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LETTER TO THE EDITOR

ON GOLDSTEIN'S VARIANCE BOUND

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Abstract

Goldstein (1974) derived an upper bound on the variance of certain non-negative functions when the first two moments of the underlying random variable are known. This bound is compared to a simple and fundamental variance bound which requires only that the range of the function be known. It is shown that Goldstein's bound frequently exceeds the simpler bound. Finally, an interpretation of such bounds in the context of economic risk analysis is given.

VARIANCE; INEQUALITY; UPPER BOUNDS; ECONOMIC RISK ANALYSIS

1. Previous work

The literature on variance bounds is primarily concerned with lower bounds. Relatively little has been written about upper bounds on the variance of functions of random variables. The case of unimodal densities and $f(X) = X$ has been studied by Jacobson (1969) and Seaman (1985). Miulwijk (1966) obtained an upper bound for the variance of a discrete random variable. His bound is a special case of the bound discussed in Section 4. Upper bounds for more general functions have been studied by Gray and Odell (1967), Goldstein (1974), and Seaman (1985).

2. The problem

Goldstein (1974) studied the problem of bounding the variance of certain functions of interest in economic risk analysis. Assuming knowledge of the first two moments of the underlying random variable, he derived the bounds presented in the next section. Goldstein failed to compare his bounds with a simple and much more widely applicable upper bound requiring only knowledge of the extremes of the function. This bound is presented in Section 4. We shall show that the bounds considered by Goldstein frequently exceed the simpler bound. In addition, we shall briefly discuss an application of such bounds in risk analysis.

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3. Goldstein's bounds

Goldstein (1974) derived upper bounds on the variance of certain non-negative functions on $[0, \delta]$, when the first two moments of the underlying random variable are known. The next two theorems summarize his results.

Theorem 3.1. Let X be a random variable on $[0, \delta]$ such that $E(X) = \mu$, and $E(X^2) = \gamma$. Let g be a function of X such that $g(0) = 0$ and $E[g^2(X)]$ exists and is finite. Define $f(x) = g(x)/x$, $0 < x \leqq \delta$. Let $\alpha = \lim_{x\to 0} f(x)$ exist and be finite. Then if f is non-increasing on $(0, \delta)$,

$$\mathrm{Var}[g(X)] \leqq \alpha^2\gamma - f^2(\delta)\mu^2. \tag{3.1}$$

Theorem 3.2. Let X, g, and f be as defined above. If f is non-increasing on $[0, \delta]$ and if g is non-decreasing on $[0, \delta]$ then

$$\mathrm{Var}[g(X)] \leqq f^2(\delta)(\gamma - 2\delta\mu + \delta^2). \tag{3.2}$$

Goldstein remarks that if X is concentrated about δ, then the bound given in (3.2) is sharp. If, in addition, $\lim_{x\to 0} g(x) = g(\delta)$, the bound in (3.1) is sharp as well. He also shows that the first bound is to be preferred to the second when $f^2(\delta) > \gamma(\sup f)^2/(\gamma + (\mu - \delta)^2)$, where $\sup f = \sup_{x\in[0,\delta]}[f(x)]$ and is assumed to be finite.

4. A fundamental inequality

If we know the extremes of a function of a random variable, we can obtain a bound on its variance. The following result provides a least upper bound on the variance of a function of a random variable. This bound is not unknown; Miulwijk's (1966) bound is a special case, and the bound is considered by Jacobson (1969) and Seaman (1985).

Theorem 4.1. Let X be a random variable on a subset Ω of the real numbers R. Let $g : \Omega \to R$ be a continuous function of X which is bounded above and below. Then, $\mathrm{Var}[g(X)] \leqq (\bar{g} - \underline{g})^2/4$, where $\bar{g} = \sup_{x\in\Omega}[g(x)]$, and $\underline{g} = \inf_{x\in\Omega}[g(x)]$.

The proof is straightforward and is omitted. We shall call this the maximum variance bound.

5. Comparing the bounds

As an example, consider the function $g(x) = 1 - \exp(-\theta x)$, $\theta > 0$, $0 \leqq x \leqq \delta$. Goldstein's bound is

$$\mathrm{Var}[g(X)] \leqq \begin{cases} \gamma\theta^2 - ((1 - \exp(-\theta\delta))/\delta)^2\mu^2 & \text{if } f^2(\delta) > c \\ ((1 - \exp(-\theta\delta))/\delta)^2(\gamma - 2\delta\mu + \delta^2) & \text{if } f^2(\delta) \leqq c, \end{cases}$$

where $c = \theta^2\gamma/(\gamma + (\mu - \delta)^2)$. The maximum variance bound is $\mathrm{Var}[g(x)] \leqq (1 - \exp(-\theta\delta))^2/4$. These bounds are compared in Table 1 for values of the parameters.

The previous example suggests that a better bound would be the minimum of Goldstein's bound and the maximum variance bound. In general, the maximum variance bound should be used as a benchmark in variance bounding studies.

6. An application

Goldstein applied his bounds to functions of interest in economic risk analysis but did not offer motivation for their use. We do so here.

TABLE 1
Values of the maximum variance bound (MVB) and Goldstein's bound (GOLD) for the function $g(x) = 1 - \exp(-\theta x)$. These results are typical of a much larger set of parameter combinations and of other functions as well

δ	θ	μ	γ	MVB	GOLD	MVB–GOLD
1·0	0·1	0·1	0·510	0·002	0·005	−0·003
1·0	0·2	0·4	0·160	0·008	0·001	0·007
1·0	3·5	0·25	0·630	0·235	1·063	−0·828
1·0	3·5	0·75	0·563	0·235	0·059	0·176
1·0	5·0	0·50	0·750	0·247	0·740	−0·493
3·0	0·5	0·50	6·750	0·151	0·855	−0·704
3·0	0·5	1·00	1·000	0·151	0·183	−0·032
3·0	0·5	2·00	5·000	0·151	0·134	0·017
3·0	0·5	2·00	5·500	0·151	0·168	−0·017
3·0	0·5	2·75	8·563	0·151	0·071	0·080
3·0	5·0	2·00	5·500	0·250	0·278	−0·028
10·0	5·0	7·00	66·000	0·250	0·260	−0·010
10·0	5·0	9·00	83·000	0·250	0·030	0·220

Agnew (1972) states that certain functional forms, including the example function in the previous section, have been suggested by economists as being reasonable utilities for use in economic risk analysis. Agnew derived bounds on expected utility and applied them to this problem.

We submit that the *variance* of a utility function has an application in risk analysis. For the moment, suppose the decision-maker knows the distribution of wealth corresponding to any decision he makes. When he selects a utility function, he turns his attention to finding that decision which will maximize expected utility. However, the greater the variability of the utility, the less meaningful is expected utility. Therefore, it is of interest to the decision-maker to consider the variance of his utility with respect to a given decision. Indeed, given two decisions yielding the same expected utility, the decision with smaller variance in utility should be preferred.

Now, the decision-maker typically does not know the distribution of wealth for a given decision. Thus, Agnew presented bounds on expected utilities and we present bounds on the variance of utilities.

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