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Minimal Dynamical Systems on Connected Odd Dimensional Spaces

Huaxin Lin

Abstract. Let $\beta: S^{2n+1} \to S^{2n+1}$ be a minimal homeomorphism ($n \geq 1$). We show that the crossed product $C(S^{2n+1}) \rtimes_\beta \mathbb{Z}$ has rational tracial rank at most one. Let Ω be a connected, compact, metric space with finite covering dimension and with $H^1(\Omega, \mathbb{Z}) = \{0\}$. Suppose that $K_i(C(\Omega)) = \mathbb{Z} \oplus G_i$, where G_i is a finite abelian group, $i = 0, 1$. Let $\beta: \Omega \to \Omega$ be a minimal homeomorphism. We also show that $A = C(\Omega) \rtimes_\beta \mathbb{Z}$ has rational tracial rank at most one and is classifiable. In particular, this applies to the minimal dynamical systems on odd dimensional real projective spaces. This is done by studying minimal homeomorphisms on $X \times \Omega$, where X is the Cantor set.

1 Introduction

Let Ω be a compact metric space and let $\alpha: \Omega \to \Omega$ be a minimal homeomorphism. We study the resulting C^*-algebra $C(\Omega) \rtimes_\alpha \mathbb{Z}$, the crossed product C^*-algebra. There are interesting connections between minimal dynamical systems and the study of C^*-algebras. A classical result of Giordano, Putnam, and Skau [7] showed that two Cantor minimal systems are strongly orbit equivalent if and only if the associated crossed product C^*-algebras are isomorphic. The C^*-algebra theoretic aspect of their result is indebted to the fact that the crossed product C^*-algebras are unital simple A$\mathbb{T}$-algebras of real rank zero and belong to the classifiable C^*-algebras; *i.e.*, they are classified up to isomorphisms by their Elliott invariant, namely, in this case, by their ordered K-theory. In turn, the Cantor minimal systems are classified up to strong orbit equivalence by their ordered K-theory. C^*-algebras of the form $C(\Omega) \rtimes_\alpha \mathbb{Z}$ are always simple when α is minimal. These C^*-algebras provide a rich source of unital separable amenable simple C^*-algebras that satisfy the so-called Universal Coefficient Theorem. On the other hand, the rapidly developed Elliott program, otherwise known as the program of classification of amenable C^*-algebras by K-theoretical invariant, provides a possible way to characterize minimal dynamical systems by their K-theoretical invariant. It is therefore important to known when $C(\Omega) \rtimes_\alpha \mathbb{Z}$ belongs to the classifiable class of amenable simple C^*-algebras (in the sense of the Elliott program). Elliott and Evans ([4]) showed that all irrational rotation algebras that are crossed product C^*-algebras from minimal dynamical systems on the circle are classifiable; in fact, they are unital $A\mathbb{T}$-algebras of real rank zero. Let $A = C(\Omega) \rtimes_\alpha \mathbb{Z}$.

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In [19], it was shown that, if X has finite covering dimension and $\rho_A(K_0(A))$, the tracial image of $K_0(A)$, is dense in $\mathrm{Aff}(T(A))$, the space of real continuous affine functions on the tracial state space, then A has tracial rank zero (the converse also holds). Consequently A is classifiable. In the case where Ω is connected and (Ω, α) is unique ergodic, this result states, for example, that if (Ω, α) has an irrational rotation number, then A is classifiable. In particular, this recovers the previously mentioned case of irrational rotations on the circle. With the development in the Elliott program ([11, 16, 29]), the classifiable C^*-algebras now include the class of C^*-algebras which are rationally of tracial rank at most one. The result of [19] were further pushed by Toms and Winter to the great generality: if projections in A separate the tracial states, then A has rational tracial rank zero. Moreover these crossed products are also classifiable by the Elliott invariant (see [27]).

