ERGODIC BEHAVIOUR OF EXTREME VALUES

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Abstract

Let \( \{X_n, n \geq 1\} \) be independent identically distributed random variables with a common non-degenerate distribution function \( F \). For each \( n \geq 1 \), denote \( M_n = \max\{X_1, \ldots, X_n\} \). Under certain conditions on \( F \), there exist constants \( a_n > 0 \) and \( b_n \in \mathbb{R} \) such that \( (M_n - b_n)/a_n \xrightarrow{d} G \). In this paper, we shall show that \( (M_n - b_n)/a_n \) exhibits ergodic behaviour under additional conditions on \( F \).


Keywords and phrases: ergodic behaviour, extreme values, logarithmic means, almost sure convergence.

1. Introduction

Let \( \{X_n, n \geq 1\} \) be a sequence of independent identically distributed random variables with \( \mathbb{E}X_1 = 0 \) and \( \mathbb{E}X_1^2 = 1 \). For each \( n \geq 1 \), set \( S_n = \sum_{i=1}^{n} X_i \). Then for \( x \in \mathbb{R} \)

\[
\lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \mathbf{1}\left( \frac{S_k}{\sqrt{k}} \leq x \right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt \quad \text{almost surely.}
\]

This is known as the almost sure central limit theorem, and was proved first by Brosamler [1] and Schatte [8]. Recently, the related problems have attracted much attention (see Lacey and Philipp [6], Schatte [9–11]).

It was shown in Cheng et al. [2] that the above phenomenon holds also for extreme values (see Lemma 2 in Section 2). In this paper we consider the general problem related to maxima (see below).

We assume throughout that \( \{X_n, n \geq 1\} \) are a sequence of independent identically distributed random variables with a common non-degenerate distribution function \( F \).
Ergodic behaviour of extreme values

For each $n \geq 1$, denote

$$M_n = \max \{ X_1, \ldots, X_n \}.$$

Suppose there exist constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$(1.1) \quad (M_n - b_n)/a_n \xrightarrow{d} G$$

where $G$ is a non-degenerate distribution function. Then we say $G$ is an extreme value distribution and $F$ is in the domain of attraction of $G$ (notation: $F \in D(G)$).

It is well known that $G$ must be one of the following three types:

$$G(x) = \begin{cases} 
\exp\{-x^{-\alpha}\} & \text{if } x \geq 0, \\
0 & \text{if } x < 0;
\end{cases}$$

$$G(x) = \Psi_\alpha(x) = \begin{cases} 
1 & \text{if } x \geq 0, \\
\exp\{-(x)\alpha\} & \text{if } x < 0;
\end{cases}$$

$$G(x) = \Lambda(x) = \exp\{-e^{-x}\} \text{ for } x \in \mathbb{R},$$

where $\alpha > 0$ (see [7, Proposition 0.3]). Furthermore $a_n$ and $b_n$ can be chosen as

$$(1.2) \quad a_n = \begin{cases} 
U(n) & \text{if } G(x) = \Phi_\alpha(x), \\
x_F - U(n) & \text{if } G(x) = \Psi_\alpha(x), \\
U(ne) - U(n) & \text{if } G(x) = \Lambda(x),
\end{cases}$$

and

$$(1.3) \quad b_n = \begin{cases} 
0 & \text{if } G(x) = \Phi_\alpha(x), \\
x_F & \text{if } G(x) = \Psi_\alpha(x), \\
U(n) & \text{if } G(x) = \Lambda(x),
\end{cases}$$

where $x_F := \sup \{ x : F(x) < 1 \}$ and $U(x) := \inf \{ y : 1/(1 - F(y)) > x \}$.

It is important to have necessary and sufficient conditions for a distribution to belong to the domain of attraction of an extreme value distribution. Some characterisation theorems can be found in [7, Chapter 1]. For example, $F \in D(\Phi_\alpha)$ if and only if

$$\lim_{t \to \infty} \frac{U(tx)}{U(t)} = x^{1/\alpha} \quad \text{for all } x > 0$$

(notation: $U \in RV_{1/\alpha}$).

