REMARKS ON QUASI-HERMITE-FEJÉR INTERPOLATION

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1. Introduction. Let

(1) $$-1 < x_1 < x_2 < \dots < x_n < 1$$

be $n+2$ distinct points on the real line and let us denote the corresponding real numbers, which are at the moment arbitrary, by

(2) $$y_0, y_1, y_2, \dots, y_n, y_{n+1}$$

The problem of Hermite-Fejér interpolation is to construct the polynomials which take the values (2) at the abscissas (1) and have preassigned derivatives at these points. This idea has recently been exploited in a very interesting manner by P. Szasz [1] who has termed quasi-Hermite-Fejér interpolation to be that process wherein the derivatives are only prescribed at the points $x_1, x_2, \dots, x_n$ and the points $-1, +1$ are left out, while the values are prescribed at all the abscissas (1). The corresponding theorems give interesting analogues to the theorems of Fejér [2], Egerváry and Turán [4], and Grünwald [3]. The interpolatory formulas have been obtained by Egerváry and Turán in a special case from a different point of view.

In the present note we extend these results by observing that analogous theorems hold true if the derivatives are prescribed at all the points except only at either of the end-points. We prove the corresponding theorems of convergence as well.

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This brings a sort of asymmetry in the formulas but the results remain similar. The terms "strongly normal" and "normal" abscissas can be defined in an analogous fashion. In the special case we are able to take the abscissas as zeros of Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, where $1 \le \alpha \le 2$ and $0 \le \beta \le 1$. To bring out the similarity with the results of P. Szasz the case $\alpha = 1$, $\beta = 0$ is treated separately in § 3.

2. Step Parabolas. We then begin by constructing the corresponding step parabolas $q(x)$ as in Fejér [2], given by the $2n+2$ conditions

$$q(-1) = Y_0, \quad q(x_i) = Y_i, \quad i = 1, 2, \ldots, n, \quad q(1) = Y_{n+1}$$

$$q'(x_i) = 0, \quad i = 1, 2, \ldots, n+1 .$$

We put $x_{n+1} = 1$. By an elementary calculation we have

$$(3) \qquad q(x) = Y_0 \cdot \frac{(1-x)^2 \omega^2(x)}{4\omega^2(-1)} + Y_{n+1} \frac{(1+x)(1+c_{n+1}(x-1))}{2\omega^2(1)} \omega^2(x) + \sum_{\nu=1}^{n} Y_\nu \cdot \left(\frac{\omega_1(x)}{(x-x_\nu)\omega_1'(x_\nu)}\right)^2 \frac{1+x}{1+x_\nu} \cdot \{1 + c_\nu(x-x_\nu)\}$$

where

$$(4) \qquad \begin{aligned} \omega(x) &= (x-x_1) \ldots (x-x_n) \\ \omega_1(x) &= (x-1)\,\omega(x) \end{aligned}$$

and

$$(5) \qquad c_\nu = -\frac{1}{1+x} - \frac{\omega_1''(x_\nu)}{\omega_1'(x_\nu)} \qquad (\nu = 1, 2, \ldots, n)$$

$$(6) \qquad c_{n+1} = -\frac{1}{2} - \frac{2\omega'(1)}{\omega(1)} .$$

The corresponding polynomials $Q(x)$ with the properties

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$$(7)\qquad Q(-1) = Q(x_\nu) = Q(1) = 0, \quad (\nu = 1, 2, \ldots, n)$$

$$Q'(x_\nu) = y'_\nu \quad (\nu = 1, 2, \ldots, n+1)$$

are given by

$$(8)\qquad \begin{aligned} Q(x) = {} & \sum_{\nu=1}^{n} y'_\nu \cdot \frac{1-x^2}{1-x_\nu^2} \cdot (x-x_\nu) \left\{ \frac{\omega(x)}{(x-x_\nu)\,\omega(x_\nu)} \right\}^2 \frac{1-x}{1-x_\nu} \\ & + y'_{n+1} \cdot \frac{(1-x^2)\,\omega^2(x)}{-2\,\omega^2(1)} . \end{aligned}$$

Then the interpolatory polynomial $s(x)$ of degree $\leq 2n+2$ which takes the values (2) at the abscissas (1) and whose derivative at the abscissas $x_1, x_2, \ldots, x_n, 1$ takes the values $y'_1, y'_2, \ldots, y'_{n+1}$ is then given by

$$s(x) = q(x) + Q(x) ,$$

where $q(x)$ and $Q(x)$ are as defined above. The verification that these polynomials are the ones sought for is left out.

