Efficient and Stable Numerical Methods for Multi-Term Time Fractional Sub-Diffusion Equations

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Abstract. Some efficient numerical schemes are proposed for solving one-dimensional (1D) and two-dimensional (2D) multi-term time fractional sub-diffusion equations, combining the compact difference approach for the spatial discretization and \textit{L}1 approximation for the multi-term time Caputo fractional derivatives. The stability and convergence of these difference schemes are theoretically established. Several numerical examples are implemented, testifying to their efficiency and confirming their convergence order.

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1. Introduction

Fractional differential equations are now frequently invoked in various scientific and engineering applications. Physical phenomena in fields such as viscoelasticity, diffusion processes, relaxation vibrations and electrochemistry are successfully described by differential equations involving derivatives of fractional order [1–8]. Moreover, some underlying processes that cannot be described by single term time fractional partial differential equations can be described by multi-term equations — e.g. the multi-term time fractional diffusion-wave equation modelling various types of viscoelastic damping [9].

In this article, we provide some numerical difference schemes to solve multi-term time fractional sub-diffusion equations of the following form [9–11]:

\[
P(\mathbf{C}D_{t})u(x,t) = \kappa \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), \quad 0 < x < L, \quad 0 < t \leq T,
\]

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where $\kappa$ is a positive diffusion constant and the multi-term fractional operator $P(\frac{\partial}{\partial t})$ here is defined by

$$P(\frac{\partial}{\partial t})v(x, t) = \left( C_0 D_t^\alpha + \sum_{i=1}^s a_i C_0 D_t^{\alpha_i} \right) v(x, t)$$

with $0 < \alpha_i < \cdots < \alpha_1 < \alpha < 1$ and $a_i \in \mathbb{R}$, $i = 1, 2, \cdots, s$, and $C_0 D_t^\alpha$ denotes the Caputo fractional derivative of order $\alpha$:

$$C_0 D_t^\alpha v(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{v'(s)}{(t - s)^{\alpha}} d s.$$  


Other authors have discussed the numerical solution of fractional partial differential equations, including the fractional anomalous diffusion equation [14–24]. Chen et al. [14] proved the stability and convergence of an implicit difference approximation scheme of the fractional sub-diffusion equation using Fourier analysis. Lynch et al. [15] studied the numerical properties of the partial differential equations of fractional order $1 < \alpha < 2$. Yuste [16, 17] presented an explicit scheme and weighted average finite difference methods for the time fractional diffusion equation, and analysed their stability by the von-Neumann method. Zhuang et al. [18] obtained an implicit numerical method to solve the sub-diffusion equation by integrating the original equation on both sides, and proved the stability and convergence of their scheme by the energy method. Zhang et al. [19] constructed a Crank-Nicolson-type difference scheme and a compact difference scheme, to solve the time fractional sub-diffusion equation with a Riemann-Liouville fractional derivative. They proved that these two difference schemes are unconditionally stable, and the numerical solution converges in the maximum norm. Zhao & Sun [20] provided a box-type scheme for solving a class of fractional sub-diffusion equation with Neumann boundary conditions. Later, Ren et al. [21] proposed a compact difference scheme for the time fractional sub-diffusion equation with Neumann boundary conditions, and proved its unconditional stability and global convergence to be $O(\tau^{2-\alpha} + h^4)$ in the discrete $L_2$ norm.

There has also been some previous work on the numerical solution of problems with multiple fractional derivatives. Ford et al. [25] introduced a numerical method for solving the space-time fractional telegraph equation. Based on a quadrature formula approximation of the Caputo fractional derivative in spatial and temporal direction respectively, they proved the scheme was conditionally stable using the Fourier method. Liu et al. [26] proposed an implicit difference scheme for modified anomalous sub-diffusion equations with a nonlinear source term, and showed its convergence is $O(\tau + h^2)$ by the energy method.
Recently, Liu et al. [28] discussed some computationally effective numerical methods for the multi-term time fractional wave-diffusion equations, and Yang et al. [29] applied the variational iteration method to obtain approximate solutions for multi-term fractional differential equations.

Since fractional derivatives are non-local operators, any low-order finite difference or finite element scheme requires a large number of operations and a large memory storage capacity, so it is very desirable to use high-order methods for the efficient numerical solution of fractional derivative problems. However, there appear to be very few previous studies on efficient numerical methods and their relevant stability and convergence analysis, for problems involving multi-term fractional partial differential equations. The main purpose of this article is to construct effective and fast numerical methods for the 1D and 2D forms of the multi-term time fractional sub-diffusion equation (1.1), and to establish corresponding error estimates. To reduce the computational burden, we adopt fourth-order compact differences for the spatial approximation [19,21,33], such that fewer grid points are required to produce accurate solutions. The $L_1$ approximation proposed by Xu [30] and Sun [31] is adopted to deal with the multi-term temporal Caputo fractional derivatives. Using the discrete energy method, we prove that our resulting compact difference scheme is unconditionally stable and globally convergent, with the convergence $O(\tau^{2-a} + h^4)$ in the maximum norm. Furthermore, another new scheme with second-order spatial accuracy is also presented, and its corresponding stability and convergence discussed.

In Section 2, we first give some auxiliary lemmas, and then derive our compact difference scheme. In Section 3, by introducing a new inner product and using the energy method, we prove the stability and convergence of the compact difference scheme. In Section 4, the second-order scheme with unconditional stability and maximum norm convergence is discussed. Some results on the two-dimensional multi-term time fractional diffusion equation are given in Sections 5 and 6, respectively. In Section 7, some numerical experiments are presented to support the theoretical analysis, and to show the efficiency of the difference scheme. Some final comments are made in the concluding section.

2. Construction of the Compact Finite Difference Scheme

2.1. Notation and auxiliary lemmas

We first give some notation and auxiliary lemmas, to be used in the construction of the compact finite difference scheme. Without loss of generality, we may take $a_1 = 1$ and $\kappa = 1$ in Eq. (1.1) and consider the following problem involving the 1D two-term time fractional sub-diffusion equation:

\begin{align}
\frac{C}{\tau}D_t^{a_1}u(x,t) + \frac{C}{\tau}D_t^{a_2}u(x,t) &= \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t), & 0 < x < L, & 0 < t \leq T, \quad (2.1)
\end{align}

\begin{align}
 u(x,0) &= \varphi(x), & 0 \leq x \leq L, \quad (2.2)
\end{align}

\begin{align}
 u(0,t) = \psi_1(t), & u(L,t) = \psi_2(t), & 0 < t \leq T, \quad (2.3)
\end{align}
where $0 < \alpha_1 < \alpha < 1$ and $\varphi(x), \psi_1(t), \psi_2(t)$ and $f(x, t)$ are known smooth functions.

For the finite difference approximation, we set $h = L/M$ and $\tau = T/N$, where $M$ and $N$ are two positive integers. The domain $[0, L] \times [0, T]$ is covered by $\Omega_h \times \Omega_\tau$, where $\Omega_h = \{x_i \mid x_i = ih, 0 \leq i \leq M\}$ and $\Omega_\tau = \{t_n \mid t_n = n\tau, 0 \leq n \leq N\}$. For any grid function $u = \{u^k_j \mid 0 \leq j \leq M, 0 \leq k \leq N\}$ defined on $\Omega_h \times \Omega_\tau$, we introduce the notation

$$
\delta_x u^k_{j+\frac{1}{2}} = \frac{1}{h} \left( u^k_j - u^k_{j-1} \right), \quad \delta_x^2 u^k_j = \frac{1}{h} \left( \delta_x u^k_{j+\frac{1}{2}} - \delta_x u^k_{j-\frac{1}{2}} \right).
$$

For convenience, we denote a discrete fractional derivative operator $D^\alpha_t$ and an average operator $A$ respectively as follows:

$$
D^\alpha_t u^k_i = \frac{1}{\mu} \left[ u^k_i - \sum_{j=1}^{k-1} (a^a_{k-j-1} - a^a_{k-j}) u^j_i - a^a_{k-i} \varphi(x_i) \right], \quad 0 \leq i \leq M, \ 1 \leq k \leq N,
$$

where $\mu = \tau^a \Gamma(2 - \alpha)$ and $a^a_{k-i} = (k + 1)^{1-a} - k^{1-a}$; and

$$
Au^k_i = \begin{cases} 
\frac{1}{12} \left( u^k_{i-1} + 10u^k_i + u^k_{i+1} \right) = \left( I + \frac{h^2}{12} \delta_x^2 \right) u^k_i, & 1 \leq i \leq M - 1, \ 0 \leq k \leq N, \\
u^k_i, & i = 0, M, \ 0 \leq k \leq N,
\end{cases}
$$

where $I$ denotes the identity operator.

The following lemmas will be used in deriving the difference schemes.

**Lemma 2.1** (cf. Ref. [31]). Suppose that $0 < \alpha < 1$, $g \in C^2[0, t_n]$, and

$$
R^\alpha(g(t_n)) = \frac{1}{\Gamma(1 - \alpha)} \int_0^{t_n} g'(s) \frac{g(s)}{(t_n - s)\alpha} ds - \frac{\pi^{-\alpha}}{2 \Gamma(2 - \alpha)} \left[ a^a_n g(t_n) - \sum_{k=1}^{n-1} (a^a_{n-k-1} - a^a_{n-k}) g(t_k) - a^a_{n-1} g(0) \right].
$$

Then

$$
|R^\alpha(g(t_n))| \leq \frac{1}{\Gamma(2 - \alpha)} \left[ \frac{1 - \alpha}{12} + \frac{2^{2-\alpha}}{2 - \alpha} - (1 + 2^{-\alpha}) \right] \max_{0 \leq t \leq t_n} |g''(t)| \pi^{2-\alpha}.
$$

**Lemma 2.2** (cf. Ref. [32]). Denote $\zeta(s) = (1-s)^3[5-3(1-s)^2]$. Then if $f(x) \in C^6[x_{i-1}, x_{i+1}]$ where $1 \leq i \leq M - 1$,

$$
\frac{1}{12} \left[ f''(x_{i-1}) + 10f''(x_i) + f''(x_{i+1}) \right] - \frac{1}{h^2} \left[ f(x_{i-1}) - 2f(x_i) + f(x_{i+1}) \right] = \frac{h^4}{360} \int_0^1 \left[ f^{(6)}(x_i - sh) + f^{(6)}(x_i + sh) \right] \zeta(s) ds.
$$
2.2. Derivation of the compact difference scheme