Assume that (1.1) holds. If there exists a positive sequence $\{ r_n, n \geq 1 \}$ with $\sum_{k=1}^{\infty} r_k = \infty$ such that

$$\lim_{n \to \infty} \frac{1}{\sum_{k=1}^{n} r_k} \sum_{k=1}^{n} r_k f \left( \frac{M_k - b_k}{a_k} \right) = \int f(x) G(dx) \quad \text{almost surely}$$
(with \( D := \{ x : 0 < G(x) < 1 \} \)) holds for a class of functions \( f \), then we may say that the sequence \( \{(M_n - b_n)/a_n\} \) has ergodic behaviour. It is natural to consider the case where \( r_n = n^{-\gamma} \) with \( 0 < \gamma \leq 1 \). Unfortunately, for \( \gamma \in (0, 1) \), the above equation is not true even for the indicator function (see Cheng et al. [2]). Hence we only consider the case \( \gamma = 1 \), that is the logarithmic means. In the present paper, the following results are obtained (proofs are given in Section 2).

**Theorem 1.** Suppose (1.1) holds for \( G = \Phi_\alpha \), \( F(0-) = 0 \), and \( a_n \) and \( b_n \) are defined by (1.2) and (1.3). Assume \( f \) is an almost everywhere continuous function which is defined on \((0, \infty)\). If there are constants \( B > 0, \beta \in (0, \alpha) \) and \( \tau > 0 \) such that

\[
|f(x)| \leq B(x^\beta + x^{-\tau}) \quad \text{for all } x > 0,
\]

then

\[
\lim_{N \to \infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} f \left( \frac{M_n - b_n}{a_n} \right) = \int_0^\infty f(x) \Phi_\alpha(dx) \quad \text{almost surely.}
\]

**Remark 1.** Condition (1.4) ensures that \( \int_0^\infty f(x) \Phi_\alpha(dx) < \infty \). Indeed, (1.5) still holds if we replace (1.4) by assuming that there are constants \( \beta \in (0, \alpha) \) and \( \tau \in (0, 1/\alpha) \) such that

\[
|f(x)| \leq B(x^\beta + e^{x^\tau}) \quad \text{for all } x > 0.
\]

**Remark 2.** Assume (1.1) holds for \( G = \Phi_\alpha \), and \( f \) is an almost sure continuous function \( f \) which is defined on \((-\infty, \infty)\). If there are constants \( B > 0 \) and \( \beta \in (0, \alpha) \) such that

\[
|f(x)| \leq B(|x| + 1)^\beta \quad \text{for } x \in \mathbb{R},
\]

then (1.5) holds.

Note that since

\[
\int_0^\infty x^{\beta} \Phi_\alpha(dx) = \Gamma(1 - \beta/\alpha),
\]

we have

\[
\lim_{N \to \infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{M_n^{\beta}}{n^{\beta}} = \Gamma(1 - \beta/\alpha) \quad \text{almost surely.}
\]
THEOREM 2. Suppose (1.1) holds for $G = \Psi_\alpha$, and $a_n$ and $b_n$ are defined by (1.2) and (1.3). Assume $g$ is an almost everywhere continuous function which is defined on $(-\infty, 0)$. If there are constants $B > 0$, $\beta \in (0, \alpha)$ and $\tau > 0$ such that

\[ |g(x)| \leq B(|x|^{-\beta} + |x|^\tau) \quad \text{for all } x < 0, \]

then

\[ \lim_{N \to \infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} g \left( \frac{M_n - b_n}{a_n} \right) = \int_{-\infty}^{0} g(x) \Psi_\alpha(dx) \quad \text{almost surely}. \]

REMARK 3. Assume (1.1) holds for $G = \Psi_\alpha$, and $g$ is an almost everywhere continuous function which is defined on $(-\infty, \infty)$. If there are constants $B > 0$ and $\beta > 0$ such that

\[ |g(x)| \leq B(|x| + 1)^\beta \quad \text{for } x \in \mathbb{R}, \]

then (1.8) holds.