On the other hand, during these developments, minimal dynamical systems on $X \times \mathbb{T}$, where X is the Cantor set and $\mathbb{T}$ is the circle, have been studied (see [14, 15]). More general cases were also studied in [25]. In both [19] and [27], Putnam algebra A_x was used as the main bridge. One may view that A_x is a large C^*-subalgebra of $A = C(\Omega) \rtimes_\alpha \mathbb{Z}$ by "taking away " one point. In [14, 15], a smaller C^*-subalgebra is used. That C^*-subalgebra may be viewed as a C^*-subalgebra by "taking away" a circle (an idea of Hiroki Matui). This method seemed to be too specialized to be useful in general case. However, it has recently been adopted by K. Strung [24] to obtain very interesting results about crossed products of certain minimal systems on the odd spheres. Strung showed that by studying the minimal systems of product type $X \times S^{2n+1}$, one can provide examples of non-unique ergodic minimal dynamical systems on the odd spheres whose crossed product C^*-algebras are classifiable. Let A be the crossed product from the minimal system on the odd sphere. It should be noted these crossed products A may not have rational tracial rank zero. In particular, their projections may not separate the tracial states. Nevertheless, A has rational tracial rank at most one; *i.e.*, $A \otimes U$ has tracial rank at most one for any UHF-algebra U of infinite type. Therefore, a more general classification result from [10] can be applied.

In [24], the minimal homeomorphisms on the odd spheres are assumed to be limits of periodic homeomorphisms constructed by "fast approximation-conjugation". In this note we will study the general minimal dynamical systems on the odd spheres as well as on the odd dimensional real projective spaces. We show that crossed product C^*-algebras from any minimal dynamical systems on odd spheres or odd dimensional real projective spaces have rational tracial rank at most one and are classifiable. It should be mentioned that there are no minimal homeomorphisms on even spheres or on the even dimensional real projective spaces because of the existence of fixed points. We actually prove much more general results (see Theorems 6.1 and 6.2). The methods we used here to study the minimal dynamical systems on the product spaces of the form $X \times \Omega$, where X is the Cantor set and Ω is a connected space, are those developed in [14, 15]. We also use a recent uniqueness theorem (see Theorem 4.2) from [12]. We continue to use the argument of Strung as well as an embedding result of Winter [30]. The classification result in [10] is also applied.

This paper is organized as follows. The next section includes some preliminary concepts. In Section 3, we study general minimal dynamical systems on the product spaces $X \times \Omega$, where X is the Cantor set and Ω is a connected compact space. Examples of minimal dynamical systems studied in Section 3 are presented in Section 4. Applications are presented in Section 5.

2 Preliminaries

Definition 2.1 Let A be a unital C^*-algebra. Denote by $U(A)$ the unitary group of A and $U_0(A)$ the normal subgroup of connected component of $U(A)$ containing the identity. Denote by $CU(A)$ the closure of the commutator subgroup of $U_0(A)$. Denote by $T(A)$ the tracial state space of A. We also use $T(A)$ for traces of the form $\tau \otimes \mathrm{Tr}$ on $M_n(A)$ for all integers n, where Tr is the standard (un-normalized) trace on M_n. Denote by $\rho_A: K_0(A) \to \mathrm{Aff}(T(A))$ the homomorphism defined by $\rho_A([p]) = \tau(p)$ for all projections in $M_n(A)$, $n = 1, 2, \ldots$.

Definition 2.2 Let $A = M_n$ and $a \in A$. We use $\det(a)$ for the usual determinant. If $A = C(X) \otimes M_n$, we will often identify A with $C(X, M_n)$. If $f \in A$, we use $\det(f)$ for the function $\det(f)(x)$ in $C(X)$.

Let A be a unital C^*-algebra with $T(A) \neq \varnothing$ and let $u \in U_0(A)$. Let

$$\{u(t) : t \in [0,1]\} \subset U_0(A)$$

be a piecewise smooth continuous path with $u(0) = u$ and $u(1) = 1_A$. Define

$$\widetilde{\Delta}_\tau(u(t)) = \Big(\frac{1}{2\pi i}\Big) \int_0^1 \tau\Big(\frac{du(t)}{dt} u(t)^*\Big)\, dt \quad \text{for all} \quad \tau \in T(A).$$

If $u \in CU(A)$, then $\widetilde{\Delta}_\tau(u(t)) \in \overline{\rho_A(K_0(A))}$. This de La Harp–Skandalis determinant (which is independent of the choice of the path) gives an injective homomorphism

$$\Delta: U_0(A)/CU(A) \to \mathrm{Aff}(T(A))/\overline{\rho_A(K_0(A))}$$

(see [26]).

If $u \in U(A)$, we will use $\overline{u}$ for its image in $U(A)/CU(A)$.

