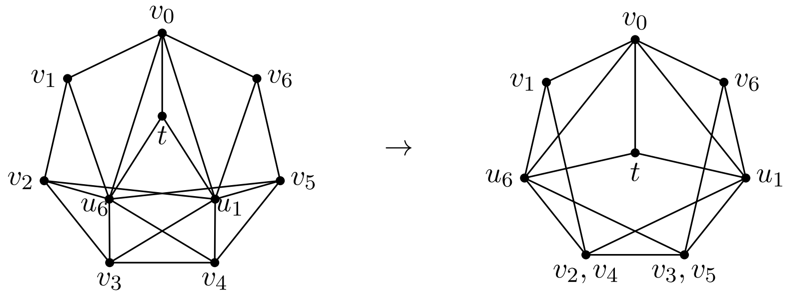
The following figures gives a homomorphism from T_0 to H_2^+: the left diagram is a labelling of the vertices of $T_{0,}$ and the right diagram shows the images of those vertices under the map.



In particular, all arrows in Figure 3 are correct. We need to show that further arrows could not be added. There are some subtleties in our notation that we now elucidate. Given a homomorphism $\varphi\colon H \to G$, we say φ is surjective or injective if the map $\varphi\colon V(H) \to V(G)$ is surjective or injective, respectively. Note that φ being injective implies that H is actually a subgraph of G. By $\varphi(H)$, we mean the graph on vertex set $\varphi(V(H))$ and edge set $\varphi(E(H))$. In particular, this is a spanning subgraph of $G[\varphi(V(H))]$ but it may not have all the edges of $G[\varphi(V(H))]$. We make frequent use of the fact that $\chi(\varphi(H)) \geqslant \chi(H)$.

We first deal with left-hand side ($H_0 \to H_1 \to H_2 \to \overline{C}_7$) of the figure: we need to show that $\overline{C}_7 \nrightarrow H_2$, $H_2 \nrightarrow H_1$ and $H_1 \nrightarrow H_0$. The arguments are very similar, making use of the fact that H_0, H_1, H_2 and $\overline{C}_7$ are all vertex-critical 4-chromatic graphs on seven vertices, so we only give the explicit proof for $H_2 \nrightarrow H_1$.

Proposition A.1. *The graph H_2 is not homomorphic to H_1.*

Proof. Suppose there is a homomorphism $\varphi\colon H_2 \to H_1$. Then, $\chi(\varphi(H_2)) \geqslant \chi(H_2) = 4$. Now, H_1 is a vertex-critical 4-chromatic graph, so φ is surjective. But H_1 and H_2 both have seven vertices so φ is injective. That is, H_1 must contain a copy of H_2, which is absurd as $e(H_1) < e(H_2)$. □

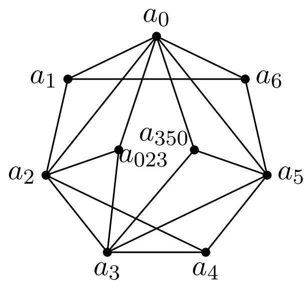
We now know that the left-hand side of Figure 3 is correct and consider how H_2^+ relates to it. It suffices to show that $H_2^+ \nrightarrow \overline{C}_7$ and $\overline{C}_7 \nrightarrow H_2^+$ (note that $H_2^+ \nrightarrow \overline{C}_7$ implies $H_2^+ \nrightarrow H_2, H_1, H_0$). That H_2^+ is not homomorphic to $\overline{C}_7$ and vice versa follows from the next lemma, which appeared in [11] – both $\overline{C}_7$ and H_2^+ are edge-maximal locally bipartite graphs and neither is a subgraph of the other ($\overline{C}_7$ has fewer vertices than H_2^+ and H_2^+ does not have seven vertices all of degree at least four).

Lemma A.2. *Let F be an edge-maximal locally bipartite graph in which no two neighbourhoods are the same. Let F be homomorphic to a locally bipartite graph G. Then, F is an induced subgraph of G.*

Next, we relate H_1^{++} to the diagram. It suffices to show that $H_1^{++} \nrightarrow \overline{C}_7$ and $H_2 \nrightarrow H_1^{++}$ (note that $H_1^{++} \nrightarrow \overline{C}_7$ implies $H_1^{++} \nrightarrow H_2, H_1, H_0$ while $H_2 \nrightarrow H_1^{++}$ implies $\overline{C}_7 \nrightarrow H_1^{++}$ and $H_2^+ \nrightarrow H_1^{++}$).

Proposition A.3. *The graph H_1^{++} is not homomorphic to $\overline{C}_7$.*

Proof. Suppose there is a homomorphism $\varphi\colon H_1^{++} \to \overline{C}_7$. Label the copy of H_1^{++} as shown below and let $A = \{a_0, a_1, \dots, a_6\}$ so $H_1^{++}[A]$ is a copy of H_1. Note that $\chi(\varphi(H_1^{++}[A])) \geqslant \chi(H_1^{++}[A]) = 4$ and $\overline{C}_7$ is a vertex-critical 4-chromatic graph, so the restriction of φ to A is a surjection onto $\overline{C}_7$ and so $\varphi(a_0), \varphi(a_1), \dots, \varphi(a_6)$ are all distinct.