Note that since for any positive integer $\beta$

\[ \int_{-\infty}^{0} x^\beta \Psi_\alpha(dx) = (-1)^{\beta} \Gamma(1 + \beta/\alpha), \]

we have

\[ \lim_{N \to \infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{(M_n - x)^\beta}{na_n^\beta} = (-1)^{\beta} \Gamma(1 + \beta/\alpha) \quad \text{almost surely}. \]

THEOREM 3. Suppose (1.1) holds for $G = \Lambda$, and $a_n$ and $b_n$ are defined by (1.2) and (1.3). Assume $h$ is an almost everywhere continuous function which is defined on $(-\infty, \infty)$. If there are constants $B > 0$ and $\beta > 0$ such that

\[ |h(x)| \leq B(|x| + 1)^\beta \quad \text{for } x \in \mathbb{R}, \]

then

\[ \lim_{N \to \infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} h \left( \frac{M_n - b_n}{a_n} \right) = \int_{-\infty}^{\infty} h(x) \Lambda(dx) \quad \text{almost surely}. \]

REMARK 4. Under the conditions of Theorem 3, if $\beta$ is a positive integer, then

\[ \lim_{N \to \infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{(M_n - b_n)^\beta}{na_n^\beta} = (-1)^{\beta} \Gamma^{(\beta)}(1) \quad \text{almost surely}, \]

where $\Gamma^{(\beta)}(1)$ denotes the $\beta$-th derivative of the gamma function at $x = 1$. 
REMARK 5. According to [7, Proposition 2.1], if (1.1) holds, then under additional conditions on the left tail of $F$, we have

$$\lim_{n \to \infty} \mathbb{E} \left( \frac{M_n - b_n}{a_n} \right)^\beta = \begin{cases} \Gamma(1 - \beta/\alpha) & \text{if } G(x) = \Phi_a(x), \\ (-1)^\beta \Gamma(1 + \beta/\alpha) & \text{if } G(x) = \Psi_a(x), \\ (-1)^\beta \Gamma(\beta)(1) & \text{if } G(x) = \Lambda(x), \end{cases}$$

where $a_n$ and $b_n$ are defined by (1.2) and (1.3), and in the last two equations, $\beta$ should be a positive integer. Thus by (1.6), (1.9) and (1.12) we have

$$\lim_{n \to \infty} \frac{1}{\log N} \sum_{n=1}^N \frac{(M_n - b_n)^\beta}{n\mathbb{E}(M_n - b_n)^\beta} = 1 \quad \text{almost surely.}$$

REMARK 6. It is obvious that (1.1) holds for constants $a'_n$ and $b'_n$ which satisfy

$$\begin{cases} a'_n/a_n \to 1 \\ (b'_n - b_n)/a_n \to 0 \end{cases}$$

as $n \to \infty$, where $a_n$ and $b_n$ are defined by (1.2) and (1.3) (see [7, Proposition 0.2]). Moreover, Remark 2, Remark 3 and Theorem 3 hold for above constants $a'_n$ and $b'_n$.

## 2. Proofs

For every measurable function $l$ let

$$S(l) = \{ x : l \text{ is continuous at } x \}.$$ 

The proofs of our theorems are mainly based on the following lemmas. The proof of Lemma 1 below is very standard and we omit it.

**Lemma 1.** Assume $\{ Z, Z_n, n \geq 1 \}$ is a sequence of random variables with distribution functions $\{ G, G_n, n \geq 1 \}$. Assume $\{ Z_n \}$ converges in distribution to $Z$, that is

$$\lim_{n \to \infty} G_n(x) = G(x), \quad \text{for } x \in S(G).$$

If $l$ is a real-valued almost everywhere continuous function with respect to $G$, that is $\Pr(Z \in S(l)) = 1$ and $(l(Z), l(Z_n), n \geq 1)$ is uniformly integrable (for definition of uniformly integrable, see [3, page 93]), then

$$\lim_{n \to \infty} \mathbb{E} l(Z_n) = \mathbb{E} l(Z).$$
LEMMA 2. Assume (1.1) holds. Then

\[
\Pr \left\{ \lim_{n \to \infty} \sup_{x} \left| \frac{1}{\log n} \sum_{k=1}^{n} \frac{1}{k} \left( \frac{M_k - b_k}{a_k} \leq x \right) - G(x) \right| = 0 \right\} = 1
\]

where \( 1(A) \) denotes the indicator function of set \( A \), and \( a_n \) and \( b_n \) are defined by (1.2) and (1.3).

