Adv. Appl. Prob. **15**, 468–469 (1983)
Printed in N. Ireland

A NOTE ON THE PROCESSOR-SHARING QUEUE IN A QUASI-REVERSIBLE NETWORK

PANTELIS TSOUCAS* AND
JEAN WALRAND,* *University of California, Berkeley*

Abstract

Consider a processor-sharing queue placed in a quasi-reversible network in equilibrium. This note explains why the expected sojourn time of a customer in such a queue is proportional to his service time.

1. The problem

The result recalled in the abstract has been observed in the literature (see [2]). This note shows that result to be a direct consequence of quasi-reversibility.

The situation is depicted in Figure 1. The network N is quasi-reversible and node 1 is processor-sharing. This network can be open, closed, or mixed. (See [1] for definitions.)

Denote by T the duration of one specific sojourn of a given customer in node 1 and by S the corresponding service time in the node.

It will be shown that $E[T \mid S] = \alpha S$, where α is a constant to be determined from the network parameters. Notice that one then has $E\{T\} = \alpha E\{S\}$, so that it is clear how to calculate α.

Assume that x and y are two positive numbers such that $\Pr\{S = x\}$ and $\Pr\{S = y\}$ are positive. If only the density of S is non-zero at x and y, then an arbitrarily small perturbation of the distribution of S would lead us to the previous situation. Also, one can assume that $x/y = m/n$ for some m and n in $\{1, 2, 3, \cdots\}$.

All that has to be shown is that

(1)
$$\frac{E[T \mid S = x]}{E[T \mid S = y]} = \frac{x}{y}.$$

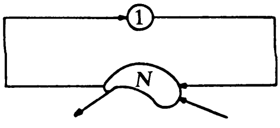

Figure 1

Received 11 February 1983.

* Postal address for both authors: Department of Electrical Engineering and Computer Sciences and the Electronics Research Laboratory, University of California, Berkeley, CA 94720, U.S.A.

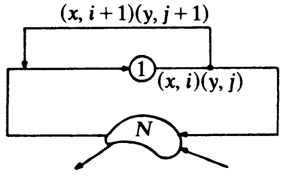

Figure 2

2. The argument

Let $x = m\varepsilon$ and $y = n\varepsilon$. A customer requiring $m\varepsilon$ seconds of service in node 1 can be thought of as being fed back $m-1$ times in that node and requiring ε seconds during each visit.

This leads to the equivalent model of Figure 2. In that network, the customers requiring a service time x are fed back $m-1$ times. This is accomplished as follows. Say that a customer is of class (x, i) if his total service time is x and he has been fed back i times ($i = 0, 1, \cdots, m-1$). With probability 1, a class (x, i) customer is fed back and becomes a class $(x, i+1)$ customer, for $i = 0, 1, \cdots, m-2$. A customer of class (x, i) requires a service time of ε in node 1.

Similarly one defines the classes (y, j) for $j = 0, 1, \cdots, n-1$, and their feedbacks.

The other customers proceed as before, and the customers of class $(x, m-1)$ or $(y, n-1)$ are not fed back.

The crucial observation is that upon being fed back, customers of class (x, i) or (y, j) find the queue in the same distribution for all i and j. This distribution is indeed the equilibrium distribution, since the network is quasi-reversible.

Hence

$$m^{-1}E[T \mid S = m\varepsilon] = n^{-1}E[T \mid S = n\varepsilon]$$

and this establishes (1).

Acknowledgments

The authors wish to thank Dr D. Mitra for bringing this problem and reference [2] to their attention.

This research was supported in part by the NSF under Grant ECS-8205428.

References

[1] Kelly, F. P. (1979) *Reversibility and Stochastic Networks.* Wiley, London.

[2] Mitra, D. (1981) Waiting time distributions from closed queueing network models of shared-processor systems. In *Performance 81*, ed. S. J. Kylstra. North-Holland, Amsterdam.