Let us now construct a compact difference scheme to solve the problem (2.1)–(2.3). On \( \Omega_h \times \Omega_\tau \), we define the grid functions

\[
U^k_i = u(x_i, t_k), \quad f^k_i = f(x_i, t_k), \quad 0 \leq i \leq M, \quad 0 \leq k \leq N.
\]

Suppose \( u(x, t) \in C^6_{x, i}([0, L] \times [0, T]) \). Considering Eq. (2.1) at the point \( (x_i, t_k) \), we have

\[
C_0 D_t^{\alpha_1} u(x_i, t_k) + C_0^2 D_t^{\alpha_2} u(x_i, t_k) = \frac{\partial^2 u(x_i, t_k)}{\partial x^2} + f(x_i, t_k), \quad 0 \leq i \leq M, \quad 1 \leq k \leq N,
\]

and on applying the average operator \( A \) to both sides:

\[
A C_0 D_t^{\alpha_1} u(x_i, t_k) + A C_0^2 D_t^{\alpha_2} u(x_i, t_k) = A \frac{\partial^2 u(x_i, t_k)}{\partial x^2} + A f(x_i, t_k), \quad 1 \leq i \leq M - 1, \quad 1 \leq k \leq N.
\]

From Lemmas 2.1 and 2.2 and recalling \( D_t^{\alpha_1} \) and \( D_t^{\alpha_2} \),

\[
AD_t^{\alpha_1} U^k_i + AD_t^{\alpha_2} U^k_i = \frac{\partial^2 U^k_i}{\partial x^2} + F^k_i + R^k_i, \quad 1 \leq i \leq M - 1, \quad 1 \leq k \leq N,
\]

where

\[
R^k_i = \frac{h^4}{360} \int_0^1 \left[ \frac{\partial^6 u}{\partial x^6}(x_i - sh, t_k) + \frac{\partial^6 u}{\partial x^6}(x_i + sh, t_k) \right] \zeta(s)ds
\]

\[
+ AR_t^{\alpha_1} (u(x_i, t_k)) + AR_t^{\alpha_2}(u(x_i, t_k)).
\]

Noting that \( 0 < \alpha_1 < \alpha < 1 \), there exists a positive constant \( c_1 \) independent of \( h \) and \( \tau \) such that

\[
|R^k_i| \leq c_1 (\tau^{2-a} + h^4), \quad 1 \leq i \leq M - 1, \quad 1 \leq k \leq N.
\]

In addition, it follows from the initial and boundary conditions (2.2) and (2.3) that

\[
U^0_i = \varphi(x_i), \quad 0 \leq i \leq M, \quad (2.6)
\]

\[
U^k_0 = \varphi_1(t_k), \quad U^k_M = \varphi_2(t_k), \quad 1 \leq k \leq N. \quad (2.7)
\]

Omitting the small term \( R^k_i \) in Eq. (2.4) and replacing the function \( U^k_i \) with its numerical approximation \( u^k_i \), on noting (2.6) and (2.7) we construct the following compact difference scheme (L1-CD):

\[
AD_t^{\alpha_1} u^k_i + AD_t^{\alpha_2} u^k_i = \frac{\partial^2 u^k_i}{\partial x^2} + f^k_i, \quad 1 \leq i \leq M - 1, \quad 1 \leq k \leq N, \quad (2.8)
\]

\[
u^0_i = \varphi(x_i), \quad 0 \leq i \leq M, \quad (2.9)
\]

\[
u^k_0 = \psi_1(t_k), \quad u^k_M = \psi_2(t_k), \quad 1 \leq k \leq N. \quad (2.10)
\]

It is easy to see that at each time level, this L1-CD scheme (2.8)–(2.10) is a linear tridiagonal system with a strictly diagonally dominant coefficient matrix, so we have the following theorem:

**Theorem 2.1.** The L1-CD scheme (2.8)–(2.10) is uniquely solvable.
3. Stability and Convergence of the L1-CD Scheme

We now proceed to investigate of the stability and convergence of the L1-CD scheme (2.8)–(2.10). Let $V_h = \{ v \mid v = (v_0, v_1, \cdots, v_{M-1}, v_M), v_0 = v_M = 0 \}$. For any $u, v \in V_h$, we define the discrete inner product, $L_2$ norm, $H^1$ semi-norm and maximum norm as follows:

$$(u, v) = h \sum_{i=1}^{M-1} u_i v_i, \quad \| u \| = \sqrt{(u, u)}.$$  

$$\langle \delta_x u, \delta_x v \rangle = h \sum_{i=0}^{M-1} \left( \delta_x u_{i+\frac{1}{2}} \right) \left( \delta_x v_{i+\frac{1}{2}} \right) - \frac{h^3}{12} \sum_{i=1}^{M-1} (\delta_x^2 u_i)(\delta_x^2 v_i), \quad \| \delta_x u \|_h = \sqrt{\langle \delta_x u, \delta_x u \rangle},$$

$$|u|_1 = \sqrt{h \sum_{i=0}^{M-1} \left( \delta_x u_{i+\frac{1}{2}} \right)^2}, \quad \| u \|_{\infty} = \max_{1 \leq i \leq M-1} |u_i|.$$  

Lemma 3.1 (cf. Refs. [14, 19]). If $0 < \gamma < 1$ and $a_k^\gamma = (k+1)^{1-\gamma} - k^{1-\gamma}$, $k = 0, 1, 2, \cdots$. Then

$$1 = a_0^\gamma > a_1^\gamma > a_2^\gamma > \cdots > a_k^\gamma > \cdots \to 0,$$

$$(1 - \gamma)(k+1)^{-\gamma} < a_k^\gamma < (1 - \gamma)k^{-\gamma}.$$  

Lemma 3.2. Let $0 < \gamma < 1$. For any grid function $u = \{ u_i^k \mid 0 \leq i \leq M, 0 \leq k \leq N \}$ defined on $\Omega_h \times \Omega_\tau$, it holds that

$$-(AD_x u^k, \delta_x^2 u^k) = \frac{1}{\mu} \left[ \| \delta_x u^k \|_h^2 - \sum_{j=1}^{k-1} (a_{k-j}^\gamma - a_{k-j}^\gamma) (\delta_x u^j, \delta_x u^k) - a_{k-1}^\gamma (\delta_x u^0, \delta_x u^k) \right],$$

where $\mu = \tau^\gamma \Gamma(2 - \gamma)$.

Proof: From the definition of $D_x^\gamma$ and the discrete Green formula,

$$- \mu (AD_x^\gamma u^k, \delta_x^2 u^k) = - \left( I + \frac{h^2}{12} \delta_x^2 \right) u^k + \sum_{j=1}^{k-1} (a_{k-j}^\gamma - a_{k-j}^\gamma) \left( I + \frac{h^2}{12} \delta_x^2 \right) u^j, \delta_x^2 u^k$$

$$+ a_{k-1}^\gamma \left( I + \frac{h^2}{12} \delta_x^2 \right) u^0, \delta_x^2 u^k$$

$$= \left( \| \delta_x u^k \|_h^2 - \frac{h^2}{12} \| \delta_x^2 u^k \|_h^2 \right) - \sum_{j=1}^{k-1} (a_{k-j}^\gamma - a_{k-j}^\gamma) \left( \delta_x u^j, \delta_x u^k \right) - \frac{h^2}{12} \left( \delta_x^2 u^j, \delta_x^2 u^k \right)$$

$$- a_{k-1}^\gamma \left( \delta_x u^0, \delta_x u^k \right) - \frac{h^2}{12} \left( \delta_x^2 u^0, \delta_x^2 u^k \right)$$

$$= \| \delta_x u^k \|_h^2 - \sum_{j=1}^{k-1} (a_{k-j}^\gamma - a_{k-j}^\gamma) \left( \delta_x u^j, \delta_x u^k \right) - a_{k-1}^\gamma \left( \delta_x u^0, \delta_x u^k \right).$$
Dividing the above equality by \( \mu \) then yields the desired result. \( \square \)

**Lemma 3.3** (cf. Ref. [33]). For any mesh function \( u \in V_h \),
\[
\frac{2}{3} |u|^2 \leq \| \delta_x u \|^2 \leq |u|^2.
\]
The semi-norm \( \| \cdot \|_h \) is equivalent to the standard \( H^1 \) semi-norm, but the semi-norm \( \| \cdot \|_h \) is more convenient than the standard \( H^1 \) semi-norm for the stability and convergence analysis. We now prove the following Theorem on an a priori estimate.

**Theorem 3.1.** Let \( \{ v^k \} \mid 0 \leq i \leq M, 0 \leq k \leq N \} \) be the solution of the difference system
\[
AD_x v^k + AD_i v^k = \delta_x v^k + \delta_i^k, \quad 1 \leq i \leq M - 1, 1 \leq k \leq N, \tag{3.1}
\]
\[
v_i^0 = v^0(x_i), \quad 0 \leq i \leq M, \tag{3.2}
\]
\[
v_k^0 = 0, \quad v_M^0 = 0, \quad 1 \leq k \leq N, \tag{3.3}
\]
where \( v^0(x_0) = v^0(x_M) = 0. \) Then
\[
\| \delta_x v^k \|^2 \leq \| \delta_x v^0 \|^2 + \frac{c_{\max}}{4} \max_{1 \leq l \leq N} \| g^l \|^2,
\]
where \( c_{\max} = \max\{ V^a \Gamma(1 - \alpha_1), V^a \Gamma(1 - \alpha) \}. \)