PROOF. See Cheng et al. [2].

Next we are going to prove our theorems. Set

\[
\Omega_1 = \left\{ \omega : \lim_{N \to \infty} \sup_{x \in \mathbb{D}} \left| \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} 1_{(-\infty, x]} \left( \frac{M_n - b_n}{a_n} \right) - G(x) \right| = 0 \right\}.
\]

From Lemma 2 we know

\[
\Pr(\Omega_1) = 1.
\]

Assume \( \{W_j, j \geq 1\} \) is a sequence of independent random variables with common distribution function \( \Phi_1 \). It is easily seen that \( \{U(1/(1 - \Phi_1(W_j))), j \geq 1\} \) is a sequence of independent random variables which have the same distributions as \( \{X_j, j \geq 1\} \). For the sake of simplicity, we assume that \( X_j = U(1/(1 - \Phi_1(W_j))) \), for \( j \geq 1 \). Using the well-known inequalities for regular variation and \( \Pi \)-variation (see Geluk and de Haan [4, Proposition 1.7.5 and Proposition 1.19.4]), we may concentrate on dealing with \( \{W_j, j \geq 1\} \) (see (2.9) and (2.13) below).

For \( 1 \leq m \leq n \), set \( W(n, m) = \max_{n-m+1 \leq j \leq n} W_j \). Obviously, \( W_{n,m}/m \) has distribution function \( \Phi_1 \), and \( M_n = U(1/(1 - \Phi_1(W(n, n))) \) for \( n \geq 1 \). We also have

\[
W(n, n) \to \infty \quad \text{almost surely as} \; n \to \infty.
\]

PROOF (of Theorem 1). Put \( \delta = (\beta/\alpha + 1)/2 \) and \( d^2 = (\alpha + \beta)/(2\beta) \). Then \( d > 1 \) and \( \delta \in (0, 1) \). Throughout the proof we use \( C \) to denote a positive constant, and we let \( O(1) \) refer to almost surely.

We write

\[
\Omega_2 = \left\{ \omega : \lim_{N \to \infty} \sup_{n} \frac{1}{\log N} \sum_{n=1}^{N} \left| f \left( \frac{M_n}{a_n} \right) \right|^d < \infty \right\}.
\]

First we show that

\[
\Pr(\Omega_2) = 1.
\]
Write \( S_N = (\log N)^{-1} \sum_{n=1}^{N} n^{-1} |f(M_n/a_n)|^d \). Then (1.4) implies
\[
S_N \leq C \left( \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} \left( \frac{M_n}{a_n} \right)^{-d \tau} + \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} \left( \frac{M_n}{a_n} \right)^{d \beta} \right)
\]
\[
= C \left( \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} \left( \frac{U(1/(1 - \Phi_1(W(n, n))))}{U(n)} \right)^{-d \tau} \right)
\]
\[
+ \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} \left( \frac{U(1/(1 - \Phi_1(W(n, n))))}{U(n)} \right)^{d \beta}
\]
\[
:= C(S_N^{(1)} + S_N^{(2)}).
\]

Since \( U \in RV_{1/\alpha} \), Potter-bound inequality (see Geluk and de Haan [4, Proposition 1.7.5]) implies that there exists \( t_0 > 0 \) such that
\[
\frac{U(tx)}{U(t)} \leq 2x^{d/\alpha}
\]
for all \( t > t_0 \) and \( x \geq 1 \). Since \( U(x) \) is non-decreasing, we have
\[
\frac{U(1/(1 - \Phi_1(W(n, n))))}{U(n)} \leq 1 + \frac{2}{(n(1 - \Phi_1(W(n, n))))^{d/\alpha}}
\]
for all \( n \geq t_0 \). Hence
\[
(2.8) \quad S_N^{(2)} = O(1) + C \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} \left( n(1 - \Phi_1(W(n, n))) \right)^{-d \beta/\alpha}
\]
\[
= O(1) + C \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} \left( n(1 - \Phi_1(W(n, n))) \right)^{-\delta}
\]
\[
= O(1) + \frac{O(1)}{\log N} \sum_{n=1}^{N} \frac{1}{n} \left( \frac{W(n, n)}{n} \right)^{\delta},
\]
since \( 1 - \Phi_1(W(n, n)) \sim (W(n, n))^{-1} \) holds almost surely from (2.6).