Definition 2.3 Let Ω be a compact metric space. Denote by $\mathrm{Homeo}(\Omega)$ the set of all homeomorphisms on Ω equipped with the topology of point-wise convergence. Let $\beta \in \mathrm{Homeo}(\Omega)$. Denote by $\widetilde{\beta}: C(\Omega) \to C(\Omega)$ the automorphism defined by $\widetilde{\beta}(f) = f \circ \beta^{-1}$ for all $f \in C(\Omega)$. If $F \subset \Omega$, denote by χ_F the characteristic function of F. When F is a clopen set, $\chi_F \in C(\Omega)$.

Lemma 2.4 ([14, Lemma 2.1]) *Let X be the Cantor set and let Ω be a connected compact metric space. Let $\beta \in \mathrm{Homeo}(X \times \Omega)$. Then there is $\gamma \in \mathrm{Homeo}(X)$ and a continuous map $\phi: X \to \mathrm{Homeo}(\Omega)$ such that $\beta(x, \xi) = (\gamma(x), \phi_x(\xi))$ for all $(x, \xi) \in X \times \Omega$.*

Proof Let $p_X: X \times \Omega \to X$ and $p_\Omega: X \times \Omega \to \Omega$ be projection maps such that $p_X(x, \xi) = x$ and $p_\Omega(z, \xi) = \xi$ for all $(x, \xi) \in X \times \Omega$. Fix $(x, \xi) \in X \times \Omega$. Then $\{x\} \times \Omega$ is the connected component of $X \times \Omega$ containing (x, ξ). The homeomorphism β maps it into a connected component containing $\beta(x, \xi) = (x_1, \xi_1)$, $x_1 = p_X(\beta(x, \xi))$ and

$\xi_1 = p_\Omega(\beta(x,\xi))$. Therefore, as just mentioned, the component is $\{x_1\}\times\Omega$. Define $\gamma(x) = p_X(\beta(x,\xi))$ for $x \in X$. Since $p_X(\beta(x,\xi)) = p_X(\beta(x,\xi'))$, it is a well defined map. Since β is a homeomorphism, we can also see that $\gamma \in \mathrm{Homeo}(X)$. Fix $x \in X$, then map $\phi_x(\xi) = p_\Omega(\beta(x,\xi))$ is a homeomorphism from Ω onto Ω. It is easy to check that $\phi\colon X \to \mathrm{Homeo}(\Omega)$ is continuous. ∎

2.1 Notation

Let X be the Cantor set and let Ω be a connected compact metric space. Then

$$K_0(C(X\times\Omega)) = C\big(X, K_0(C(\Omega))\big) \quad \text{and} \quad K_1\big(C(X\times\Omega)\big) = C\big(X, K_1(C(\Omega))\big),$$

where the group $K_i(C(\Omega))$ is viewed as a discrete space, $i = 0,1$. Moreover,

$$\begin{aligned} K_0(C(X\times\Omega))_+ &= \big\{ f \in C\big(X, K_0(C(\Omega))\big) : f(x) \in K_0(C(\Omega))_+ \big\} \\ &= C\big(X, K_0(C(\Omega))\big)_+ . \end{aligned}$$

Let (X,α) be a Cantor minimal system. Following [7], denote

$$K^0(X,\alpha) = C(X,\mathbb{Z})/\{ f - f\circ\alpha^{-1} : f \in C(X,\mathbb{Z})\}.$$

Define $\widetilde{\phi}_x\colon C(\Omega) \to C(\Omega)$ to be the isomorphism defined by $\widetilde{\phi}_x(f) = f\circ\phi_x$ for all $f \in C(\Omega)$. Denote by $(\widetilde{\phi}_x)_{*i}\colon K_i(C(\Omega)) \to K_i(C(\Omega))$ the induced isomorphism, $i = 0,1$.