Taking $f(x) = 1$, we get the identity

$$(9)\qquad \begin{aligned} & \frac{(1-x)^2\,\omega^2(x)}{4\,\omega^2(-1)} + \frac{(1+x)(1+c_{n+1}(x-1))}{2\,\omega^2(1)}\,\omega^2(x) \\ & + \sum_{\nu=1}^{n} \left(\frac{\omega_1(x)}{(x-x_\nu)\,\omega'_1(x_\nu)} \right)^2 \cdot \frac{1+x}{1+x_\nu} \cdot (1+c_\nu(x-x_\nu)) = 1, \end{aligned}$$

from which we can get a partial fraction decomposition of $\dfrac{1}{(1-x^2)\,\omega^2(x)}$. Formula (9) is useful in the sequel.

3. We shall now prove the following theorem:

THEOREM 1. _If_ $f(x)$ _is continuous for_ $-1 \leq x \leq +1$,

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and if x_ν $(\nu = 1, 2, \ldots, n+1)$ denote the zeros of $P_n^{(1,0)}(x)$, then the generalized quasi-step parabola $S_n(x)$ of degree $2n+2$ coinciding with $f(x)$ at -1, x_ν $(\nu = 1, 2, \ldots, n+1)$ and with $S_n'(x_\nu) = y_\nu'$ $(\nu = 1, 2, \ldots, n+1)$, then $S_n(x)$ converges uniformly to $f(x)$ as $n \to \infty$ in the interval $[-1, 1]$ provided

$$|y_\nu'| \leq c \quad (\nu = 1, 2, \ldots, n+1) .$$

Remark. A similar theorem holds for zeros of $P_n^{(0,1)}(x)$ as abscissas. But the result and proof are analogous.

In the case of the x_ν being zeros of $P_n^{(1,0)}(x)$, the formula we get from (3), is

$$q(x) \equiv R_n(x) = \frac{(1-x)^2 \left(P_n^{(1,0)}(x)\right)^2}{4} f(-1)$$

$$+ \frac{(1+x)\left(1 - \frac{(n+1)^2}{2}(x-1)\right)\left(P_n^{(1,0)}(x)\right)^2}{2(n+1)^2} f(1)$$

$$+ \sum_{\nu=1}^{n} \frac{1-x^2}{1-x_\nu^2} \cdot \frac{1-x}{1-x_\nu} \cdot \ell_\nu^2(x) f(x_\nu) ,$$

where $$\ell_\nu(x) = \frac{P_n^{(1,0)}(x)}{(x-x_\nu) P_n'^{(1,0)}(x_\nu)} .$$

We shall base the proof of this theorem on two lemmas.

LEMMA 1. $f(x) - R_n(x)$ tends to 0 uniformly in $-1 \leq x \leq 1$, as $n \to \infty$.

From the corresponding form of the identity (9), we have on simplifying the first two terms,

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$$\frac{2-n(n+2)(x-1)}{2(n+1)^2}\left(P_n^{(1,0)}(x)\right)^2$$

(10)

$$+\sum_{\nu=1}^{n}\frac{1-x^2}{1-x_\nu^2}\,\frac{1-x}{1-x_\nu}\,\ell_\nu^2(x)=1\,.$$

Now

$$f(x)-R_n(x)=\frac{(1-x)^2\left(P_n^{(1,0)}(x)\right)^2}{4}\{f(x)-f(-1)\}$$

$$+\frac{(1+x)\left\{1-\frac{(n+1)^2}{2}(x-1)\right\}\left(P_n^{(1,0)}(x)\right)^2}{2(n+1)^2}\{f(x)-f(1)\}$$

$$+\sum_{\nu=1}^{n}\frac{1-x^2}{1-x_\nu^2}\cdot\frac{1-x}{1-x_\nu}\cdot\ell_\nu^2(x)\,\{f(x)-f(x_\nu)\}$$

$$=I_1+I_2+I_3\,.$$

Now $I_1 \to 0$ uniformly in $-1 \le x \le 1$ since it vanishes at $x=-1, +1$ and is continuous while $P_n^{(1,0)}(x)(1-x)=P_n(x)-P_{n+1}(x)\to 0$ uniformly in $-1+\delta \le x \le 1-\delta$ for any $\delta>0$, since $P_n(x)$, the Legendre polynomial of degree n, is bounded in $[-1,1]$ and tends to 0 uniformly in $[-1+\delta, 1-\delta]$. By the same reasoning $I_2 \to 0$ uniformly in $[-1,1]$.