**Proof.** Taking the inner product of Eq. (3.1) with \( -\delta_x^2 v^k \), we have
\[
-(AD_x a^k v^k, \delta_x^2 v^k) - (AD_i a^k v^k, \delta_i^k v^k) = -(\delta_x^2 v^k, \delta_x^2 v^k) - (g^k, \delta_x^2 v^k), \quad 1 \leq k \leq N; \tag{3.4}
\]
and hence from Lemma 3.2
\[
-(AD_x a^k v^k, \delta_x^2 v^k) = \frac{1}{\mu_1} \| \delta_x v^k \|^2 - \frac{1}{\mu_1} \sum_{j=1}^{k-1} (a_{k-j-1}^a - a_{k-j}^a) (\delta_x v^j, \delta_x v^k) - \frac{a_{k-1}^a}{\mu_1} (\delta_x v^0, \delta_x v^k), \tag{3.5}
\]
\[
-(AD_i a^k v^k, \delta_i^k v^k) = \frac{1}{\mu_0} \| \delta_x v^k \|^2 - \frac{1}{\mu_0} \sum_{j=1}^{k-1} (a_{k-j}^a - a_{k-j-1}^a) (\delta_x v^j, \delta_x v^k) - \frac{a_{k-1}^a}{\mu_0} (\delta_x v^0, \delta_x v^k). \tag{3.6}
\]
Substituting Eqs. (3.5) and (3.6) into Eq. (3.4), from Lemma 3.1 and the Cauchy-Schwarz inequality we consequently deduce that
\[
\left( \frac{1}{\mu_1} + \frac{1}{\mu_0} \right) \| \delta_x v^k \|^2_h \leq \sum_{j=1}^{k-1} \left[ \frac{1}{\mu_1} (a_{k-j-1}^a - a_{k-j}^a) + \frac{1}{\mu_0} (a_{k-j}^a - a_{k-j-1}^a) \right] \| \delta_x v^j \|^2_h + \frac{a_{k-1}^a}{\mu_1} \left( \| \delta_x v^0 \|^2_h + \frac{\mu_1}{4 a_{k-1}^a} \| g^k \|^2 \right) + \frac{a_{k-1}^a}{\mu_0} \left( \| \delta_x v^0 \|^2_h + \frac{\mu_0}{4 a_{k-1}^a} \| g^k \|^2 \right). \tag{3.7}
\]
Applying Lemma 3.1 and noting that $0 < \alpha_1 < \alpha < 1$, we also obtain

\[
\begin{align*}
\frac{\mu_1}{4 \alpha_k} &< \frac{k^{\alpha_1} \tau^{\alpha_1} \Gamma(2-\alpha_1)}{4(1-\alpha_1)} \leq \frac{T \tau^{1-\alpha}}{4}, \\
\frac{\mu_0}{4 \alpha_k} &< \frac{k^{\alpha_1} \tau^{\alpha_1} \Gamma(2-\alpha)}{4(1-\alpha)} \leq \frac{T \tau^{1-\alpha}}{4}.
\end{align*}
\]

Letting

\[ B = \|\delta_x v^0\|^2_h + \frac{c_{\max}}{4} \|g\|^2 \]

and substituting the above estimates into Eq. (3.7), we have

\[
\left( \frac{1}{\mu_1} + \frac{1}{\mu_0} \right) \|\delta_x v^k\|^2_h \leq \sum_{j=1}^{k-1} \left[ \frac{1}{\mu_1} \left( a_{k-j-1} - a_{k-j} \right) + \frac{1}{\mu_0} \left( a_{k-j} - a_{k-j} \right) \right] \|\delta_x v^j\|^2_h + \left( \frac{a_{k-1}^0}{\mu_1} + \frac{a_k^0}{\mu_0} \right) B, \quad 1 \leq k \leq N.
\]

Using the same arguments as in [20, 21], we obtain the desired result. \hfill \Box

We obtain the following stability statement from Theorem 3.1.

**Theorem 3.2.** The L1-CD scheme (2.8)–(2.10) is unconditionally stable to the initial value $\varphi$ and the inhomogeneous term $f$.

Let us now consider the convergence of the L1-CD scheme (2.8)–(2.10). Writing

\[ e_i^k = U_i^k - u_i^k, \quad 0 \leq i \leq M, \quad 0 \leq k \leq N, \]

and subtracting Eqs. (2.8)–(2.10) from Eq. (2.4) and Eqs. (2.6)–(2.7) respectively, we get the error equations

\[
\begin{align*}
AD_{\tau} e_i^k + AD_{\tau}^2 e_i^k &= \delta_x^2 e_i^k + R_i^k, \quad 1 \leq i \leq M - 1, \quad 1 \leq k \leq N, \\
e_0^0 &= 0, \quad 0 \leq i \leq M, \\
e_0^k &= 0, \quad k = 0, \quad 1 \leq k \leq N.
\end{align*}
\]

From Eq. (2.5) and Theorem 3.1 we have

\[
\|\delta_x e_i^k\|^2_h \leq \frac{c_{\max}}{4} \max_{1 \leq i \leq N} \|R_i^k\|^2 \leq \frac{c_{\max} L c_1^2}{4} (\tau^{2-\alpha} + h^4)^2,
\]

and then applying the embedding inequality $\|u\|_\infty \leq \frac{\sqrt{\pi}}{2} |u|_1$ (cf. [34]) and Lemma 3.3 we obtain the following convergence result.

**Theorem 3.3.** Assume that $u(x, t) \in C_{\infty}([0, L] \times [0, T])$ is the solution of Eqs. (2.1)–(2.3) and $\{u_i^k | 0 \leq i \leq M, 0 \leq k \leq N\}$ is the solution of the L1-CD scheme Eqs. (2.8)–(2.10). Then

\[
\|e^k\|_\infty \leq \frac{c_1 L}{8} \sqrt{\delta c_{\max} (\tau^{2-\alpha} + h^4)}, \quad 0 \leq k \leq N.
\]
4. A Second-Order Finite Difference Scheme

We now present a finite difference scheme second-order in the spatial direction that is computationally efficient when the storage is inexpensive. Thus for the problem (2.1)–(2.3) we construct the L1-SOD scheme

\[
D_t^\alpha u_i^k + D_{xx}^\alpha u_i^k = \delta_x^2 u_i^k + f_i^k, \quad 1 \leq i \leq M - 1, 1 \leq k \leq N, \quad (4.1)
\]

\[
u_i^0 = \varphi(x_i), \quad 0 \leq i \leq M, \quad (4.2)
\]

\[
u_0 = \psi_1(t_k), \quad u_N = \psi_2(t_k), \quad 1 \leq k \leq N. \quad (4.3)
\]

Lemma 4.1. Suppose \(u(x, t) \in C^{4,2}_{x,t}([0, L] \times [0, T])\) is the solution of the problem (2.1)–(2.3). Then the truncation error of the L1-SOD scheme (4.1)–(4.3) satisfies

\[
|\hat{R}_i^k| \leq \hat{c}_R(\tau^{2-a} + h^2), \quad 0 \leq i \leq M, \quad 1 \leq k \leq N,
\]

where \(\hat{c}_R\) is a positive constant independent of \(\tau\) and \(h\).

Using the analytical method of the compact difference scheme in Section 3, we can also prove that the L1-SOD scheme (4.1)–(4.3) is stable to the initial value \(\varphi\) and the forcing term \(f\), and convergent with the convergence order of \(O(\tau^{2-a} + h^2)\) in the maximum norm.

Theorem 4.1. Suppose \(\{u_i^k\} | 0 \leq i \leq M, 0 \leq k \leq N\} is the solution of the L1-SOD scheme (4.1)–(4.3). Then

\[
\|\delta_x u_i^k\|^2 \leq \|\delta_x u_0^0\|^2 + \frac{c_{\text{max}}}{4} \max_{1 \leq i \leq N} \|f_i^i\|^2.
\]

Theorem 4.2. If the problem (2.1)–(2.3) has smooth solution \(u(x, t) \in C^{4,2}_{x,t}([0, L] \times [0, T])\) and \(\{u_i^k\} | 0 \leq i \leq M, 0 \leq k \leq N\} is the solution of the L1-SOD scheme (4.1)–(4.3), then

\[
\|U^k - u^k\|_\infty \leq \frac{\hat{c}_R L}{4} \sqrt{\max} (\tau^{2-a} + h^2), \quad 0 \leq k \leq N.
\]

5. Extension to 2D Multi-Term Time-Fractional Sub-Diffusion Equations

Let is now consider the numerical solution of the following problem involving the 2D multi-term fractional sub-diffusion equations:

\[
\frac{\partial}{\partial t} D_t^{\alpha_1} u(x, y, t) + \frac{\partial}{\partial t} D_t^{\alpha_2} u(x, y, t) = \Delta u(x, y, t) + f(x, y, t), \quad (x, y) \in \Omega, \quad 0 < t \leq T, \quad (5.1)
\]

\[
u(x, y, 0) = \varphi(x, y), \quad (x, y) \in \hat{\Omega} = \Omega \cup \partial \Omega, \quad (5.2)
\]

\[
u(x, y, t) = \psi(x, y, t), \quad (x, y) \in \partial \Omega, \quad 0 < t \leq T, \quad (5.3)
\]

where \(\Omega = (0, L_1) \times (0, L_2)\), \(\partial \Omega\) is the boundary, \(0 < \alpha_1 < \alpha < 1\), and \(\varphi(x, y), \psi(x, y, t)\) and \(f(x, y, t)\) are known smooth functions.
Set \( x_i = ih_1 \) and \( y_j = jh_2 \) with \( h_1 = L_1/M_1 \) and \( h_2 = L_2/M_2 \), where \( M_1 \) and \( M_2 \) are positive integers. Define \( \Omega_{h_1} = \{ x_i \mid 0 \leq i \leq M_1 \} \) and \( \Omega_{h_2} = \{ y_j \mid 0 \leq j \leq M_2 \} \), \( \Omega_h = \Omega_{h_1} \times \Omega_{h_2}, \Omega = \hat{\Omega}_h \cup \Omega \) and \( \partial\Omega_h = \hat{\Omega}_h \cup \partial\Omega \). The definitions of \( \tau, t_k \) are the same in Section 2. For any grid function \( v = \{ v_{ij} \mid 0 \leq i \leq M_1, 0 \leq j \leq M_2 \} \), we denote

\[
\begin{align*}
\delta_x v_{i-\frac{1}{2},j} &= \frac{1}{h_1} \left( v_{ij} - v_{i-1,j} \right), \\
\delta_x^2 v_{ij} &= \frac{1}{h_1} \left( \delta_x v_{i+\frac{1}{2},j} - \delta_x v_{i-\frac{1}{2},j} \right), \\
\delta_y \delta_x v_{i-\frac{1}{2},j} &= \frac{1}{h_2} \left( \delta_x v_{i,\frac{1}{2}+j} - \delta_x v_{i,\frac{1}{2}-j} \right), \\
\delta_y \delta_x v_{i,j-\frac{1}{2}} &= \frac{1}{h_2} \left( \delta_x v_{i-\frac{1}{2},j} - \delta_x v_{i+\frac{1}{2},j} \right).
\end{align*}
\]