Define $\widetilde{\phi}^{-1}_{*i}(f\circ\alpha^{-1})(x) = (\widetilde{\phi}_x)^{-1}_{*i}(f\circ\alpha^{-1}(x))$ for $x \in X$, $i = 0,1$, where $f \in C(X, K_i(C(\Omega)))$. Denote

$$\begin{aligned} &K^i\big(X, \alpha\times\phi, K_0(C(\Omega))\big) = \\ &\qquad C\big(X, K_i(C(\Omega))\big)/\big\{ f - (\widetilde{\phi})^{-1}_{*i}(f\circ\alpha^{-1}) : f \in C\big(X, K_i(C(\Omega))\big)\big\} \end{aligned}$$

with $K^0(X, \alpha\times\phi, K_0(C(\Omega)))$ equipped with the positive cone

$$K^0(X,\alpha\times\phi,K_0(C(\Omega)))_+ = \{[f] : f \in C(X, K_0(C(\Omega))_+\}.$$

Denote

$$\begin{aligned} \ker(\mathrm{id}_{X\times\Omega} - (\alpha\times\phi)^{-1})_{*i} &= \big\{ f \in C\big(X, K_i(C(\Omega))\big) : f - (\widetilde{\phi})^{-1}_{*i}(f\circ\alpha^{-1}) = 0\big\}, \\ & i = 0,1. \end{aligned}$$

When Ω is connected, $\rho_{C(\Omega)}(K_0(C(\Omega))) = \mathbb{Z}$. This is also used in the following computation.

Proposition 2.5 *Let (X,α) be a Cantor minimal system, let Ω be a connected compact metric space, and let $\phi\colon X \to \mathrm{Homeo}(\Omega)$. Let $A = C(X\times\Omega)_{\alpha\times\phi}\mathbb{Z}$ and $C = C(X)\rtimes_\alpha\mathbb{Z}$. Then there are short exact sequences*

$$0 \to K^0\big(X,\alpha\times\phi,K_0(C(\Omega))\big) \to K_0(A) \to \ker\big(\mathrm{id}_{X\times\Omega} - (\alpha\times\phi)^{-1}_{*1}\big) \to 0, \tag{2.1}$$

$$0 \to K^1\big(X,\alpha\times\phi,K_1(C(\Omega))\big) \to K_1(A) \to \ker\big(\mathrm{id}_{X\times\Omega} - (\alpha\times\phi)^{-1}_{*0}\big) \to 0. \tag{2.2}$$

Moreover,

$$\rho_A\big(K^0\big(X,(\alpha\times\phi)^{-1},K_0(C(\Omega))\big)\big) = \rho_C(K^0(X,\alpha)), \tag{2.3}$$

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Furthermore, if $(\phi_x)_{*i} = \mathrm{id}_{K_i(C(\Omega))}$, $i = 0, 1$, *then*

$$0 \to K^0\big(X, \alpha \times \mathrm{id}, K_0(C(\Omega))\big) \to K_0(A) \to K_1(C(\Omega)) \to 0,$$
$$0 \to K^1\big(X, \alpha \times \mathrm{id}, K_1(C(\Omega))\big) \to K_1(A) \to K_0(C(\Omega)) \to 0.$$

Proof It is clear that (2.1) and (2.2) follow directly from the Pimsner–Voiculescu six-term exact sequence. To see the second statement, we note that Ω is connected and all tracial states on $K_0(C(\Omega))$ agree with the rank. It follows that $\rho_{C(\Omega)}(K_0(C(\Omega))) = \mathbb{Z}$. Moreover, $\rho_{C(\Omega)}(f \circ (\phi)_{*0}^{-1})) = \rho_{C(\Omega)}(f)$ for any $f \in K_0(C(\Omega))$, since ϕ does not change the rank of any projections of $M_n(C(\Omega))$. Therefore, any $(\alpha \times \phi)_{*0}^{-1}$-invariant state on $K_0(C(X) \otimes C(\Omega))$ can be viewed as an α_{*0}-invariant state on $C(X, \mathbb{Z})$. We then check that (2.3) holds.

To see the last statement, we note that

$$K_i\big(C(X \times \Omega)\big) = C\big(X, K_i(C(\Omega))\big), \quad i = 0, 1.$$

In the case $\phi_{*i} = \mathrm{id}_{K_i(C(\Omega))}$, $i = 0, 1$, (only) constant elements are invariant under $(\alpha \otimes \phi)_{*i}^{-1}$. It follows that

$$\ker\big(\mathrm{id}_{X \times \Omega} - (\alpha \times \phi)_{*i}^{-1}\big) = K_{i+1}(C(\Omega)), \quad i = 0, 1. \qquad \blacksquare$$