We now examine I_3. We have then

$$|I_3| \le \sum_{|x_\nu - x|\le\delta}|\,\Sigma\,| + \sum_{|x_\nu - x|>\delta}|\,\Sigma\,| = \Sigma_1+\Sigma_2\,.$$

Since $f(x)$ is continuous, for any $\varepsilon>0$, there exists a δ such that $|f(x)-f(y)|<\varepsilon$ for $|x-y|\le\delta$.

Then

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$$\Sigma_1 \le \sum_{|x_\nu - x| \le \delta} \frac{1-x^2}{1-x_\nu^2} \cdot \frac{1-x}{1-x_\nu} \cdot \ell_\nu^2(x) \le \varepsilon .$$

This follows from the identity (10).

Let $|f(x)| \le M$ for $-1 \le x \le 1$. Then

$$\Sigma_2 \le \frac{2M}{\delta^2} (1-x^2) \left(P_n^{(1,0)}(x)\right)^2 (1-x) . \sum_{\nu=1}^{n} \frac{1}{1-x^2} \cdot \frac{1}{1-x_\nu} \cdot \frac{1}{\left(P_n^{'(1,0)}(x_\nu)\right)^2}$$

$$\le \frac{2M}{\delta^2} (1-x^2) \left(P_n^{(1,0)}(x)\right)^2 (1-x) .$$

This follows from the identity

$$(11) \qquad \sum_{\nu=1}^{n} \frac{1}{(1-x_\nu^2)(1-x_\nu)\left(P_n^{'(1,0)}(x_\nu)\right)^2} = \frac{n(n+2)}{2(n+1)^2}$$

which follows from comparing the coefficients of the highest power of x in (10). Now $P_n^{(1,0)}(x) = \dfrac{P_n(x) - P_{n+1}(x)}{1-x}$ and using Stieltjes's inequality

$$|P_n(x)| < \frac{c}{\sqrt{n}} \cdot \frac{1}{\sqrt[4]{1-x^2}} , \quad -1 < x < 1$$

where c is a numerical constant, we have

$$\Sigma_2 \le \frac{2M}{\delta^2} \cdot \frac{c_1^2}{n} \cdot \sqrt{\frac{1+x}{1-x}}$$

$$\le \frac{2M}{\delta^2} \frac{c_2}{\sqrt{\varepsilon_1}} \cdot \frac{1}{n} \quad \text{for } -1+\varepsilon_1 \le x \le 1-\varepsilon_1 .$$

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For n sufficiently large this can be made less than ε. From the fact that $(1-x^2)(1-x)\left|P_n^{(1,0)}(x)\right|^2$ is continuous and bounded in $[-1,1]$ and vanishes at $x=+1$, we can choose ε_1 such that for $1-\varepsilon_1 \leq x \leq 1$, $\left|\frac{2M}{\delta^2}(1-x^2)(1-x)\left(P_n^{(1,0)}(x)\right)^2\right| < \varepsilon$.

Thus we have $\Sigma_2 < 2\varepsilon$. This establishes uniform convergence to zero in $-1 \leq x \leq 1$ of $f(x) - P_n(x)$.

LEMMA 2.

$$Q(x) = \sum_{\nu=1}^{n} y'_\nu \frac{1-x^2}{1-x_\nu^2} \cdot \frac{1-x}{1-x_\nu} \cdot (x-x_\nu)\, \ell_\nu^2(x)$$

$$+ y'_{n+1} \frac{(1-x^2)\left(P_n^{(1,0)}(x)\right)^2}{-2(n+1)^2}$$

<u>tends to zero uniformly in</u> $-1 \leq x \leq 1$ if $|y'_\nu| \leq \Delta$.