Similarly, the notations \( \delta_y v_{i,j-\frac{1}{2}}, \delta_y^2 v_{ij}, \delta_x^2 v_{i-\frac{1}{2},j}, \delta_x^2 \delta_y v_{ij} \) can be defined. The discrete Laplace operator is denoted \( \Delta_h v_{ij} = \delta_x^2 v_{ij} + \delta_y^2 v_{ij} \), and the spatial average difference operators are defined as

\[
\begin{align*}
H_x v_{ij} &= \begin{cases} 
\frac{1}{12} (v_{i-1,j} + 10v_{ij} + v_{i+1,j}) = \left( I + \frac{h_2^2}{12} \delta_y^2 \right) v_{ij}, & 1 \leq i \leq M_1 - 1, 0 \leq j \leq M_2, \\
v_{ij}, & i = 0 \text{ or } M_1, 0 \leq j \leq M_2,
\end{cases} \\
H_y v_{ij} &= \begin{cases} 
\frac{1}{12} (v_{i,j-1} + 10v_{ij} + v_{i,j+1}) = \left( I + \frac{h_2^2}{12} \delta_y^2 \right) v_{ij}, & 1 \leq j \leq M_2 - 1, 0 \leq i \leq M_1, \\
v_{ij}, & j = 0 \text{ or } M_2, 0 \leq i \leq M_1.
\end{cases}
\]

We introduce the space of grid functions on \( \hat{\Omega}_h \):

\[
V_h^* = \{ v \mid v = \{ v_{ij} \mid (x_i, y_j) \in \hat{\Omega}_h \} \text{ and } v_{ij} = 0 \text{ if } (x_i, y_j) \in \partial\Omega_h \}.
\]

For any grid functions \( u, v \in V_h^* \), we define the discrete inner product

\[
(u, v) = h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2-1} u_{ij} v_{ij}
\]

and denote \( \| v \| = \sqrt{(v, v)} \). Similarly, we define \( \| \delta_x^2 v \|, \| \delta_y^2 v \| \) and \( \| \delta_x^2 \delta_y^2 v \| \); and denote

\[
\begin{align*}
\| \delta_x v \| &= \sqrt{h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2} \left| \delta_x v_{i-\frac{1}{2},j} \right|^2}, \\
\| \delta_x \delta_y v \| &= \sqrt{h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2} \left| \delta_x \delta_y v_{i-\frac{1}{2},j} \right|^2}, \\
\| \delta_y \delta_x v \| &= \sqrt{h_1 h_2 \sum_{i=1}^{M_1-1} \sum_{j=1}^{M_2} \left| \delta_y \delta_x v_{i,j-\frac{1}{2}} \right|^2},
\end{align*}
\]
and $\|\delta_y v\|$, $\|\delta_x \delta_y v\|$ similarly. The discrete $H^1$ semi-norm and $H^1$ norm are

$$
\|\nabla_h v\| = \sqrt{\|\delta_x v\|^2 + \|\delta_y v\|^2}, \quad \|v\|_{H^1} = \sqrt{\|v\|^2 + \|\nabla_h v\|^2}.
$$

Finally, for any grid functions $u, v \in V_h^*$ we denote

$$(\nabla_h u, \nabla_h v)_{h^*} = (\delta_x u, \delta_x v) + (\delta_y u, \delta_y v) - \frac{h_2^2}{12} \left[ (\delta^2_x u, \delta^2_x v) + (\delta_x \delta_y u, \delta_x \delta_y v) \right]
- \frac{h_2^2}{12} \left[ (\delta^2_y u, \delta^2_y v) + (\delta_x \delta_y u, \delta_x \delta_y v) \right] + \frac{h_2^2 h_3^2}{144} \left[ (\delta^2_x \delta_y u, \delta^2_y \delta_y v) + (\delta_x \delta^2_y u, \delta_x \delta^2_y v) \right]
$$

and

$$
\|\nabla_h u\|_{h^*} = \sqrt{(\nabla_h u, \nabla_h u)_{h^*}}.
$$

5.1. Derivation of the compact ADI difference scheme

We suppose that $u(x, y, t) \in C^{6,6,2}_{x,y,t}(\Omega \times [0, T])$, and define the grid functions

$$
U^n_{ij} = u(x_i, y_j, t_n), \quad f^n_{ij} = f(x_i, y_j, t_n), \quad (x_i, y_j) \in \Omega_h, \quad 1 \leq n \leq N.
$$

Considering Eq. (5.1) at the point $(x_i, y_j, t_n)$, we have

$$
\frac{\partial}{\partial t} C^\alpha_0 D^\alpha_t u(x_i, y_j, t_n) + \frac{\partial}{\partial t} C^\alpha_0 D^\alpha_t u(x_i, y_j, t_n) = \Delta u(x_i, y_j, t_n) + f(x_i, y_j, t_n),
$$

$$(x_i, y_j) \in \Omega_h, \quad 1 \leq n \leq N.
$$

Applying the average operator $H_x H_y$ on both sides of this equation,

$$
H_x H_y \left( u_{x,x}(x_i, y_j, t_n) + u_{y,y}(x_i, y_j, t_n) \right) + H_x H_y f(x_i, y_j, t_n) = H_x H_y \left( \frac{\partial}{\partial x} U^n_{ij} + \frac{\partial}{\partial y} U^n_{ij} \right),
$$

$$(x_i, y_j) \in \Omega_h, \quad 1 \leq n \leq N.
$$

From Lemmas 2.1 and 2.2,

$$
H_x H_y D^a_t U^n_{ij} + H_x H_y D^a_t U^n_{ij} = H_x \delta^2_x U^n_{ij} + H_y \delta^2_y U^n_{ij} + H_x H_y f^n_{ij} + \hat{R}^n_{ij},
$$

$$(x_i, y_j) \in \Omega_h, \quad 1 \leq n \leq N, \quad (5.4)
$$

where

$$
\hat{R}^n_{ij} = \frac{h_3^4}{360} \int_0^1 \left[ \frac{\partial^6 u}{\partial x^6}(x_i - \lambda h_1, y_j, t_n) + \frac{\partial^6 u}{\partial x^6}(x_i + \lambda h_1, y_j, t_n) \right] \zeta(\lambda) d\lambda
$$

$$
+ \frac{h_3^4}{360} \int_0^1 \left[ \frac{\partial^6 u}{\partial y^6}(x_i, y_j - \lambda h_2, t_n) + \frac{\partial^6 u}{\partial y^6}(x_i, y_j + \lambda h_2, t_n) \right] \zeta(\lambda) d\lambda
$$

$$
+ H_x H_y \left( u(x_i, t_k) + H_x H_y R^a(u(x_i, t_k)) \right).
$$
Adding a small term \( \frac{\mu_1 \mu_0 \tau}{\mu_1 + \mu_0} \delta_x^2 \delta_y^2 \delta_t U_{ij}^{n-\frac{1}{2}} \) into Eq. (5.4), we then have

\[
H_x H_y D_{\tau}^{\alpha_1} U_{ij}^n + H_x H_y D_{\tau}^{\alpha_2} U_{ij}^n + \frac{\mu_1 \mu_0 \tau}{\mu_1 + \mu_0} \delta_x^2 \delta_y^2 \delta_t U_{ij}^{n-\frac{1}{2}}
= H_x \delta_x^2 U_{ij}^n + H_x \delta_y^2 U_{ij}^n + H_x H_y f_{ij}^n + R_{ij}^n, \quad (x_i, y_j) \in \Omega_h, \quad 1 \leq n \leq N, \tag{5.5}
\]

where

\[
R_{ij}^n = \tilde{R}_{ij}^n + \frac{\mu_1 \mu_0 \tau}{\mu_1 + \mu_0} \delta_x^2 \delta_y^2 \delta_t U_{ij}^{n-\frac{1}{2}},
\]

so there exists a positive constant \( c_2 \) independent of \( h_1, h_2 \) and \( \tau \) such that

\[
|R_{ij}^{n-\frac{1}{2}}| \leq c_2 (\tau_{\text{min}}[1+\alpha,2] + h_1^2 + h_2^2), \quad (x_i, y_j) \in \Omega_h, \quad 1 \leq n \leq N. \tag{5.6}
\]

In addition, on noting the initial and boundary value conditions (5.2)–(5.3) we find that

\[
U_{ij}^0 = \varphi(x_i, y_j), \quad (x_i, y_j) \in \Omega_h, \tag{5.7}
\]

\[
U_{ij}^0 = \phi(x_i, y_j, t_n), \quad (x_i, y_j) \in \partial \Omega_h, \quad 1 \leq n \leq N. \tag{5.8}
\]

Omitting small terms \( R_{ij}^n \) in Eq. (5.5), and replacing the function \( U_{ij}^n \) with its numerical approximation \( U_{ij}^n \) in Eqs. (5.5), (5.7) and (5.8), we obtain the following compact difference scheme (L1-CADI):

\[
H_x H_y D_{\tau}^{\alpha_1} U_{ij}^n + H_x H_y D_{\tau}^{\alpha_2} U_{ij}^n + \frac{\mu_1 \mu_0 \tau}{\mu_1 + \mu_0} \delta_x^2 \delta_y^2 \delta_t U_{ij}^{n-\frac{1}{2}}
= H_x \delta_x^2 U_{ij}^n + H_x \delta_y^2 U_{ij}^n + H_x H_y f_{ij}^n, \quad (x_i, y_j) \in \Omega_h, \quad 1 \leq n \leq N, \tag{5.9}
\]

\[
U_{ij}^0 = \varphi(x_i, y_j), \quad (x_i, y_j) \in \Omega_h, \tag{5.10}
\]

\[
U_{ij}^0 = \phi(x_i, y_j, t_n), \quad (x_i, y_j) \in \partial \Omega_h, \quad 1 \leq n \leq N. \tag{5.11}
\]