Lemma 2.6 *Let* Ω *be a compact metric space with* $U(C(\Omega)) = U_0(C(\Omega))$ *and* $\beta \in \mathrm{Homeo}(\Omega)$. *Then*

$$\rho_B(K_0(B)) = \rho_B\big(\imath_{*0}\big(K_0(C(\Omega))\big)\big),$$

where $B = C(\Omega) \rtimes_\beta \mathbb{Z}$ *and* $\imath: C(\Omega) \to B$ *is the natural embedding. Consequently, when* Ω *is connected,* $\rho_B(K_0(B)) = \mathbb{Z}$, *and in Proposition 2.5, if* $H^1(\Omega, \mathbb{Z}) = \{0\}$, *then*

$$\rho_A(K_0(A)) = \rho_C(K^0(X, \alpha)),$$

where $C = C(X) \rtimes_\alpha \mathbb{Z}$.

Proof The first part follows from [5, Chapter VI] (see also [1, 10.10.5]). For the second part, we note that X has zero dimension, so $H^1(X \times \Omega, \mathbb{Z}) = \{0\}$. Therefore, by (2.3),

$$\rho_A(K_0(A)) = \rho_A\big(K^0\big(X, (\alpha \times \phi)^{-1}, K_0(C(\Omega))\big)\big) = \rho_C(K^0(X, \alpha)),$$

where $C = C(X) \rtimes \alpha\mathbb{Z}$. $\blacksquare$

Definition 2.7 Denote by $\mathcal{A}$ the class of unital, $\mathcal{Z}$-stable, separable, simple, amenable C^*-algebras that satisfy the Universal Coefficient Theorem and have rational tracial rank at most one, *i.e.*, $A \otimes U$ has tracial rank at most one, where U is any infinite dimensional UHF-algebra (see [10]). This class of C^*-algebras are classifiable in the sense that if A, $B \in \mathcal{A}$, then $A \cong B$ if and only if they have the isomorphic Elliott invariants ([10]). This class contains all unital simple AH-algebras with no dimension growth as well as the Jiang-Su algebra. A description of the range of the invariant is presented in [17].

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3 The C^*-subalgebra A_x

Definition 3.1 Let (X,α) be a Cantor minimal system and let Ω be a connected, finite dimensional compact, metric space. Fix $x\in X$. Denote by A_x the C^*-subalgebra of $A=C(X\times\Omega)\times_{\alpha\times\phi}\mathbb{Z}$ generated by $C(X\times\Omega)$ and $uC_0(X\setminus\{x\}\times\Omega)$, where u is the unitary in A that implements the action $\alpha\times\phi$.

Theorem 3.2 *Let X and Ω be as above. The C^*-algebra A_x is isomorphic to a unital simple AH-algebra with slow dimension growth.*

Proof Part of the proof is known. As in the proof of [14, Proposition 3.3(5)], using groupoid C^*-algebra, A_x is simple since we assume that $\alpha\times\phi$ is minimal.

We now assume that the dimension of Ω is d. To show that A_x is locally AH, we use an argument of I. Putnam ([22, 3.1]) and proceed as in the proof of [14, Proposition 3.3]. Let

$$\mathcal{P}_n=\left\{X(n,v,k):v\in V_n,k=1,2,\ldots,k(v)\right\}$$

be a sequence of Kakutani–Rohlin partition which gives a Bratteli–Vershik model for (X,σ) (see [8, Theorem 4.2] or [20, Sect. 2])). We also assume the roof sets

$$R(\mathcal{P}_n)=\bigcup_{v\in V}X(n,v,h_n(v))$$

shrink to a single point x. Let A_n be the C^*-algebra generated by $C(X\times\Omega)$ and $uC(\mathcal{R}(\mathcal{P}_n)^c\times\Omega)$. Since $R(\mathcal{P}_{n+1})\subset R(\mathcal{P}_n)$, $A_n\subset A_{n+1}$, $n=1,2,\ldots$. It is easy see that A_x is the norm closure of the union of all $A_n's$. By using a similar argument to [22, Lemma 3.1], it can be shown that A_n is isomorphic to

$$\bigoplus_{v\in\mathcal{V}_n}M_{h_n(v)}\otimes C\Big(X\big(n,v,h_n(v)\big)\Big)\otimes C(\Omega)\cong\bigoplus_{v\in\mathcal{V}_n}M_{h_n(v)}\big(C(Y_{n,v})\big),\tag{3.1}$$