Now, a_0 has degree 6 while $\varphi(a_0) \in \overline{C}_7$ only has degree 4. The four neighbours of $\varphi(a_0)$ are $\varphi(a_1), \varphi(a_2), \varphi(a_5), \varphi(a_6)$ and so $\varphi(a_{023})$ is one of these. Also, a_2 has degree 5 while $\varphi(a_2)$ only has degree 4. The four neighbours of $\varphi(a_2)$ are $\varphi(a_0), \varphi(a_1), \varphi(a_3), \varphi(a_4)$ and so $\varphi(a_{023})$ is one of these. Hence, $\varphi(a_{023}) = \varphi(a_1)$. Similarly, considering the neighbourhoods of a_0 and a_5 shows that $\varphi(a_{350}) = \varphi(a_6)$. Then, $\Gamma(\varphi(a_3))$ contains

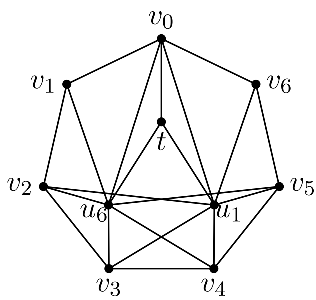
$$\{\varphi(a_2), \varphi(a_{023}), \varphi(a_{350}), \varphi(a_4), \varphi(a_5)\} = \{\varphi(a_2), \varphi(a_1), \varphi(a_6), \varphi(a_4), \varphi(a_5)\},$$

which has size 5. This contradicts the 4-regularity of $\overline{C}_7$. □

Proposition A.4. *The graph H_2 is not homomorphic to H_1^{++}.*

Proof. Suppose there is a homomorphism $\varphi\colon H_2 \to H_1^{++}$. Now, $\chi(\varphi(H_2)) \geqslant \chi(H_2) = 4$ and any 6-vertex subgraph of H_1^{++} is 3-colourable (it is homomorphic to some 6-vertex subgraph of H_2^{+}) so φ must be injective. Thus, H_1^{++} contains H_2. But H_1^{++} only has 4 vertices of degree at least 4 while H_2 has 5 vertices of degree 4. □

Finally, we relate T_0 to the diagram. It suffices to show that $H_0 \nrightarrow T_0$, $T_0 \nrightarrow \overline{C}_7$ and $T_0 \nrightarrow H_1^{++}$ (note that $H_0 \nrightarrow T_0$ implies that no other graph in the diagram is homomorphic to T_0 while $T_0 \nrightarrow \overline{C}_7$ implies that $T_0 \nrightarrow H_0, H_1, H_2$). We use the following labelling of the copy of T_0 in all three proofs.



Proposition A.5. *The graph H_0 is not homomorphic to T_0.*

Proof. We first claim that any 7-vertex subgraph of T_0 is 3-colourable. Let F be a 7-vertex subgraph of T_0. If F contains all the v_i, then F is a subgraph of a 7-cycle and so is 3-colourable. Otherwise, F is a subgraph of $T_0 - v_i$ for some i. This graph is 3-colourable: 2-colour the remaining v_j with colours 1 and 2, give u_1 and u_6 colour 3 and then give t colour 1 or 2 (opposite to the colour of v_0 if it is present).

Suppose there is a homomorphism $\varphi\colon H_0 \to T_0$. Then, $\varphi(H_0)$ is a subgraph of T_0 with at most 7 vertices, so is 3-colourable. But then, $3 \geqslant \chi(\varphi(H_0)) \geqslant \chi(H_0) = 4$. □

Proposition A.6. *The graph T_0 is not homomorphic to $\overline{C}_7$.*

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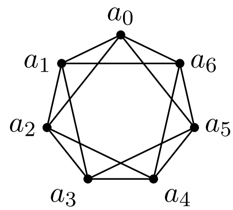
Proof. Suppose $\varphi\colon T_0 \to \overline{C}_7$ is a homomorphism. Label the copy of $\overline{C}_7$ as follows.



Without loss of generality, we may assume $\varphi(u_1) = a_0$. The common neighbourhood $\Gamma(u_1, u_6)$ contains the edge tv_0, so $\Gamma(\varphi(u_1), \varphi(u_6))$ contains an edge and so $\varphi(u_6) \in \{a_0, a_3, a_4\}$. By symmetry, we may assume $\varphi(u_6) \in \{a_0, a_3\}$.

First suppose that $\varphi(u_6) = a_0$. Then,

$$\varphi(\{v_0, v_1, \dots, v_6\}) \subset \varphi(\Gamma(u_1) \cup \Gamma(u_6)) \subset \Gamma(\varphi(u_1)) \cup \Gamma(\varphi(u_6)) = \Gamma(a_0).$$

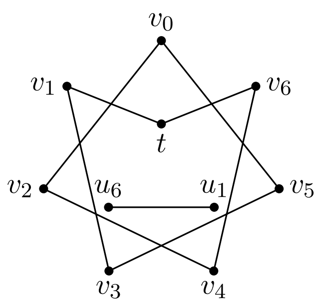
However, $v_0v_1 \dots v_6$ form a 7-cycle which is 3-chromatic, while $\Gamma(a_0)$ is a path of length 3 (which is bipartite).