To efficiently solve this formulation, the following techniques can be used. Introducing the intermediate variable \( u_{ij}^n \), we obtain the D’Yakonov-type ADI scheme:

\[
\begin{cases}
H_x - \frac{\mu_1 \mu_0}{\mu_1 + \mu_0} \delta_x^2 \left[ u_{ij}^n - \frac{1}{\mu_1 \mu_0} \delta_x^2 \delta_y^2 \delta_t U_{ij}^{n-\frac{1}{2}} \right]
+ \frac{\mu_0 \alpha_1}{\mu_1 + \mu_0} H_x H_y \psi_{ij} + \frac{\mu_1 \mu_0}{\mu_1 + \mu_0} \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) H_x H_y u_{ij}^k
+ \frac{\mu_0 \alpha_1}{\mu_1 + \mu_0} H_x H_y \psi_{ij} + \frac{\mu_1 \mu_0}{\mu_1 + \mu_0} \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) H_x H_y u_{ij}^k
= H_x - \frac{\mu_1 \mu_0}{\mu_1 + \mu_0} \delta_y^2 \left[ u_{ij}^n - \frac{1}{\mu_1 \mu_0} \delta_x^2 \delta_y^2 \delta_t U_{ij}^{n-\frac{1}{2}} \right]
+ \frac{\mu_0 \alpha_1}{\mu_1 + \mu_0} H_x H_y \psi_{ij} + \frac{\mu_1 \mu_0}{\mu_1 + \mu_0} \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) H_x H_y u_{ij}^k
+ \frac{\mu_0 \alpha_1}{\mu_1 + \mu_0} H_x H_y \psi_{ij} + \frac{\mu_1 \mu_0}{\mu_1 + \mu_0} \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) H_x H_y u_{ij}^k
\end{cases}
\]

\[
[ H_x - \frac{\mu_1 \mu_0}{\mu_1 + \mu_0} \delta_x^2 ] u_{ij}^n = u_{ij}^n. \]
We determine \( \{u^n_i\} \) by solving two sets of independent one-dimensional problems. Since the coefficient matrices are strictly diagonally dominant, at each time level one can run the \( x \)-sweep of the procedure to compute \( \{u^n_{ij}(x_i, y_j) \in \Omega_h \} \) with the known data \( \{u^{n-1}_{ij}(x_i, y_j) \in \Omega_h \} \) using the fast tridiagonal (Thomas) algorithm — cf. Ref. [35] for example. Then once the solution \( \{u^n_{ij}(x_i, y_j) \in \Omega_h \} \) is available, the \( y \)-sweep can readily be performed to calculate the desired unique solution \( \{u^n_{ij}(x_i, y_j) \in \Omega_h \} \) to the compact ADI difference scheme (5.9)–(5.11).

### 5.2. Stability and convergence of the L1-CADI scheme

To analyse the stability and convergence of the L1-CADI scheme (5.9)–(5.11), we introduce some important lemmas.

**Lemma 5.1.** If \( 0 < \gamma < 1 \), for any grid function \( \{u^n_{ij} \mid (x_i, y_j) \in \Omega_h, 0 \leq n \leq N \} \) defined on \( \tilde{\Omega}_h \times \Omega \), we have

\[
-(H_x H_y D^2_i u^n, \Delta_h u^n) = \frac{1}{\mu} \left[ \| \nabla_h u^n \|_{h^2}^2 - \sum_{k=1}^{n-1} (a^\gamma_{n-k-1} - a^\gamma_{n-k}) (\nabla_h u^k, \nabla_h u^n)_{h^2} \right]
\]

**Proof:** The result follows from the definition of \( \| \cdot \|_{h^2} \) and the discrete Green formula. \( \square \)

**Lemma 5.2.** For any grid function \( u \in V_h^s \),

\[
\left( (H_y \delta^2_{x} + H_x \delta^2_{y}) u, \Delta_h u \right) \geq \frac{2}{3} \| \Delta_h u \|^2.
\]

**Proof:** From the discrete Green formulation,

\[
\left( (H_y \delta^2_{x} + H_x \delta^2_{y}) u, \Delta_h u \right) = \left( \left( I + \frac{h_x^2}{12} \delta^2_{y} \right) \delta^2_{x} u + \left( I + \frac{h_y^2}{12} \delta^2_{x} \right) \delta^2_{y} u, \Delta_h u \right)
\]

\[
= \| \Delta_h u \|^2 - \frac{h_x^2}{12} \left( \| \delta_x \delta^2_{x} u \|^2 + \| \delta^2_{x} \delta_y u \|^2 \right) - \frac{h_y^2}{12} \left( \| \delta_y \delta^2_{y} u \|^2 + \| \delta^2_{y} \delta_x u \|^2 \right)
\]

\[
\geq \frac{2}{3} \| \Delta_h u \|^2,
\]

where we have used

\[
\| \delta_x \delta_y u \| \leq 2h_x^{-1} \| \delta_y u \|, \quad \| \delta^2_{x} u \| \leq 2h_x^{-1} \| \delta_x u \|,
\]

\[
\| \delta_x \delta_y u \| \leq 2h_y^{-1} \| \delta_x u \|, \quad \| \delta^2_{y} u \| \leq 2h_y^{-1} \| \delta_y u \|.
\]

\( \square \)
Lemma 5.3. For any grid function \( u \in V_h^n \), we have
\[
\frac{1}{3} \| \nabla_h u \|^2 \leq \| \nabla_h u \|_{h^r}^2 \leq \| \nabla_h u \|^2.
\]

Proof: The result readily follows from the definition of \( \| \cdot \|_{h^r} \) and inverse estimates. \( \square \)

We may now obtain an \( \text{a priori} \) estimate in the following Theorem.

Theorem 5.1. Let \( \{ v_{ij}^n \mid (x_i, y_j) \in \Omega_h, \ 0 \leq n \leq N \} \) be the solution of the difference system
\[
H_x H_y D_t^{\alpha_1} v_{ij}^n + H_x H_y D_t^{\alpha_2} v_{ij}^n + \frac{\mu_1 \mu_0 \tau}{\mu_1 + \mu_0} \delta_x^2 \delta_y^2 \delta_{t} v_{ij}^{n-\frac{1}{2}} = H_x \delta_x^2 v_{ij}^n + H_x \delta_y^2 v_{ij}^n + g_{ij}^n, \quad (x_i, y_j) \in \Omega_h, \ 1 \leq n \leq N,
\]
\[
\begin{align*}
\tau \sum_{k=1}^{n} \| \nabla_h v^k \|_{h^r}^2 \leq & \ c_{\text{max}} \left\{ \frac{\mu_1 \mu_0 T}{\mu_1 + \mu_0} \left( \| \delta_x \delta_y^2 v^0 \|^2 + \| \delta_x^2 \delta_y v^0 \|^2 \right) \right. \\
+ & \left. \left( \frac{\tau^{1-a_1}}{1(2-a_1)} + \frac{\tau^{1-a}}{1(2-a)} \right) \| \nabla_h v^0 \|_{h^r}^2 + \frac{3}{4} \tau \sum_{k=1}^{n} \| g^k \|_0^2 \right) \\
\end{align*}
\]

Proof: Taking the inner product of Eq. (5.12) with \( -\Delta_h v^n \), we have
\[
- \left( H_x H_y D_t^{\alpha_1} v^n, \Delta_h v^n \right) - \left( H_x H_y D_t^{\alpha_2} v^n, \Delta_h v^n \right) - \frac{\mu_1 \mu_0 \tau}{\mu_1 + \mu_0} \left( \delta_x^2 \delta_y^2 \delta_{t} v^{n-\frac{1}{2}}, \Delta_h v^n \right)
\]
\[
= - \left( H_x \delta_x^2 + H_x \delta_y^2 v^n, \Delta_h v^n \right) - \left( g^k, \Delta_h v^n \right), \quad 1 \leq k \leq N. \tag{5.15}
\]

On using the discrete Green formula and Cauchy-Schwarz inequality, we rewrite the third term on the left-hand side of Eq. (5.15) as
\[
\begin{align*}
&- \tau \left( \delta_x^2 \delta_y^2 \delta_{t} v^{n-\frac{1}{2}}, \Delta_h v^n \right) \\
&= - \left( \delta_x^2 \delta_y^2 v^n, \Delta_h v^n \right) + \left( \delta_x^2 \delta_y^2 v^{n-1}, \Delta_h v^n \right) \\
&= \left( \| \delta_x^2 \delta_y v^n \|^2 + \| \delta_x \delta_y^2 v^n \|^2 \right) - \left[ \left( \delta_x^2 \delta_y v^{n-1}, \delta_x \delta_y^2 v^n \right) + \left( \delta_x^2 \delta_y v^n, \delta_x \delta_y^2 v^{n-1} \right) \right] \\
&\geq \left( \| \delta_x^2 \delta_y v^n \|^2 + \| \delta_x \delta_y^2 v^n \|^2 \right) - \frac{1}{2} \left[ \left( \| \delta_x^2 \delta_y v^{n-1} \|^2 + \| \delta_x \delta_y^2 v^n \|^2 \right) \right. \\
&\left. + \left( \| \delta_x^2 \delta_y v^n \|^2 + \| \delta_x \delta_y^2 v^{n-1} \|^2 \right) \right] \\
&\geq \frac{1}{2} \left[ \left( \| \delta_x^2 \delta_y v^n \|^2 + \| \delta_x \delta_y^2 v^n \|^2 \right) - \left( \| \delta_x^2 \delta_y v^{n-1} \|^2 + \| \delta_x \delta_y^2 v^{n-1} \|^2 \right) \right]. \tag{5.16}
\end{align*}
\]
On substituting Eq. (5.16) into Eq. (5.15), and using Lemmas 5.1 and 5.2 together with the Cauchy-Schwarz inequality, we deduce that

\[
\begin{align*}
\left( \frac{1}{\mu_1} + \frac{1}{\mu_0} \right) \| \nabla_h v^n \|^2_{h^*} + \frac{2}{3} \| \Delta_h v^n \|^2 + \frac{\mu_1 \mu_0}{2(\mu_1 + \mu_0)} \left[ \left( \| \delta_x^2 \delta_y v^n \|^2 + \| \delta_y^2 v^n \|^2 \right) \\
- \left( \| \delta_x^2 \delta_y v^{n-1} \|^2 + \| \delta_y^2 v^{n-1} \|^2 \right) \right] \\
\leq \frac{1}{\mu_1} \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}^1) (\nabla_h v^k, \nabla_h v^n)_{h^*} + \frac{a_{n-1}^0}{\mu_1} (\nabla_h v^0, \nabla_h v^n)_{h^*} \\
+ \frac{1}{\mu_0} \sum_{k=1}^{n-1} (a_{n-k-1}^0 - a_{n-k}) (\nabla_h v^k, \nabla_h v^n)_{h^*} + \frac{a_{n-1}^0}{\mu_0} (\nabla_h v^0, \nabla_h v^n)_{h^*} \\
- \left( g^n, \nabla_h v^n \right) \\
\leq \frac{1}{2\mu_1} \sum_{k=1}^{n-1} (a_{n-k-1}^1 - a_{n-k}^1) \left( \| \nabla_h v^k \|_{h^*}^2 + \| \nabla_h v^n \|_{h^*}^2 \right) + \frac{a_{k-1}^1}{2\mu_1} \| \nabla_h v^0 \|_{h^*}^2 + \frac{a_{n-1}^1}{\mu_1} \| \nabla_h v^n \|_{h^*}^2 \\
+ \frac{1}{2\mu_0} \sum_{k=1}^{n-1} (a_{n-k-1}^0 - a_{n-k}^0) \left( \| \nabla_h v^k \|_{h^*}^2 + \| \nabla_h v^n \|_{h^*}^2 \right) + \frac{a_{n-1}^0}{2\mu_0} \| \nabla_h v^0 \|_{h^*}^2 + \frac{a_{n-1}^0}{\mu_0} \| \nabla_h v^n \|_{h^*}^2 \\
+ \frac{3}{8} \| g^n \|^2 + \frac{2}{3} \| \Delta_h v^n \|^2 ,
\end{align*}
\]