where $Y_{n,v}$ is a compact metric space of covering dimension d. In fact, let

$$p_v=\sum_{k=1}^{h_n(v)}\chi_{X(n,v,k)}$$

for each $v\in V$. It is easy to check that $p_vu(1-\chi_{R(\mathcal{P}_n)})=u(1-\chi_{R(\mathcal{P}_n)})p_v$. It follows that p_v is central in A_n. Put $e_{i,j}(n,v)=u^{i-j}\chi_{X(n,v,j)\times\Omega}$. One observes that, for each n and v, $\{e_{i,j}(n,v)\}_{i,j}$ $(1\le i,\ j\le h_n(v))$ form matrix units in A_n with

$$\sum_{i=1}^{h_n(v)}e_{i,i}=\sum_{i=1}^{h(v)}\chi_{X(n,v,i}=p_{n,v}.$$

Note that $\chi_{X(n,v,h_n(v))}A_n\chi_{X(n,v,h_n(v))}=C(X(n,v,h_n(v))\times\Omega)$. One checks that the C^*-subalgebra generated by $\{e_{i,j}(n,v)\}$ and $\chi_{X(n,v,h_n(v)}A_n\chi_{X(n,v,h_n(v)}$ is isomorphic to $M_{h_n(v)}(C(X(n,v,h_n(v))\times\Omega))$. Therefore, (3.1) holds. It follows from [13, Theorem 1.1] that A_x is a unital simple AH-algebra with no dimension growth. ∎

Proposition 3.3 *Let (X,α) be a Cantor minimal system, let $x \in X$, let Ω be a connected finite dimensional compact metric space, and let $\phi\colon X \to \mathrm{Homeo}(\Omega)$ be a continuous map. Then*

$$K_i(A_x) \cong K_i(C(X\times\Omega))/\big\{f - (\widetilde{\phi})_{*i}^{-1}(f\circ\alpha^{-1}) : f(x)=0, f\in C\big(X, K_i(C(\Omega))\big)\big\}. \tag{3.2}$$

Moreover, the embedding $\iota\colon A_x \to A$ gives an affine homeomorphism $\iota_\sharp\colon T(A)\to T(A_x)$ and gives an order isomorphism $\rho_{A_x}(K_0(A_x)) = \rho_C(K^0(X,\alpha))$, where $C = C(X)\rtimes_\alpha \mathbb{Z}$. Moreover, if $q \in C(X\times\Omega)$ is a projection, then uqu^* and q are equivalent in A_x.

Proof Let A_n be as in the proof of Theorem 3.2; *i.e*, A_n is generated by $C(X\times\Omega)$ and $uC(R(\mathcal{P})^c\times\Omega)$. There is a natural homomorphism from $K_i(C(X\times\Omega))$ to $K_i(A_n)$ (by the embedding of $C(X\times\Omega)$). By (3.1), this homomorphism is surjective. For $i=0,1$, the kernel is

$$\big\{f - (\widetilde{\phi})_{*i}^{-1}(f\circ\alpha^{-1}) : f(y)=0 \quad \text{for all} \quad y\in R\big(\mathcal{P}_n\big),\, f\in C(X, K_0(C(\Omega)))\big\}.$$

Thus (3.2) holds.

We now prove that for any projection $q\in C(X\times\Omega)$, uqu^* and q are equivalent in A_x. If $q(x,\omega)=0$ for any $\omega\in\Omega$, then $uq\in A_x$. It follows that $qu^*\in A_x$. Therefore uqu^* and q are equivalent in A_x. Suppose that $q(x,\omega)\neq 0$ for some $\omega\in\Omega$. Since q is a projection in $C(X\times Om)$ and Ω is connected, $q(x,\omega)=1$ for all ω. It suffices to show that

$$[q]-[q\circ(\alpha\times\phi)^{-1}] = 0 \quad \text{in} \quad K_0(A_x).$$

Let $f(y,\omega) = [q(y,\omega)] - [1_{C(X\times\Omega)}(y,\omega)]$ for all $(y,\omega)\in X\times\Omega$. It follows that

$$[q]-[q\circ(\alpha\times\phi)^{-1}] = f - f\circ(\alpha\times\phi)^{-1} + [1_{C(X\times\Omega)}] - [1_{C(X\times\Omega)}\circ(\alpha\times\phi)^{-1}].$$

Note that since $f(x,\omega)=0$ for all ω and $1_{C(X\times\Omega)} - 1_{C(X\times\Omega)}\circ(\alpha\times\phi)^{-1} = 0$, w f may be represented by an element $F\in C(X, K_0(C(\Omega)))$ with $F(x)=0$. From (3.2), this implies that $[q]-[q\circ(\alpha\times\phi)^{-1}]=0$ in $K_0(A_x)$. This proves that q and uqu^* are equivalent in A_x.

