Some Properties of Parabolic Curves.

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If the tangent at a point P on the parabolic curve $cy = x^n$ meet the axis of x at M, it is a well-known property that the area between the radius vector OP and the arc OP is n times that between the arc OP and the two tangents OM, MP, O being the origin and $n > 1$. The converse is also true; for taking any point O on a curve as origin and the tangent at O as axis of x, let us seek for the locus of P if the area between OP and the arc OP be n times the area between the arc OP and the tangents OM, MP.

The area between the chord OP and the arc OP is

$$\tfrac{1}{2}xy - \int_0^x ydx$$

and the area between arc and tangents is

$$\int_0^x ydx - \frac{y^2}{2p}$$

where $p = dy/dx$. Hence

$$\tfrac{1}{2}xy - \int_0^x ydx = n\int_0^x ydx - \frac{ny^2}{2p}$$

Differentiating with respect to x, the differential equation of the curve will be

$$\frac{ny^2}{p^2}\ \frac{dp}{dx} = xp - y$$

This may be written

$$n\frac{d}{dx}\left(\frac{1}{p}\right) = \frac{d}{dx}\left(\frac{x}{y}\right)$$

$$\therefore \frac{n}{p} = \frac{x}{y} + C$$

$$i.e.\ \frac{dx}{dy} - \frac{1}{ny}x = \frac{C}{n}$$

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the integral of which is

$$x = \mathrm{D}y^{\frac{1}{n}} + \frac{\mathrm{C}}{n-1}y$$

or $$y = (\mathrm{A}x + \mathrm{B}y)^n$$

If $\mathrm{B}=0$, we have the form $cy = x^n$; and in general if $\mathrm{A}x + \mathrm{B}y = 0$ be taken as axis of y in a system of oblique coordinates, the equation takes the same form $cy = x^n$.

If n were a positive proper fraction, the axes would simply be interchanged.

Consider more particularly the parabola $x^2 = 4ay$. In this case the area between OP and the curve is $b^3/24a$ if b is the abscissa of P, while the area between the arc and the tangents is $b^3/48a$. It will be noticed that $b^3/48a$ is the area between the chord OP′ and the arc OP′ of the parabola $x^2 + ay = 0$ where $b/2$ is the abscissa of P′. But $b/2$ is the abscissa of M while the ordinate of $x^2 + ay = 0$ for the abscissa $b/2$ is $-b^2/4a$, that is, the intercept made by the tangent at P on the axis of y. In fact $x^2 + ay = 0$ is the locus of a point which has for coordinates the intercepts made by the tangent at P on the axis of x and y. (Compare Forsyth's *Diff. Equations*, p. 41 ex. 9.) How far does this property hold for the general parabola? In other words what is the solution of the following problem:—A curve is referred to the tangent and normal at a point O as axis of x and y and the tangent at P cuts the axis of x at M and that of y at N; if the point P′ be taken having OM, ON for coordinates what will be the equations of the loci of P and of P′ if the area between the chord OP′ and the arc OP′ be n times the area between the arc OP and the tangents OM, MP?

Let (x, y) (ξ, η) be the coordinates of P and P′ and denote dy/dx by p; then

$$\xi = x - y/p, \ \eta = y - px.$$

The area between the arc OP and the tangents OM, PM is

$$\int_0^x xyd - \frac{y^2}{2p}$$

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The area cut off by the chord OP′ from the locus of P is

$$\int_0^\xi \eta d\xi + \frac{y^2}{2p} + \tfrac{1}{2}px^2 - xy$$

both areas being positive. Hence

$$n\int_0^x y dx - \frac{ny^2}{2p} = \int_0^\xi \eta d\xi + \frac{y^2}{2p} + \tfrac{1}{2}px^2 - xy$$

Differentiating with respect to x and noting that

$$\frac{d\xi}{dx} = \frac{y}{p^2}\frac{dp}{dx}$$

we get
$$\left(\frac{n-1}{2}\frac{y^2}{p^2} + \frac{xy}{p} - \tfrac{1}{2}x^2\right)\frac{dp}{dx} = 0$$

$dp/dx = 0$ gives no solution. Hence the equation of the locus of P is given by

$$p^2x^2 - 2xyp - (n-1)y^2 = 0$$

or
$$xp = y(1 \pm \sqrt{n})$$

the integral of which is $cy = x^{1 \pm \sqrt{n}}$, giving only one solution, $cy = x^2$ when $n = 1$.

If n be not a square each curve is transcendental, but if $n = m^2$, we have $cy = x^{m+1}$ or $cy = x^{1-m}$. The solution $cy = x^{1-m}$ evidently does not satisfy the conditions of the problem, the axis of x not being the tangent at 0, but obviously the other solution $cy = x^{m+1}$ does.

To find the locus of P′ we have

$$\xi = \frac{m}{m+1}x, \quad \eta = -\frac{m}{c}x^{m+1}$$

and therefore
$$\xi^{m+1} + \frac{cm^m}{(m+1)^{m+1}}\eta = 0$$

These are parabolic curves which for $m = 1$ reduce to the ordinary parabola.

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With regard to the solution $cy = x^{1-m}$, it may be noted that when m is greater than two the axes are asymptotes and a similar proposition holds for the two loci. Using the form $x^{m-1}y = k$ as the equation to the locus of P we find for the locus of P′ the equation

$$\xi^{m-1}\eta = \frac{km^m}{(m-1)^{m-1}}$$

The area bounded by the tangent PM, the part of the axis of x from M to $+\infty$ and the arc from P to the same end of the axis of x is

$$\frac{mxy}{2(m-1)(m-2)}$$

On the other hand the area bounded by the line OP′, the positive part of the axis of x and the arc of the locus of P′ from P′ to the positive end of the axis of x is

$$\frac{m\xi\eta}{2(m-2)} = \frac{m^3xy}{2(m-1)(m-2)}$$

and is therefore m^2, *i.e.*, n times the former area.

When m is less than 1 the tangent at the origin to the curve $cy = x^{1-m}$ is the axis of y and a similar proposition to that given for the curve $cy = x^{m+1}$ holds if M and N be taken on the axes of y and x respectively, while if m be greater than 1 but less than 2 the same change in M, N gives a result analogous to that for the curve $x^{m-1}y = k$ when m is greater than 2.

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