OSCILLATIONS OF DELAY DIFFERENTIAL EQUATIONS

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Abstract

Sufficient conditions are established for all solutions of the linear system
\[ \frac{dy_i(t)}{dt} + \sum_{j=1}^{n} q_{ij} y_j(t - \tau_{ij}) = 0, \quad i = 1, 2, \ldots, n, \]
to be oscillatory, where \( q_{ij} \in (-\infty, \infty), \tau_{ij} \in (0, \infty), i, j = 1, 2, \ldots, n \).

1. Introduction

Consider the system of delay differential equations
\[ \frac{dy_i(t)}{dt} + \sum_{j=1}^{n} q_{ij} y_j(t - \tau_{ij}) = 0, \quad i = 1, 2, \ldots, n \]
where the coefficients are real numbers and the delays are positive real numbers. We say that a solution
\[ y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix} \]
of (1) oscillates if for some \( i \in (1, 2, \ldots, n) \), \( y_i(t) \) has arbitrarily large zeros. A solution \( y(t) \) of (1) is said to be nonoscillatory if there exists a \( t_0 \geq 0 \) such that for each \( i = 1, 2, \ldots, n \), \( y_i(t) \neq 0 \) for \( t \geq t_0 \). The aim of this brief paper is to derive a set of sufficient conditions for all solutions

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of (1) to oscillate. Our result is an extension of a result of Gopalsamy in [2], where only bounded solutions of systems like (1) have been considered. For references concerning the oscillation of systems, the reader is referred to the references in [2].

2. Sufficient conditions for oscillation

The following lemma will be useful in the proof of our theorem below.

**Lemma 1.** Assume that (1) has a nonoscillatory solution (2). Then there are numbers
\[ \delta_i \in \{-1, 1\} \quad \text{for } i = 1, 2, \ldots, n \]
such that the system
\[ \frac{dz_i(t)}{dt} + \sum_{j=1}^{n} p_{ij} z_j(t - \tau_{ij}) = 0 \]  
(3)
where
\[ p_{ij} = \frac{\delta_i q_{ij}}{\delta_j} \quad \text{for } i, j = 1, 2, \ldots, n \]  
(4)
has a nonoscillatory solution \([z_1(t), z_2(t), \ldots, z_n(t)]^T\) with eventually positive components \(z_i(t), i = 1, 2, \ldots, n\).

**Proof.** The components \(y(t)\) of (2) are positive or negative eventually. That is, there exists a \(T > 0\) such that \(y_i(t) \neq 0\) for \(t > T\) and \(i = 1, 2, \ldots, n\). Set \(\delta_i = \text{sign}[y_i(t)], i = 1, 2, \ldots, n\) and \(t > T\). It is now easy to see that
\[ z(t) = [\delta_1 y_1(t), \delta_2 y_2(t), \ldots, \delta_n y_n(t)]^T \]  
(5)
satisfies (3) and \(\delta_i y_i(t) > 0\) for \(i = 1, 2, \ldots, n\) and \(t > T\).

The next result is concerned with the asymptotic behaviour of nonoscillatory solutions of (1).

**Lemma 2.** Consider the system (1) and suppose that the constant coefficients of (1) satisfy
\[ q = \min_{1 \leq i \leq n} \left[ q_{ii} - \sum_{j=1}^{n} |q_{ji}| \right] > 0. \]  
(6)
Then every nonoscillatory solution \(y(t) = (y_1, y_2, \ldots, y_n)\) satisfies
\[ \lim_{t \to \infty} y_i(t) = 0. \]
PROOF. Clearly (6) is also satisfied with the $q_{ij}$ replaced by the respective $p_{ij}$ of (4). From this and (5) it suffices to prove the lemma for nonoscillatory solutions of (2) with eventually positive components. Let us assume that there is a $t_0 \geq 0$ such that $y_i(t) > 0$ for $t \geq t_0$, $i = 1, 2, \ldots, n$. If we let

$$w(t) = \sum_{j=1}^{n} y_j(t), \quad t \geq t_0$$

then

$$\frac{dw(t)}{dt} + \sum_{i=1}^{n} \sum_{j=1}^{n} q_{ij} y_j(t - \tau_{jj}) = 0$$

or

$$\frac{dw(t)}{dt} + \sum_{i=1}^{n} q_{ii} y_i(t - \tau_{ii}) = -\sum_{i=1}^{n} \sum_{j=1}^{n} q_{ij} y_j(t - \tau_{jj})$$

$$\leq \sum_{i=1}^{n} \sum_{j=1}^{n} |q_{ji}| y_i(t - \tau_{ii}).$$

It follows from (8) that

$$\frac{dw(t)}{dt} + \sum_{i=1}^{n} \left( q_{ii} - \sum_{j=1}^{n} |q_{ji}| \right) y_i(t - \tau_{ii}) \leq 0.$$  