implying that

\[
\begin{align*}
\left( \frac{1}{\mu_1} + \frac{1}{\mu_0} \right) \| \nabla_h v^n \|^2_{h^*} + \frac{\mu_1 \mu_0}{\mu_1 + \mu_0} \left[ \left( \| \delta_x^2 \delta_y v^n \|^2 + \| \delta_y^2 v^n \|^2 \right) \\
- \left( \| \delta_x^2 \delta_y v^{n-1} \|^2 + \| \delta_y^2 v^{n-1} \|^2 \right) \right] \\
\leq \sum_{k=1}^{n-1} \left[ \frac{1}{\mu_1} (a_{n-k-1}^1 - a_{n-k}^1) + \frac{1}{\mu_0} (a_{n-k-1}^0 - a_{n-k}^0) \right] \| \nabla_h v^k \|_{h^*}^2 + \frac{a_{n-1}^0}{\mu_1} \| \nabla_h v^0 \|_{h^*}^2 + \frac{3}{4} \| g^n \| .
\end{align*}
\]

(5.17)

On denoting

\[
G^n = \sum_{k=1}^{n-1} \left( \frac{a_{n-k}^1}{\mu_1} + \frac{a_{n-k}^0}{\mu_0} \right) \| \nabla_h v^k \|_{h^*}^2 + \frac{\mu_1 \mu_0}{\mu_1 + \mu_0} \left( \| \delta_x^2 \delta_y v^n \|^2 + \| \delta_y^2 v^n \|^2 \right) , \quad 1 \leq n \leq N
\]

and

\[
G^0 = \frac{\mu_1 \mu_0}{\mu_1 + \mu_0} \left( \| \delta_x^2 \delta_y v^0 \|^2 + \| \delta_y^2 v^0 \|^2 \right) ,
\]
Eq. (5.17) can be written as
\[
G^n \leq G^{n-1} + \left( \frac{a_{n-1}^a}{\mu_1} + \frac{a_{n-1}^a}{\mu_0} \right) \| \nabla_h v^0 \|_{h^*}^2 + \frac{3}{4} \| g^n \|_2^2 \\
\leq G^0 + \sum_{k=1}^{n} \left( \frac{a_{k-1}^a}{\mu_1} + \frac{a_{k-1}^a}{\mu_0} \right) \| \nabla_h v^0 \|_{h^*}^2 + \frac{3}{4} \sum_{k=1}^{n} \| g^k \|_2^2.
\] (5.18)

On the one hand, noting that \( a_{n-k}^\gamma \geq (1-\gamma)(n-k+1)^{1-\gamma} \geq (1-\gamma)N^{-\gamma} \) where \( \gamma = \alpha_1, \alpha \) we arrive at
\[
a_{n-k}^1 = \frac{a_{n-k}^1}{\mu_1} = \frac{1}{\tau \alpha_1 \Gamma(2-\alpha_1)} \geq \frac{(1-\alpha_1) N^{-\alpha_1}}{\Gamma(1-\alpha_1)} = \frac{1}{T \alpha_1 \Gamma(1-\alpha_1)},
\]
\[
a_{n-k}^a = \frac{a_{n-k}^a}{\mu_0} = \frac{1}{\tau \alpha_1 \Gamma(2-\alpha)} \geq \frac{(1-\alpha) N^{-\alpha}}{\Gamma(1-\alpha)} = \frac{1}{T \alpha \Gamma(1-\alpha)}.
\]

On the other hand, noting that \( a_{k}^\gamma = (k+1)^{1-\gamma} - k^{1-\gamma} \) we have
\[
\tau \sum_{k=1}^{n} \frac{a_{k-1}^1}{\mu_1} \leq \frac{\tau n^{1-\alpha_1}}{\tau \alpha_1 \Gamma(2-\alpha_1)} \leq \frac{1}{\Gamma(1-\alpha_1)} \leq T^{1-\alpha_1},
\]
\[
\tau \sum_{k=1}^{n} \frac{a_{k-1}^a}{\mu_0} \leq \frac{\tau n^{1-\alpha}}{\tau \alpha \Gamma(2-\alpha)} \leq \frac{1}{\Gamma(1-\alpha)} \leq T^{1-\alpha}.
\]

Multiplying Eq. (5.18) by \( \tau \) and invoking the above inequality yields the desired result. \( \square \)

From Theorem 5.1, we obtain the following stability statement.

**Theorem 5.2.** The L1-CADI scheme (5.9)–(5.11) is unconditionally stable to the initial value \( \varphi \) and the inhomogeneous term \( f \).

Let us now consider the convergence of the L1-CADI scheme (5.9)–(5.11).

**Theorem 5.3.** Assume that \( u(x, y, t) \in C^{6,6,2}_{0,0,0}(\Omega \times [0, T]) \) is the solution of Eqs. (5.1)–(5.3), and \( \{u^n_j \mid (x_i, y_j) \in \Omega_h, \ 0 \leq n \leq N \} \) is the solution of the L1-CADI scheme (5.9)–(5.11). Then
\[
\sqrt{\tau \sum_{k=1}^{n} \| \nabla_h e^k \|_2^2} \leq \frac{3c_2}{2} \sqrt{TL_1 L_2} \left( \tau^{\min[1+a,2-a]} + h_1^4 + h_2^4 \right), \quad 1 \leq n \leq N.
\]

**Proof.** Let
\[
e^n_{ij} = U^n_{ij} - u^n_{ij}, \quad (x_i, y_j) \in \Omega_h, \quad 0 \leq n \leq N.
\]
Subtracting Eqs. (5.9)–(5.11) from Eqs. (5.5), (5.7) and (5.8) respectively, we obtain the error equations
\[
H_xH_y D_{\tau} e^n_{ij} + H_x H_y D_{\tau} e^n_{ij} + \frac{\mu_1 \mu \tau}{\mu_1 + \mu} \delta_x \delta_y e^n_{ij} = H_y \delta_x e^n_{ij} + H_x \delta_y e^n_{ij} + R^n_{ij}, \quad (x_i, y_j) \in \Omega_h, \quad 1 \leq n \leq N,
\]
\[
e^n_{ij} = 0, \quad (x_i, y_j) \in \Omega_h,
\]
\[
e^n_{ij} = 0, \quad (x_i, y_j) \in \partial \Omega_h, \quad 0 \leq n \leq N.
\]
where \( D_{ij}^x e_{ij}^n = D_{ij}^x (U_{ij}^n - u_{ij}^n) \) and \( \gamma = \alpha, \alpha_1 \). Using inequality (5.6), it follows from Theorem 5.1 and Lemma 5.3 that

\[
\tau \sum_{k=1}^{n} \|\nabla_h e^k\|^2 \leq \frac{9\tau}{4} \sum_{k=1}^{n} \|R^k\|^2 \leq \frac{9\tau}{4} \sum_{k=1}^{n} L_1 L_2 c_z^2 (\tau^{\min[1+\alpha,2-\alpha]} + h_1^4 + h_2^4)^2
\]

\[
\leq \frac{9T}{4} c_z^2 L_1 L_2 (\tau^{\min[1+\alpha,2-\alpha]} + h_1^4 + h_2^4)^2, \quad 1 \leq n \leq N.
\]

\( \square \)

6. A Second-Order ADI Finite Difference Scheme for the 2D Problem

We consider a difference scheme L1-ADI for the problem defined by Eqs. (5.1)–(5.3) as follows:

\[
D_{ij}^{\alpha_1} u_{ij}^n + D_{ij}^{\alpha_2} u_{ij}^n + \frac{\mu_1 \mu_0 \tau}{\mu_1 + \mu_0} \delta_x^2 \delta_y^2 u_{ij}^{n-1} + \mu_1 \mu_0 \sum_{k=1}^{n-1} (a_{n-k-1}^{\alpha_1} - a_{n-k}^{\alpha_1}) u_{ij}^k
\]

\[
- \Delta_3 u_{ij}^n + f_{ij}^n, \quad (x_i, y_j) \in \bar{\Omega}_h, \quad 1 \leq n \leq N, \quad (6.1)
\]

\[
u_{ij}^{n+1} = \varphi(x_i, y_j), \quad (x_i, y_j) \in \bar{\Omega}_h, \quad (6.2)
\]

\[
u_{ij}^{n+1} = \phi(x_i, y_j, t_n), \quad (x_i, y_j) \in \partial \Omega_h, \quad 1 \leq n \leq N. \quad (6.3)
\]

The difference scheme for Eq. (6.1) can be written in the ADI form

\[
\begin{bmatrix}
I - \frac{\mu_1 \mu_0}{\mu_1 + \mu_0} \delta_x^2 u_{ij}^n + \left( \frac{\mu_1 \mu_0}{\mu_1 + \mu_0} \right)^2 \delta_x^2 \delta_y^2 u_{ij}^{n-1} + \frac{\mu_0}{\mu_1 + \mu_0} \sum_{k=1}^{n-1} (a_{n-k-1}^{\alpha_1} - a_{n-k}^{\alpha_1}) u_{ij}^k
\end{bmatrix}
\]

\[
+ \frac{\mu_0 a_{n-1}^{\alpha_1}}{\mu_1 + \mu_0} \psi_{ij} + \frac{\mu_1}{\mu_1 + \mu_0} \sum_{k=1}^{n-1} (a_{n-k-1}^{\alpha_1} - a_{n-k}^{\alpha_1}) u_{ij}^k + \frac{\mu_0 a_{n-1}^{\alpha_1}}{\mu_1 + \mu_0} \psi_{ij} + \frac{\mu_1 \mu_0}{\mu_1 + \mu_0} f_{ij}^n,
\]

\[
\begin{bmatrix}
I - \frac{\mu_1 \mu_0}{\mu_1 + \mu_0} \delta_y^2 u_{ij}^n = u_{ij}^n
\end{bmatrix}
\]

Lemma 6.1. Suppose that \( u(x, y, t) \in C^{4,3,2}_{x,y,t}(\Omega \times [0, T]) \) is the solution of the problem (5.1)–(5.3). Then the truncation error of the L1-ADI scheme (6.1)–(6.3) satisfies

\[
|\hat{R}_{ij}^k| \leq \tilde{c}_R \left( \tau^{\min[1+\alpha,2-\alpha]} + h_1^2 + h_2^2 \right), \quad (x_i, y_j) \in \bar{\Omega}_h, \quad 1 \leq n \leq N,
\]

where \( \tilde{c}_R \) is a positive constant independent of \( \tau \) and \( h_1, h_2 \).