Now suppose that $\varphi(u_6) = a_3$. The edge tv_0 is in $\Gamma(u_1, u_6)$ so $\varphi(t)\varphi(v_0)$ must be an edge in $\Gamma(a_0, a_3)$. In particular, $\{\varphi(t), \varphi(v_0)\} = \{a_1, a_2\}$. By symmetry, we may assume that $\varphi(v_0) = a_1$. Now $v_1 \in \Gamma(v_0, u_6)$, so $\varphi(v_1) \in \Gamma(a_1, a_3)$ and so $\varphi(v_1) = a_2$. Next, $v_2 \in \Gamma(u_1, u_6, v_1)$, so $\varphi(v_2) \in \Gamma(a_0, a_3, a_2)$ and so $\varphi(v_2) = a_1$. Working in this way round the outer 7-cycle gives $\varphi(v_3) = a_2$, $\varphi(v_4) = a_1$ and $\varphi(v_5) = a_2$. Finally, $v_6 \in \Gamma(u_1, v_0, v_5)$ and so $\varphi(v_6) \in \Gamma(a_0, a_1, a_2) = \varnothing$, which is a contradiction. □

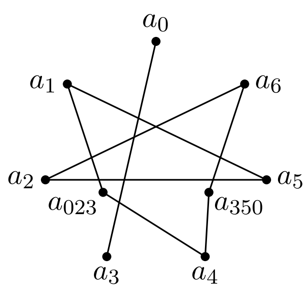
Proposition A.7. *The graph T_0 is not homomorphic to H_1^{++}.*

Proof. Suppose $\varphi\colon T_0 \to H_1^{++}$ is a homomorphism. If x, y is a dense pair (see Definition 3.5) of vertices in T_0, then $\Gamma(x, y)$ contains an edge, so $\Gamma(\varphi(x), \varphi(y))$ contains an edge and so either $\varphi(x) = \varphi(y)$ or $\varphi(x), \varphi(y)$ is a dense pair in H_1^{++}. □

For a graph G, let $\mathcal{D}G$ be the graph with vertex set $V(G)$ and with vertices x and y adjacent if x, y is a dense pair in G. The previous paragraph shows that φ maps a connected set of vertices in $\mathcal{D}T_0$ to a connected set in $\mathcal{D}H_1^{++}$. The graphs $\mathcal{D}T_0$ and $\mathcal{D}H_1^{++}$ are displayed below (we have used the same labelling of the vertices of H_1^{++} as in Proposition A.3).



$\mathcal{D}T_0$



$\mathcal{D}H_1^{++}$

Let C be the 8-cycle $v_0v_2v_4v_6tv_1v_3v_5$ of $\mathcal{D}T_0$ and so $\varphi(C)$ is connected in $\mathcal{D}H_1^{++}$. If $\varphi(C)$ meets $\{a_0, a_3\}$, then $|\varphi(C)| \leqslant 2$ and so $|\varphi(T_0)| \leqslant 4$ while $\chi(\varphi(T_0)) \geqslant \chi(T_0) = 4$ so $\varphi(T_0)$ is a 4-clique which is absurd as H_1^{++} is K_4-free. Hence, $\varphi(C) \subset H_1^{++} - \{a_0, a_3\}$.

Now, both $H_1^{++} - a_0$ and $H_1^{++} - a_3$ are 3-colourable (they are both 2-degenerate) and $\chi(\varphi(T_0)) \geqslant 4$, so $a_0, a_3 \in \varphi(T_0)$. In particular, $\varphi(\{u_1, u_6\}) = \{a_0, a_3\}$. By symmetry, we may assume that $\varphi(u_1) = a_0$ and $\varphi(u_6) = a_3$.

In T_0, the path $v_2v_3v_4v_5$ lies in the common neighbourhood of u_1 and u_6. While, in H_1^{++}, the common neighbourhood of $a_0 = \varphi(u_1)$, $a_3 = \varphi(u_6)$ consists of two disconnected edges a_2a_{023} and a_5a_{350}. Thus, $\{\varphi(v_2), \varphi(v_5)\}$ is either $\{a_2, a_{023}\}$ or $\{a_5, a_{350}\}$.

Back in $\mathcal{D}T_0$, $v_2v_0v_5$ is a path, so $\varphi(v_2)$, $\varphi(v_5)$ are within distance two of each other in $\mathcal{D}H_1^{++}$. But this is inconsistent with $\{\varphi(v_2), \varphi(v_5)\}$ being either $\{a_2, a_{023}\}$ or $\{a_5, a_{350}\}$. $\square$

Cite this article: Illingworth F (2022). The chromatic profile of locally colourable graphs. *Combinatorics, Probability and Computing* **31**, 976–1009. https://doi.org/10.1017/S0963548322000050