Note that for each \( N \geq 2 \), there exists \( m \geq 2 \) such that \( 2^{m-1} \leq N < 2^m \), and
\[
(2.9) \quad \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} \left( \frac{W(n, n)}{n} \right)^{\delta}
\]
\[
\leq \frac{1}{(m - 1) \log 2} \sum_{n=1}^{2^m} \frac{1}{n} \left( \frac{W(n, n)}{n} \right)^{\delta}
\]
\[
\leq \frac{1}{(m - 1) \log 2} \sum_{j=1}^{m} \sum_{n=2^{j-1}}^{2^j} \frac{1}{n} \left( \frac{W(n, n)}{n} \right)^{\delta}
\]
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\[
\begin{align*}
&\leq \frac{2^{\delta+1}}{m-1} \sum_{j=1}^{m} \left( \frac{W(2^{j}, 2^{j})}{2^{j}} \right)^{\delta} \\
&\leq \frac{2^{\delta+1}}{m-1} \sum_{j=1}^{m} \frac{1}{2^{j\delta}} \left( \sum_{i=1}^{j} (W(2^{i}, 2^{i-1}))^{\delta} + (W(1, 1))^{\delta} \right) \\
&= \frac{2^{\delta+1}}{m-1} \left( \sum_{i=1}^{m} (W(2^{i}, 2^{i-1}))^{\delta} + (W(1, 1))^{\delta} \right) \sum_{j=1}^{m} \frac{1}{2^{j\delta}} \\
&\leq \frac{2^{\delta}}{2^{\delta}-1} \frac{1}{m-1} \left( \sum_{i=1}^{m} \left( \frac{W(2^{i}, 2^{i-1})}{2^{i-1}} \right)^{\delta} + (W(1, 1))^{\delta} \right) .
\end{align*}
\]

Since \( \{(W(2^{i}, 2^{i-1}))^{\delta}, i \geq 1\} \) is a sequence of identical and independent random variables with finite means \( \mathbb{E} W_{1}^{\delta} \), by the strong law of large numbers we have

\[
(2.10) \quad \frac{1}{m-1} \sum_{i=1}^{m} \left( \frac{W(2^{i}, 2^{i-1})}{2^{i-1}} \right)^{\delta} \rightarrow \mathbb{E} W_{1}^{\delta} \quad \text{almost surely.}
\]

Therefore, by (2.8), (2.9) and (2.10) we have \( S_{N}^{(2)} = O(1) \). In order to prove (2.7), we only need to show that

\[
(2.11) \quad S_{N}^{(1)} = O(1).
\]

Using Potter-bound inequality, for some \( t_{1} > 0 \) and \( C > 0 \)

\[
(2.12) \quad \frac{U(tx)}{U(t)} \geq Cx^{d/\alpha}
\]

holds for all \( t > t_{1}, tx > t_{1} \) and \( x \leq 1 \). From (2.6), \( 1 - \Phi_{1}(W(n, n)) \rightarrow 0 \) almost surely. Hence,

\[
\Pr \left( 1 - \Phi_{1}(W(n, n)) \geq \frac{1}{t_{1}}, \text{ infinitely often} \right) = 0.
\]