The proofs for the stability and convergence for the L1-ADI scheme (6.1)–(6.3) are essentially the same as for Theorems 5.1 and 5.3, so we omit the details and simply state the relevant results.
Theorem 6.1. The L1-ADI scheme (6.1)–(6.3) is unconditionally stable to the initial value \( \varphi \) and the inhomogeneous term \( f \) — i.e.

\[
\tau \sum_{k=1}^{n} \| \nabla h u_k \|^2 \leq c_{\max} \left\{ \frac{\mu_1 \mu_0 T}{\mu_1 + \mu_0} \left( \| \delta_x \delta_y^2 u_0 \|^2 + \| \delta_x^2 \delta_y u_0 \|^2 \right) \right. \\
+ \left. \left( \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} \right) \| \nabla h u_0 \|^2 + \frac{3}{4} \tau \sum_{k=1}^{n} \| f_k \|^2 \right\}.
\]

Theorem 6.2. Suppose \( u(x,y,t) \) is the smooth solution of (5.1)–(5.3) and \( \{ u_{ij}^n \mid (x_i, y_j) \in \tilde{\Omega}_h, 0 \leq n \leq N \} \) is the solution of the L1-ADI scheme (6.1)–(6.3), respectively. Then

\[
\sqrt{\tau \sum_{k=1}^{n} \| \nabla h e_k \|^2} \leq \frac{\tilde{c}_h}{2} \sqrt{3T L_1 L_2 \left( \tau^{\min \{1+\alpha, 2-\alpha\}} + h_1^2 + h_2^2 \right)}, \quad 1 \leq n \leq N.
\]

7. Numerical Experiments

We now present numerical solutions of the problem using the numerical methods previously discussed, where the errors involved are measured by comparing the numerical solutions with exact solutions.

Example 7.1. Let \( L = \pi, \ T = 1 \). In order to test the convergence rate of the proposed methods, we refer to the exact solution of the problem (2.1)–(2.3) — viz.

\[ u(x,t) = t^{1+\alpha} \sin x. \]

It can readily be checked that the corresponding source term \( f(x,t) \) and the initial and boundary conditions are respectively

\[ f(x,t) = \left( \frac{\Gamma(2+\alpha_1+\alpha)}{\Gamma(2+\alpha)} t^{2+\alpha} + \frac{\Gamma(2+\alpha)}{\Gamma(2+\alpha_1)} t^{2+\alpha_1} + t^{+1+\alpha_1+\alpha} \right) \sin x, \]

and

\[ \varphi(x) = 0, \quad \psi_1(t) = 0, \quad \psi_2(t) = 0. \]

We compute the maximum norm errors of the numerical solution

\[ e_\infty(h, \tau) = \max_{0 \leq k \leq N} \| U^k - u^k \|_\infty, \]

and respectively characterise the temporal convergence order and the spatial convergence order as

\[ \text{Order1} = \log_2 \left( \frac{e_\infty(h, 2\tau)}{e_\infty(h, \tau)} \right), \quad \text{Order2} = \log_2 \left( \frac{e_\infty(2h, \tau)}{e_\infty(h, \tau)} \right). \]

The first computational investigation concerns the temporal errors and convergence orders. In order to find the temporal convergence order, the space step \( h \) should be chosen.
Table 1: Numerical convergence of the L1-CD and L1-SOD schemes in the temporal direction.

<table>
<thead>
<tr>
<th>$\alpha_1$, $\alpha$</th>
<th>$\tau$</th>
<th>the L1-CD scheme</th>
<th>the L1-SOD scheme</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$e_{\infty}(h, \tau)$</td>
<td>Order1</td>
</tr>
<tr>
<td>$\alpha_1 = 0.15$, $\alpha = 0.95$</td>
<td>1/10</td>
<td>3.776e-2</td>
<td>*</td>
</tr>
<tr>
<td></td>
<td>1/20</td>
<td>1.836e-2</td>
<td>1.040</td>
</tr>
<tr>
<td></td>
<td>1/40</td>
<td>8.890e-3</td>
<td>1.047</td>
</tr>
<tr>
<td></td>
<td>1/80</td>
<td>4.296e-3</td>
<td>1.049</td>
</tr>
<tr>
<td></td>
<td>1/160</td>
<td>2.075e-3</td>
<td>1.050</td>
</tr>
<tr>
<td>$\alpha_1 = 0.35$, $\alpha = 0.65$</td>
<td>1/10</td>
<td>1.242e-2</td>
<td>*</td>
</tr>
<tr>
<td></td>
<td>1/20</td>
<td>4.817e-3</td>
<td>1.366</td>
</tr>
<tr>
<td></td>
<td>1/40</td>
<td>1.858e-3</td>
<td>1.374</td>
</tr>
<tr>
<td></td>
<td>1/80</td>
<td>7.159e-4</td>
<td>1.376</td>
</tr>
<tr>
<td></td>
<td>1/160</td>
<td>2.759e-4</td>
<td>1.375</td>
</tr>
</tbody>
</table>

Table 2: Numerical convergence of the L1-CD scheme in the spatial direction with $\tau = 1/200000$.

<table>
<thead>
<tr>
<th>$\alpha_1$, $\alpha$</th>
<th>$h$</th>
<th>the L1-CD scheme</th>
<th>Order2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\pi/2$</td>
<td>6.413e-3</td>
<td>*</td>
</tr>
<tr>
<td>$\alpha_1 = 0.35$, $\alpha = 0.65$</td>
<td>$\pi/4$</td>
<td>3.788e-4</td>
<td>4.082</td>
</tr>
<tr>
<td></td>
<td>$\pi/8$</td>
<td>2.328e-5</td>
<td>4.024</td>
</tr>
<tr>
<td></td>
<td>$\pi/16$</td>
<td>1.464e-6</td>
<td>3.991</td>
</tr>
<tr>
<td>$\alpha_1 = 0.45$, $\alpha = 0.55$</td>
<td>$\pi/2$</td>
<td>6.450e-3</td>
<td>*</td>
</tr>
<tr>
<td></td>
<td>$\pi/4$</td>
<td>3.809e-4</td>
<td>4.082</td>
</tr>
<tr>
<td></td>
<td>$\pi/8$</td>
<td>2.340e-5</td>
<td>4.025</td>
</tr>
<tr>
<td></td>
<td>$\pi/16$</td>
<td>1.461e-6</td>
<td>4.002</td>
</tr>
</tbody>
</table>

sufficiently small to prevent the effect of the spatial discretization error entering into the calculation. The computational results of the L1-CD scheme (2.8)–(2.10) and the L1-SOD scheme (4.1)–(4.3) with $h = \pi/100$ and $h = \pi/1000$ are presented in Table 1, respectively. It is observed that both schemes generate $2 - \alpha$ temporal convergence order, consistent with our theoretical analysis.

Secondly, we test the spatial errors and convergence orders of the two schemes by letting $h$ vary, but fixing the time step $\tau$ sufficiently small to avoid contamination of the spatial error. Tables 2 and 3 show the maximum norm errors and spatial convergence orders of the L1-CD scheme and the L1-SOD scheme with different $\alpha_1, \alpha$. As predicted by our theoretical estimates, the L1-CD scheme attains fourth-order spatial accuracy while the L1-SOD scheme has second-order spatial accuracy.

Next, in order to quantify some features of the computational efficiencies of the L1-CD scheme more precisely, we investigate the CPU time of both schemes. As mentioned before, since fractional derivatives are non-local operators they require a large memory storage capacity if low-order finite difference methods are employed for the spatial approximation. For the L1-SOD scheme, the optimal step sizes satisfy $\tau^{2-\alpha} \approx h^2$, or $N \approx [M^{2/\alpha}]$. For the L1-CD scheme, the optimal step sizes satisfy $\tau^{2-\alpha} \approx h^4$, or $N \approx [M^{4/\alpha}]$.

From Table 4, it is clear that the two schemes provide almost the same accuracy for

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Table 3: Numerical convergence of the L1-SOD scheme in the spatial direction with $\tau = 1/20000$.

<table>
<thead>
<tr>
<th>$\alpha_1, \alpha$</th>
<th>$h$</th>
<th>$e_\infty(h, \tau)$</th>
<th>Order2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_1 = 0.35, \alpha = 0.65$</td>
<td>$\pi/2$</td>
<td>4.612e-2</td>
<td>*</td>
</tr>
<tr>
<td></td>
<td>$\pi/4$</td>
<td>1.188e-2</td>
<td>1.956</td>
</tr>
<tr>
<td></td>
<td>$\pi/8$</td>
<td>2.993e-3</td>
<td>1.990</td>
</tr>
<tr>
<td></td>
<td>$\pi/16$</td>
<td>7.497e-4</td>
<td>1.997</td>
</tr>
<tr>
<td>$\alpha_1 = 0.45, \alpha = 0.55$</td>
<td>$\pi/2$</td>
<td>4.640e-2</td>
<td>*</td>
</tr>
<tr>
<td></td>
<td>$\pi/4$</td>
<td>1.195e-2</td>
<td>1.957</td>
</tr>
<tr>
<td></td>
<td>$\pi/8$</td>
<td>3.009e-3</td>
<td>1.990</td>
</tr>
<tr>
<td></td>
<td>$\pi/16$</td>
<td>7.537e-4</td>
<td>1.997</td>
</tr>
</tbody>
</table>

Table 4: The maximum norm error and CPU time of the L1-CD and L1-SOD schemes.