We have

$$\left| Q(x) - y'_{n+1} \frac{(1-x^2)\left(P_n^{(1,0)}(x)\right)^2}{-2(n+1)^2} \right| \leq J_1 + J_2$$

where $|J_1| \leq \Delta \cdot \sum_{|x_\nu - x| \leq \varepsilon}$

$$\leq \Delta\varepsilon \cdot \sum_{\nu=1}^{n} \frac{1-x^2}{1-x_\nu^2} \cdot \frac{1-x}{1-x_\nu}\, \ell_\nu^2(x)$$

$$\leq \Delta\varepsilon \text{ from the identity (10)}$$

and

$$|J_2| \le \sum_{|x_\nu - x| > \varepsilon}$$

$$\le \frac{1}{\varepsilon} \cdot (1-x^2)(1-x) \left| P_n^{(1,0)}(x) \right|^2$$

$\le \varepsilon_1$ by repeating the argument at the end of Lemma 1, for $-1 \le x \le 1$. Hence the Lemma.

The last term $y'_{n+1} \dfrac{(1-x^2)\left|P_n^{(1,0)}(x)\right|^2}{2(n+1)^2}$ is taken care of separately as it vanishes at $x = \pm 1$ and is continuous and bounded in $-1 < x < 1$ and tends to zero uniformly in $-1 + \delta \le x \le 1 - \delta$.

By combining the results of Lemma 1 and 2 we have the proof of Theorem 1.

4. A-Quasi-Normal Point Systems. We shall say that a point system

$$-1 < x_1 < x_2 < \ldots < x_n < 1$$

is an a-quasi-normal system if the inequalities

$$1 + c_\nu (x - x_\nu) > 0$$

hold for $-1 < x < 1$ $(\nu = 1, 2, \ldots, n+1)$ where c_ν are given by (5) and (6). In other words the points

$$x_\nu - \frac{1}{c_\nu} \qquad (\nu = 1, 2, \ldots, n+1)$$

do not lie in the open interval $(-1, +1)$.

For Jacobi polynomials which satisfy the differential equation

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$$(1-x^2)\,\omega''(x) + [\beta - \alpha - (\alpha+\beta+2)x]\,\omega'(x) + n(n+\alpha+\beta+1)\,\omega(x) = 0$$

we get

$$c_\nu = \frac{(\beta-\alpha+1)-(\alpha+\beta-1)x_\nu}{1-x_\nu^2}$$

$$\alpha > -1, \quad \beta > -1 .$$

Now $[1 + c_\nu(x-x_\nu)]_{x=1} = 1 + \dfrac{(\beta-\alpha+1)-(\alpha+\beta-1)x_\nu}{1+x_\nu}$

$$\geq 2 - \alpha \text{ if } \beta \geq 0 .$$

Also $[1+c_\nu(x-x_\nu)]_{x=-1} = 2 - \alpha - \beta - \dfrac{2-2\alpha}{1-x_\nu}$,

$$\geq 2 - \beta - 1 \text{ if } \alpha \geq 1 ,$$

$$= 1 - \beta .$$

We thus require $0 \leq \beta \leq 1$ and $1 \leq \alpha \leq 2$.

Also, $[1 + c_{n+1}(x-1)]_{x=1} = 1$, and

$$[1 + c_{n+1}(x-1)]_{x=-1} = 1 - 2c_{n+1} = 2 + \frac{n(n+\alpha+\beta+1)}{\alpha+1} > 2 .$$

In particular for $\beta = 0$, $\alpha = 1$, $c_\nu = 0$ and $c_{n+1} = \dfrac{(n+1)^2}{2}$.

As a simple consequence of identity (9) and the notion of a-quasi-normality, we can state the following theorem:

THEOREM 2. <u>If the roots</u> $x_1, x_2, \ldots, x_n$ <u>of</u> $\omega(x)$ <u>and</u> 1 <u>form an a-quasi-normal point system over</u> $-1 < x < 1$, <u>then</u>

$$(1-x)\,|\omega(x)| < 2\,|\omega(-1)| \quad \text{for } -1 < x < 1 .$$

The proof follows immediately on writing (9) as follows:

$$\frac{(1-x)^2\,\omega^2(x)}{4\,\omega^2(-1)} = 1 - \frac{(1+x)[1 + c_{n+1}(x-1)]}{2\,\omega^2(1)}\;\omega^2(x)$$

$$- \sum_{\nu=1}^{n} \frac{\omega_1^2(x)}{(x-x_\nu)^2\,\omega_1'^2(x_\nu)} \cdot \frac{1+x}{1+x_\nu}\,\{1+c_\nu(x-x_\nu)\}\,.$$