An integration of both sides of (9) leads to

$$w(t) + q \int_{t_0 + \tau}^{t} \sum_{i=1}^{n} y_i(s - \tau_{ii}) \, ds \leq w(t_0 + \tau)$$

where $\tau = \max_{1 \leq j \leq n} \tau_{jj}$. A consequence of (10) is that $w$ is bounded and $y_i \in L_1(t_0 + \tau, \infty)$ for $i = 1, 2, \ldots, n$. From the boundedness of $w$ one can conclude that of $y_i$ since $w(t) = \sum_{i=1}^{n} y_i(t)$ and $y_i(t) > 0$ eventually. It will now follow from (1) that $\dot{y}_i$ is bounded for $t \geq \tau$, and therefore $y_i$ is uniformly continuous on $[0, \infty)$. The uniform continuity of $y_i$ on $[0, \infty)$, the eventual positivity of $y_i$ and the integrability of $y_i$ on a half-line together with a lemma of Barbalat [1], will imply that $\lim_{t \to \infty} y_i(t) = 0$, $i = 1, 2, \ldots, n$ and this completes the proof.

THEOREM. Let $q_{ij} \in (-\infty, \infty)$, $\tau_{jj} \in (0, \infty)$, $i, j = 1, 2, \ldots, n$. If

$$q \tau_* > \frac{1}{e} \quad \text{where} \quad q = \min_{1 \leq i \leq n} \left( q_{ii} \sum_{j=1}^{n} |q_{ji}| \right), \quad \tau_* = \min_{1 \leq i \leq n} \tau_{ii}$$

then every solution of (1) oscillates.
Proof. Assume for the sake of a contradiction that (1) has a nonoscillatory solution (2). In view of Lemma 1 we can assume that the components of $y_i(t)$ are eventually positive for $i = 1, 2, \ldots, n$. We have directly from (1) that

$$\sum_{i=1}^{n} \frac{dy_i(t)}{dt} + \sum_{j=1}^{n} \sum_{i=1}^{n} q_{ij} y_j(t - \tau_{jj}) = 0$$

which satisfies

$$\sum_{i=1}^{n} \left[ \frac{dy_i(t)}{dt} \right] + \sum_{i=1}^{n} \left( q_{ii} - \sum_{j=1}^{n} |q_{ji}| \right) y_i(t - \tau_{ii}) \leq 0. \quad (12)$$

We have from (12) that $w(t) = \sum_{i=1}^{n} y_i(t)$ satisfies

$$\frac{dw(t)}{dt} + \sum_{i=1}^{n} \left[ q_{ii} - \sum_{j=1}^{n} |q_{ji}| \right] y_i(t - \tau_{ii}) \leq 0. \quad (13)$$

Integrating both sides of (13) over $(t, \infty)$ and using the fact

$$w(t) \to 0 \quad \text{as} \quad t \to \infty \quad \text{(since} \quad y_i(t) \to 0, \quad i = 1, 2, \ldots, n)$$

we derive that

$$-w(t) + q \int_{t}^{\infty} \sum_{i=1}^{n} y_i(s - \tau_{ii}) \leq 0 \quad (14)$$

and this leads to

$$w(t) \geq q \int_{t}^{\infty} \sum_{i=1}^{n} y_i(s - \tau_{ii}) ds. \quad (15)$$

It is found from (15) that

$$w(t) \geq q \int_{t-\tau_*}^{\infty} \sum_{i=1}^{n} y_i(s) ds, \quad \tau_* = \min_{1 \leq i \leq n} \tau_{ii} \quad (16)$$

or

$$w(t) \geq q \int_{t-\tau_*}^{\infty} w(s) ds. \quad (17)$$

Now we let

$$F(t) = \int_{t-\tau_*}^{\infty} w(s) ds \quad (18)$$
and derive from (17) and (18) that
\[
\frac{dF(t)}{dt} = -w(t - \tau_*) \\
\leq -qF(t - \tau_*); \quad t > 2\tau_*.
\]  
(19)

It follows from (19) that \(F\) is an eventually positive solution of
\[
\frac{dF(t)}{dt} + qF(t - \tau_*) \leq 0; \quad t > 2\tau_*.
\]  
(20)

But it is well known (from Ladas and Stavroulakis [3]) that when (11) holds, (20) cannot have an eventually positive solution and this contradiction completes the proof.

References