<table>
<thead>
<tr>
<th>$\alpha_1, \alpha$</th>
<th>$N$</th>
<th>the L1-CD scheme</th>
<th>the L1-SOD scheme</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>$h$</td>
<td>$e_\infty(h, \tau)$</td>
</tr>
<tr>
<td>$\alpha_1 = 0.1, \alpha = 0.3$</td>
<td>225</td>
<td>10</td>
<td>1.678e-5</td>
</tr>
<tr>
<td></td>
<td>585</td>
<td>15</td>
<td>3.286e-6</td>
</tr>
<tr>
<td></td>
<td>1151</td>
<td>20</td>
<td>1.044e-5</td>
</tr>
<tr>
<td></td>
<td>1947</td>
<td>25</td>
<td>4.262e-7</td>
</tr>
<tr>
<td>$\alpha_1 = 0.1, \alpha = 0.5$</td>
<td>464</td>
<td>10</td>
<td>2.159e-5</td>
</tr>
<tr>
<td></td>
<td>1368</td>
<td>15</td>
<td>4.229e-6</td>
</tr>
<tr>
<td></td>
<td>2947</td>
<td>20</td>
<td>1.343e-6</td>
</tr>
<tr>
<td>$\alpha_1 = 0.1, \alpha = 0.7$</td>
<td>1194</td>
<td>10</td>
<td>3.070e-5</td>
</tr>
<tr>
<td></td>
<td>4157</td>
<td>15</td>
<td>6.024e-6</td>
</tr>
<tr>
<td></td>
<td>10073</td>
<td>20</td>
<td>1.916e-6</td>
</tr>
<tr>
<td></td>
<td>20015</td>
<td>25</td>
<td>7.829e-7</td>
</tr>
</tbody>
</table>

the same temporal grid size, but the L1-CD scheme needs fewer spatial grid points and less CPU time. Thus the L1-CD scheme reduces both the storage requirement and the necessary CPU time successfully.

Finally, we compute the long time behaviour of the L1-CD and the L1-SOD schemes. The optimal step sizes again satisfy $\tau^{2-a} \approx h^2$ for the L1-SOD scheme and $\tau^{2-a} \approx h^4$ for the L1-CD scheme, on fixing $T = 10$ and $M = 4, 6, 8, \cdots, 24$. Fig. 1 shows the maximum error and CPU time of the L1-CD scheme and the L1-SOD scheme for $t = 0, 1, 2, \cdots, 10$ when $\alpha_1 = 0.1, \alpha = 0.2$, and the compact difference scheme efficiency.

Example 7.2. Let $T = 1$, $\Omega = (0, \pi) \times (0, \pi)$. As before, we refer to the exact solution of the problem (5.1)–(5.3) — i.e. in this case

$$u(x, y, t) = t^{3+\alpha_1+a} \sin x \sin y.$$  

It is again not difficult to obtain the corresponding forcing term $f(x, y, t)$, and the initial and boundary conditions $\varphi(x, y)$ and $\psi(x, y, t)$.

In order to test the convergence rate of the proposed methods, we use the same spacing $h$ in each direction ($h = h_1 = h_2$), and compute the maximum norm errors of the numerical
Firstly, the numerical accuracy in time is verified. For fixed space step sizes \( h = \pi/20 \) and \( h = \pi/200 \) respectively, and varying the temporal step size \( \tau \), the computational results we obtain are displayed in Table 5. From this data, we conclude that there is \( \min\{1 + \alpha, 2 - \alpha\} \)-order convergence in time.

\[
E_\infty(h, \tau) = \max_{(x_i, y_j) \in \Omega_h, \ 0 \leq n \leq N} |u(x_i, y_j, t_n) - u_{ij}^n|,
\]

via

\[
\text{Order3} = \log_2 \left( \frac{E_\infty(h, 2\tau)}{E_\infty(h, \tau)} \right), \quad \text{Order4} = \log_2 \left( \frac{E_\infty(2h, \tau)}{E_\infty(h, \tau)} \right).
\]

Figure 1: Error and CPU time of the L1-CD and L1-SOD schemes.

Table 5: Numerical convergence of the L1-CADI and L1-ADI schemes in the temporal direction.

<table>
<thead>
<tr>
<th>( \alpha_1, \alpha )</th>
<th>( \tau )</th>
<th>the L1-CADI scheme ( E_\infty(h, \tau) ) Order3</th>
<th>the L1-ADI scheme ( E_\infty(h, \tau) ) Order3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha_1 = 0.2, \alpha = 0.9 )</td>
<td>1/10</td>
<td>7.038e-2 *</td>
<td>7.038e-2 *</td>
</tr>
<tr>
<td></td>
<td>1/20</td>
<td>3.452e-2 1.028</td>
<td>3.453e-2 1.027</td>
</tr>
<tr>
<td></td>
<td>1/40</td>
<td>1.657e-2 1.059</td>
<td>1.658e-2 1.058</td>
</tr>
<tr>
<td></td>
<td>1/80</td>
<td>7.857e-3 1.077</td>
<td>7.862e-3 1.076</td>
</tr>
<tr>
<td></td>
<td>1/160</td>
<td>3.698e-3 1.087</td>
<td>3.703e-3 1.086</td>
</tr>
<tr>
<td>( \alpha_1 = 0.5, \alpha = 0.7 )</td>
<td>1/10</td>
<td>4.140e-2 *</td>
<td>4.141e-2 *</td>
</tr>
<tr>
<td></td>
<td>1/20</td>
<td>1.757e-2 1.236</td>
<td>1.758e-2 1.236</td>
</tr>
<tr>
<td></td>
<td>1/40</td>
<td>7.294e-3 1.269</td>
<td>7.299e-3 1.268</td>
</tr>
<tr>
<td></td>
<td>1/80</td>
<td>2.985e-3 1.289</td>
<td>2.990e-3 1.287</td>
</tr>
<tr>
<td></td>
<td>1/160</td>
<td>1.212e-3 1.301</td>
<td>1.217e-3 1.297</td>
</tr>
</tbody>
</table>
Numerical Methods for Multi-Term Time Fractional Sub-Diffusion Equations

Table 6: Numerical convergence of the L1-CADI scheme in the spatial direction with $\tau = 1/200000$.

<table>
<thead>
<tr>
<th>$\alpha_1$, $\alpha$</th>
<th>$h$</th>
<th>$E_{\infty}(h, \tau)$</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_1 = 0.1$, $\alpha = 0.2$</td>
<td>$\pi/2$</td>
<td>4.074e-3</td>
<td>*</td>
</tr>
<tr>
<td></td>
<td>$\pi/4$</td>
<td>2.413e-4</td>
<td>4.078</td>
</tr>
<tr>
<td></td>
<td>$\pi/8$</td>
<td>1.482e-5</td>
<td>4.025</td>
</tr>
<tr>
<td></td>
<td>$\pi/16$</td>
<td>9.222e-7</td>
<td>4.006</td>
</tr>
<tr>
<td>$\alpha_1 = 0.2$, $\alpha = 0.3$</td>
<td>$\pi/2$</td>
<td>2.768e-3</td>
<td>*</td>
</tr>
<tr>
<td></td>
<td>$\pi/4$</td>
<td>1.641e-4</td>
<td>4.076</td>
</tr>
<tr>
<td></td>
<td>$\pi/8$</td>
<td>1.009e-5</td>
<td>4.024</td>
</tr>
<tr>
<td></td>
<td>$\pi/16$</td>
<td>6.334e-7</td>
<td>3.993</td>
</tr>
</tbody>
</table>

Table 7: Numerical convergence of the L1-ADI scheme in the spatial direction with $\tau = 1/10000$.

<table>
<thead>
<tr>
<th>$\alpha_1$, $\alpha$</th>
<th>$h$</th>
<th>$E_{\infty}(h, \tau)$</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_1 = 0.1$, $\alpha = 0.2$</td>
<td>$\pi/4$</td>
<td>2.314e-2</td>
<td>*</td>
</tr>
<tr>
<td></td>
<td>$\pi/8$</td>
<td>5.769e-3</td>
<td>2.004</td>
</tr>
<tr>
<td></td>
<td>$\pi/16$</td>
<td>1.436e-3</td>
<td>2.007</td>
</tr>
<tr>
<td></td>
<td>$\pi/32$</td>
<td>3.528e-4</td>
<td>2.025</td>
</tr>
<tr>
<td>$\alpha_1 = 0.2$, $\alpha = 0.3$</td>
<td>$\pi/4$</td>
<td>2.124e-2</td>
<td>*</td>
</tr>
<tr>
<td></td>
<td>$\pi/8$</td>
<td>5.306e-3</td>
<td>2.001</td>
</tr>
<tr>
<td></td>
<td>$\pi/16$</td>
<td>1.324e-3</td>
<td>2.003</td>
</tr>
<tr>
<td></td>
<td>$\pi/32$</td>
<td>3.287e-4</td>
<td>2.010</td>
</tr>
</tbody>
</table>

Secondly, we test the spatial errors and convergence orders of the two schemes, by letting $h$ vary and fixing the time step $\tau$ sufficiently small in order to avoid significant contamination from the spatial error. Tables 6 and 7 give the maximum norm errors and spatial convergence orders for the two schemes. As predicted by the theoretical estimates, the L1-CADI scheme attains fourth-order spatial accuracy whereas the L1-ADI scheme has second-order spatial accuracy. In Table 8, we display some CPU time results for the L1-CADI and L1-ADI schemes. It is clear that the two schemes generate almost the same accuracy for the same temporal grid size, while the L1-CADI scheme needs fewer spatial grid points and less CPU time, so it requires less storage and CPU time.

Similar to Example 7.1, we compute the problem for a longer time by fixing $T = 10$ and $M = 4, 5, 6, \cdots, 14$, and still choosing the optimal step size $\tau_{\text{min}}^{[1+\alpha,2-\alpha]} \approx h^2$ for the L1-ADI scheme and $\tau_{\text{min}}^{[1+\alpha,2-\alpha]} \approx h^4$ for the L1-CADI scheme, respectively. Fig. 2 shows the maximum error and CPU time of both schemes for $t = 0, 1, 2, \cdots, 10$ when $\alpha_1 = 0.1$, $\alpha = 0.2$, and also the efficiency of the L1-CADI scheme.

8. Conclusions

In this article, we discuss some computationally effective numerical methods for solving 1D and 2D multi-term time fractional sub-diffusion equations. Based on the $L1$ approximation for the multi-term time Caputo fractional derivative in the temporal direction, and in order to reduce the storage requirement, a compact difference method for spatial...
approximation is derived that has fourth-order spatial accuracy \([19, 21, 33]\). Using some novel techniques, the unique solvability, unconditional stability and global convergence are proved rigorously for both the 1D and 2D cases. Numerical examples verify the effectiveness of the compact difference scheme. These methods and techniques could be extended to other kinds of multi-term time-space fractional equations and to equations with a nonlinear source term. Future work may concentrate on studying the effective numerical schemes for the multi-term fractional time-space equations.
Acknowledgments

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References