By the condition of a-quasi-normality we have

$$1 + c_{n+1}(x-1) > 0 \quad \text{and} \quad 1 + c_\nu(x-x_\nu) > 0$$

and so the left side is < 1 whence the result. Thus in particular we have for $P_n^{(\alpha,\beta)}(x)$, $0 \le \beta \le 1$ and $1 \le \alpha \le 2$, the inequality

$$(1-x)\,|P_n^{(\alpha,\beta)}(x)| \le 2\,|P_n^{(\alpha,\beta)}(-1)|\,.$$

Remark. We may say that a point-system $x_1, x_2, \ldots, x_n, 1$ is strongly a-quasi-normal if

$$\text{(A)} \qquad 1 + c_\nu(x-x_\nu) \ge \rho > 0 \quad \text{and} \quad 1 + c_{n+1}(x-1) \ge \rho > 0\,,$$

$$(\nu = 1, 2, \ldots, n)\,,$$

where c_ν, c_{n+1} are given by (5), (6). We see that the zeros of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ (with $0 \le \beta < 1$ and $1 \le \alpha < 2$) form a strongly a-quasi-normal point system.

It is possible to formulate a theorem analogous to Theorem 5 of Szasz [1], but we shall not do so here.

The identity (10), after the substitution $x = \cos \frac{z}{n}$ and on using the Mehler-Heine type relation [Szegő [5] p. 165]

$$\lim_{n\to\infty} n^{-\alpha} P_n^{(\alpha,\beta)}\left(\cos\frac{z}{n}\right) = \left(\frac{z}{2}\right)^{-\alpha} J_\alpha(z) \text{ uniformly for } |z| \le R,$$

R fixed, leads to the following interesting identity where ν_k is a zero of $J_1(z)$ (which can be otherwise proved by use of Mittag-Leffler's theorem applied to $J_1^2(z) - \frac{4}{z^2}$):

$$J_1^2(z) + \frac{4}{z^2} J_1^2(z) + 4 \sum_{k=1}^{\infty} \frac{z^2 J_1^2(z)}{(z^2-\gamma_k^2)^2 J_1'^2(\gamma_k)} = 1 .$$

[See [8] p. 105, formula (65) which gives the above identity on observing that $J_1'(\gamma_\mu) = J_0(\gamma_\mu)$ since

$$zJ_1'(z) + J_1(z) = zJ_0(z) \quad ([8] \text{ p. } 11).]$$

Formula (9) can yield other such identities too.

5. Egerváry and Turán [6] have defined an interpolation process to be stable for the interval $[0,\infty]$ and weight function e^{-x}, if for $0 \le x_1 < x_2 < \ldots < x_n$ and polynomials $r_\nu(x)$, $(\nu = 1,2,\ldots,n)$ we have

$$0 \le e^{-x}\left[\sum_{\nu=1}^{n} y_\nu r_\nu(x) - \sum_{\nu=1}^{n} y_\nu^* r_\nu(x)\right] \le \max_\nu |y_\nu - y_\nu^*| e^{-x_\nu}$$

for all $x \ge 0$, where

$$(12) \qquad r_\nu(x_j) = \begin{cases} 1, & j = \nu \\ 0, & j \ne \nu . \end{cases}$$

If we are not concerned about an interpolation process being "most economical" then we can still get a stable interpolation process in the following manner. As Egerváry and Turán have shown, for the above condition to be satisfied we must have for the $r_\nu(x)$ the following further conditions:

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$$(13)\qquad r_\nu'(x_j) = 0 \quad j \neq \nu, \quad r_\nu'(x_\nu) = r_\nu(x_\nu) = 1 .$$