It is easy to check from (2.12) that

\[
\begin{align*}
S_{N}^{(1)} &= \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} \left( \frac{U(1/(1 - \Phi_{1}(W(n, n))))}{U(n)} \right)^{\delta} \left( \frac{1}{1 - \Phi_{1}(W(n, n))} \right) ^{\leq t_{1}} \\
&\quad + \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} \left( \frac{U(1/(1 - \Phi_{1}(W(n, n))))}{U(n)} \right)^{\delta} \left( \frac{1}{1 - \Phi_{1}(W(n, n))} \right) ^{> t_{1}} \\
&= O(1) + \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} (n(1 - \Phi_{1}(W(n, n))))^{\delta + \alpha}
\end{align*}
\]
\[ = O(1) + \frac{O(1)}{\log N} \sum_{n=1}^{N} \frac{1}{n} \left( \frac{n}{W(n, n)} \right)^{d^{2}/\alpha}. \]

For \( N \geq 2, 2^{m-1} \leq N < 2^{m} \), it may easily be proved that

\[ (2.13) \quad \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} \left( \frac{n}{W(n, n)} \right)^{d^{2}/\alpha} \leq \frac{C}{m - 1} \sum_{j=1}^{m} \left( \frac{2^{j-1}}{W(2^{j}, 2^{j-1})} \right)^{d^{2}/\alpha}, \]

which is bounded almost surely by the strong law of large numbers since \( \{(n/W(n, n))^{d^{2}/\alpha}\} \) is a sequence of identical and independent random variables with finite means \( E(W_i)^{-d^{2}/\alpha} \). Thus, (2.11) is proved. This completes the proof of (2.7).

Set \( \Omega = \Omega_1 \cap \Omega_2 \). From (2.5) and (2.7) we have \( \Pr(\Omega) = 1 \). Put \( K(N) = \sum_{n=1}^{N} 1/n \). Fix \( \omega \in \Omega \) and write

\[ F_{N}(x) = \frac{1}{K(N)} \sum_{n=1}^{N} \frac{1}{n} 1_{(-\infty, x]} \left( \frac{M_n}{a_n} \right), \quad x \in \mathbb{R}. \]

Then \( \{F_{N}\} \) is a sequence of distribution functions. Let \( Z_N \) have distribution \( F_N \) and \( Z \) has distribution \( \Phi_\alpha \). Since \( K(N)/\log N \to 1 \) as \( N \to \infty \), we have

\[ \lim_{N \to \infty} \sup_{x} |F_{N}(x) - \Phi_\alpha(x)| = 0. \]

Note that

\[ \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} f \left( \frac{M_n}{a_n} \right) = \frac{K(N)}{\log N} \int_{0}^{\infty} f(x) dF_{N} = \frac{K(N)}{\log N} E f(Z_N). \]

By the definition of \( \Omega \) we know that \( \{f(Z), f(Z_N), N \geq 1\} \) is uniformly integrable. Thus by Lemma 1

\[ \lim_{N \to \infty} \frac{1}{\log N} \sum_{n=1}^{N} \frac{1}{n} f \left( \frac{M_n}{a_n} \right) = \int_{0}^{\infty} f(x) \Phi_\alpha(dx). \]

This proves (1.5). \( \square \)

\textbf{PROOF (of Theorem 2).} Put \( Y_j = 1/(x_F - X_j) \) for \( j \geq 1 \). Then \( \max_{1 \leq j \leq n} Y_j = 1/(x_F - M_n) \) and

\[ \frac{\max_{1 \leq j \leq n} Y_j}{a_n^{-1}} \to \Phi_\alpha \text{ in distribution}. \]

Put \( f(x) = g(-x^{-1}) \) for \( x > 0 \). Then (1.4) is satisfied because of (1.7). Using Theorem 1 we have (1.8). \( \square \)
Proof (of Theorem 3). Note that (1.1) implies

$$
\lim_{t \to \infty} \frac{U(tx) - U(t)}{U(te) - U(t)} = \log x \quad \text{for all } x > 0
$$

(see de Haan [5, Theorem 2.4.1]), using the known inequality for $\Pi$-function (see Geluk and de Haan [4, Proposition 1.19.4]), for every $\epsilon > 0$, there exist $C > 0$ and $t_2 > 0$ such that

$$
\frac{|U(tx) - U(t)|}{|U(te) - U(t)|} \leq C(x^\epsilon + x^{-\epsilon})
$$

for all $t \geq t_2$ and $tx \geq t_2$. Following the lines of proof of Theorem 1, we have (1.11).

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