We then find a polynomial satisfying (12) and (13) and we take the x_ν' s to be the zeros of Laguerre polynomials $L_n^{(\alpha)}(x)$, $\alpha > -1$. We also suppose that the value of the interpolatory polynomial $R_n(x)$ is prescribed at $x = 0$. Then it is easy to see that

$$(14)\qquad R_n(x,f) = R_n(x) = y_1\left(\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)}\right)^2 + \sum_{\nu=1}^{n} y_\nu \frac{x[x_\nu + \alpha(x-x_\nu)]}{x_\nu^2}\ell_\nu^2(x)$$

We have

$$U(x) = e^x - \left(\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)}\right)^2 - \sum_{\nu=1}^{n} e^{x_\nu}\cdot \frac{x[x_\nu + \alpha(x-x_\nu)]}{x_\nu^2}\ell_\nu^2(x) \geq 0$$

for $x \geq 0$.

For $U(0) = 0$, $U(x_\nu) = 0$, $U'(x_\nu) = 0$ $(\nu = 1,2,\ldots,n)$, where

$$\ell_\nu(x) = \frac{L_n^{(\alpha)}(x)}{(x-x_\nu)L_n'^{(\alpha)}(x_\nu)},$$

and so $U(x)$ has $2n+1$ zeros. If $U(x)$ vanishes for one more value of $x \geq 0$, then by Rolle's theorem for some $\xi \geq 0$, $U^{(2n+1)}(\xi) = 0$. But $U^{(2n+1)}(x) = e^x$. Hence $U(x) \geq 0$ for $x \geq 0$.

We need in fact the weaker assertion that for $0 \leq x < \infty$ we have

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$$e^{x} - \left(\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)} \right)^2 \geq \sum_{\nu=1}^{n} \frac{x[x_\nu + \alpha(x-x_\nu)]}{x_\nu^2} \ell_\nu^2(x) .$$

We shall now prove the following

THEOREM 3. If $f(x)$ is continuous and bounded for $0 \leq x < \infty$, and ω is an arbitrarily large positive number, then the sequence $R_n(x, f)$ in (14) converges to $f(x)$ in $(0, \infty)$ and uniformly in $[0, \omega]$ where $0 \leq \alpha < 1$.

For $\alpha = 0$ we get the case treated by Balázs and Turán [7].

We shall need the known inequality for modulus of continuity, viz.

$$\Lambda_{2\omega}(\lambda\delta) \leq (\lambda+1)\, \Lambda_{2\omega}(\delta)$$

where $\lambda\delta \leq 2\omega$ and $\Lambda_{2\omega}$ is the modulus of continuity of $f(x)$ with respect to the interval $[0, 2\omega]$.

We shall base the proof on the following lemmas:

LEMMA 4. For $0 \leq x \leq \omega$ and for $n = 2, 3, 4, \ldots$ we have

$$\left| \sum_{\nu=0}^{n} r_\nu(x) - 1 \right| < c_1 n^{-1/4}$$

where

$$r_\nu(x) = \frac{x(x_\nu + \alpha(x-x_\nu))}{x_\nu^2} \ell_\nu^2(x), \quad \nu = 1, 2, \ldots, n$$

$$r_0(x) = \left(\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)} \right)^2$$

and c_1 is a numerical constant depending only on ω .

Proof. We have, from the Hermite-Fejér formula of interpolation, the identity

$$(15) \qquad 1 = \sum_{\nu=1}^{n} \frac{x_\nu(x_\nu - \alpha) + x(\alpha+1-x_\nu)}{x_\nu} \ell_\nu^2(x) .$$

Also putting $x = 0$ in the above and multiplying both sides suitably we have

$$(16) \qquad \left(\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)}\right)^2 = \sum_{\nu=1}^{n} \frac{x_\nu - \alpha}{x_\nu^2} \left(\frac{L_n^{(\alpha)}(x)}{L_n'^{(\alpha)}(x_\nu)}\right)^2 .$$

Then we have, using first (15) and then (16), the following:

$$\sum_{\nu=0}^{n} r_\nu(x) - 1$$

$$= \sum_{\nu=1}^{n} \left\{\frac{x}{x_\nu} + \frac{\alpha x(x-x_\nu)}{x_\nu^2}\right\} \ell_\nu^2(x) + \left(\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)}\right)^2$$

$$- \sum_{\nu=1}^{n} \left\{\frac{x}{x_\nu} + \frac{\alpha(x-x_\nu)}{x_\nu} - (x-x_\nu)\right\} \ell_\nu^2(x)$$

$$= \sum_{\nu=1}^{n} \left\{\frac{1}{x_\nu} + \frac{1}{x-x_\nu}\right\} \left(\frac{L_n^{(\alpha)}(x)}{L_n'^{(\alpha)}(x_\nu)}\right)^2$$

$$= \sum_{\nu=1}^{n} \frac{x}{x_\nu(x-x_\nu)} \left(\frac{L_n^{(\alpha)}(x)}{L_n'^{(\alpha)}(x_\nu)}\right)^2 .$$

Then

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$$e^{-x}\left|\sum_{\nu=0}^{n} r_\nu(x) - 1\right| \le \sum_{\nu=1}^{n} \frac{x}{x_\nu |x-x_\nu|}\left(\frac{L_n^{(\alpha)}(x)}{L_n'^{(\alpha)}(x_\nu)}\right)^2 \le S_1 + S_2$$

where

$$S_1 = e^{-x} \sum_{|x-x_\nu| \le n^{-1/4}} \frac{x}{x_\nu |x-x_\nu|} \left(\frac{L_n^{(\alpha)}(x)}{L_n'^{(\alpha)}(x_\nu)}\right)^2$$

$$\le e^{-x} \cdot n^{-1/4} \sum_{\nu=1}^{n} \frac{x}{x_\nu} \ell_\nu^2(x)$$

$$\le \frac{e^{-x} \cdot n^{-1/4}}{1-\alpha} \sum_{\nu=1}^{n} \frac{x \cdot x_\nu (1-\alpha)}{x_\nu^2} \ell_\nu^2(x) \quad \text{since} \quad 0 \le \alpha < 1$$

$$\le \frac{e^{-x} \cdot n^{-1/4}}{1-\alpha} \sum_{\nu=1}^{n} \frac{x\{x_\nu + \alpha(x-x_\nu)\}}{x_\nu^2} \ell_\nu^2(x)$$

$$\le \frac{n^{-1/4}}{1-\alpha}\left[1 - e^{-x}\left(\frac{L_n^{(\alpha)}(x)}{L_n'^{(\alpha)}(0)}\right)^2\right]$$

$$\le \frac{n^{-1/4}}{1-\alpha} \quad ,$$

and

$$S_2 = e^{-x} \cdot x \left(L_n^{(\alpha)}(x)\right)^2 \sum_{|x-x_\nu| > n^{-1/4}} \frac{1}{x_\nu |x-x_\nu|} \cdot \frac{1}{\left(L_n'^{(\alpha)}(x_\nu)\right)^2}$$

$$\le n^{1/4} e^{-x} \cdot x \left(L_n^{(\alpha)}(x)\right)^2 \sum_{\nu=1}^{n} \frac{1}{x_\nu \left(L_n'^{(\alpha)}(x_\nu)\right)^2} \ .$$

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Since from Szegö [5] (p. 176 formula (7.6.12)) we know that

$$(17) \quad n^{-\alpha/2+1/4} \max_{0\le x\le \omega} e^{-x/2} x^{\alpha/2+1/4} \left| L_n^{(\alpha)}(x) \right| \le c \text{ for } \alpha \ge -\frac{1}{2}$$

and since again we know ([5] p. 342 formula (14.7))

$$(18) \quad \sum_{\nu=1}^{n} \frac{1}{x_\nu} \cdot \frac{1}{\left| L_n'^{(\alpha)}(x_\nu) \right|^2} = \frac{\Gamma(n+1)\Gamma(\alpha+1)}{\Gamma(n+\alpha+1)} \sim n^{-\alpha}$$

we have

$$S_2 < cn^{-1/4} .$$

The Lemma is then proved.

LEMMA 5. <u>For</u> $0 \le x \le \omega$, <u>we have</u>

$$\sum_{\nu=1}^{n} |x-x_\nu| \, r_\nu(x) < c_2 n^{-1/4} .$$

We have

$$e^{-x} \sum_{\nu=1}^{n} |x-x_\nu| \, r_\nu(x)$$

$$= e^{-x} \cdot x \left(\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)}(0)} \right)^2 + e^{-x} \sum_{\nu=1}^{n} \frac{x[x_\nu + \alpha(x-x_\nu)] \left(L_n^{(\alpha)}(x) \right)^2}{x_\nu^2 \left(L_n'^{(\alpha)}(x_\nu) \right)^2 |x-x_\nu|} .$$

Observe that with $0 \le \alpha < 1$, $x_\nu + \alpha(x-x_\nu) > 0$ for $x \ge 0$. The first term in the sum on the right is taken care of by the inequality (17) if $\alpha \ge -\frac{1}{2}$ while the other term involving summation on the right can be broken into two parts - one with $|x-x_\nu| \le n^{-1/4}$, and the other with $|x-x_\nu| > n^{-1/4}$. The

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first part of this decomposition has already been treated in Lemma 1. The second part can be written as

$$e^{-x} \cdot x\left(L_n^{(\alpha)}(x)\right)^2 \left[\sum_{|x-x_\nu|>n^{-1/4}} x_\nu^{-1} |x-x_\nu|^{-1} \left(L_n'^{(\alpha)}(x_\nu)\right)^{-2} \right.$$

$$\left. + \alpha \sum_{|x-x_\nu|>n^{-1/4}} x_\nu^{-2} \left(L_n'^{(\alpha)}(x_\nu)\right)^{-2} \right]$$

$$\leq e^{-x} \cdot x\left(L_n^{(\alpha)}(x)\right)^2 \left[n^{1/4} \sum_{\nu=1}^{n} x_\nu^{-1}\left(L_n'^{(\alpha)}(x_\nu)\right)^{-2} \right.$$

$$\left. + \alpha \sum_{\nu=1}^{n} x_\nu^{-2}\left(L_n'^{(\alpha)}(x_\nu)\right)^{-2} \right]$$

From (16), we have

$$\left(L_n^{(\alpha)}(0)\right)^{-2} = \sum_{\nu=1}^{n} \left(x_\nu^{-1} - \alpha x_\nu^{-2}\right)\left(L_n'^{(\alpha)}(x_\nu)\right)^{-2} .$$

Combining this identity with (18) we get for $0 \leq \alpha$

$$(19) \quad \alpha \sum_{\nu=1}^{n} x_\nu^{-2}\left(L_n'^{(\alpha)}(x_\nu)\right)^{-2} = \left(L_n^{(\alpha)}(0)\right)^{-1} - \left(L_n^{(\alpha)}(0)\right)^{-2} \sim n^{-\alpha} .$$

This estimate helps us to get the lemma.

LEMMA 6. We have

$$\sum_{x_\nu > 2\omega} r_\nu(x) \leq cn^{-1/2}, \qquad 0 \leq x \leq \omega .$$

The proof follows as above, on observing that for $0 \leq x \leq \omega$ and $x_\nu > 2\omega$, we have $|x-x_\nu| > \omega$, so that

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$$e^{-x} \sum_{x_\nu > 2\omega} r_\nu(x) < \frac{\alpha}{\omega}\left[\sum_{\nu=1}^{n} \left[x_\nu L_n'^{(\alpha)}(x_\nu)\right]^{-2}\right.$$

$$\left. + \omega^{-2} \sum_{\nu=1}^{n} x_\nu^{-1} \left(L_n'^{(\alpha)}(x_\nu)\right)^{-2}\right] e^{-x} x \left(L_n^{(\alpha)}(x)\right)^2 .$$

Using the estimates (17), (18) and (10) we get the required result.

Proof of the Theorem. Since $f(x)$ is continuous and bounded, we have

$$|R_n(x) - f(x)| \leq \sum_{\nu=0}^{n} |f(x_\nu) - f(x)| \, r_\nu(x)$$

$$+ |f(x)| \cdot \left|1 - \sum_{\nu=0}^{n} r_\nu(x)\right|$$

$$\leq \sum_{|x_\nu| \leq 2\omega} |f(x_\nu) - f(x)| \, r_\nu(x) + 2M \sum_{x_\nu > 2\omega} r_\nu(x) + M \left|1 - \sum_{\nu=0}^{n} r_\nu(x)\right| .$$

Since

$$|f(x) - f(x_\nu)| \leq \Lambda_{2\omega}(|x - x_\nu|) \leq \Lambda_{2\omega}(n^{-1/4})\{n^{1/4}|x - x_\nu| + 1\},$$

the rest of the proof follows on using the lemmas.

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