To show $T(A_x)=T(A)$, it suffices to show that every tracial state $\tau\in T(A_x)$ can be extended to tracial state of A. Let U be a clopen neighborhood of x such that $U, \alpha^{-1}(U),\dots,\alpha^{-n}(U)$ are mutually disjoint. Let $p=\chi_{U\times\Omega}$. We have shown that p, upu^*, and u^npu^{*n} are mutually equivalent in A_x and mutually orthogonal. The proof that $T(A_x)=T(A)$ can then be proceed exactly the same way as that of [14, Proposition 3.3(4)].

To show that $\rho_{A_x}(K_0(A_x)) = \rho_C(K^0(X,\alpha))$ ($C=C(X)\rtimes_\alpha\mathbb{Z}$), we first note that we have just proved that the map sending $[\chi_O]$ to $[\chi_O]$ (for clopen sets $O\subset X$) is an embedding from $K^0(X,\alpha,\mathbb{Z})$ into $K_0(A_x)$. Since Ω is connected, the subgroup $K^0(X,\alpha)$ injectively maps into $K^0(X,\alpha\times\phi,K_0(C(\Omega)))\subset K_0(A)$. ∎

4 Tracial Rank

Definition 4.1 Let T be a compact Choquet simplex. Suppose that Y is a compact metric space and let $L: C(Y)_{\text{s.a.}} \to \operatorname{Aff}(T)$ be an affine map. We say L is unital and strictly positive if $L(1_{C(Y)})(\tau) = 1$ for all $\tau \in T$ and $L(f)(\tau) > 0$ for all τ if $f \neq 0$ and $f \geq 0$.

Suppose that $L: C(Y)_{\text{s.a.}} \to \operatorname{Aff}(T)$ is a strictly positive affine homomorphism. Let $f \neq 0$ and $f \geq 0$. Then since $T(A)$ is compact,

$$\inf\{L(f)(\tau) : \tau \in T(A)\} > 0.$$

For each open subset $O \subset Y$, let

$$d(O) = \inf_{\tau \in T}\big\{ \sup\{L(f)(\tau) : 0 \leq f \leq 1, \operatorname{supp}(f) \subset O\}\big\}.$$

Then, for any non-empty open subset $O \subset Y$, $d(O) > 0$. For each $a \in (0,1)$, let $\{y_1, y_2, \ldots, y_m\} \subset Y$ be an $a/4$-dense subset. Define

$$D(a,i) = d(B(x_i, a/4)), \quad i = 1, 2, \ldots, m.$$

Put

$$\triangledown_0(a) = \min\{D(a,i) : i = 1, 2, \ldots, m\}.$$

Then $\triangledown_0: (0,1) \to (0,1)$ is non-decreasing. For any $y \in Y$, there exists i such that $B(x,a) \supset B(x, a/4)$. Thus, $d(B(x,a)) \geq (3/4)\Delta_0(a)$. Put $\triangledown(a) = (3/4)\triangledown_0(a)$ for all $a \in (0,1)$. Now let A be a unital separable simple C^*-algebra with $T(A) = T$ and let $\phi: C(Y) \to A$ be a unital monomorphism. Then $\phi_\sharp: C(Y)_{\text{s.a.}} \to \operatorname{Aff}(T(A))$ defined by

$$\phi_\sharp(f)(\tau) = \tau \circ \phi(f) \quad \text{for all} \quad f \in C(Y)_{\text{s.a.}}$$

is a unital strictly positive affine homomorphism. It is easy to check that, for any $1 > \sigma > 0$, there is a finite subset $\mathcal{H} \subset C(Y)_{\text{s.a.}}$ and $\eta > 0$ such that

$$\mu_{\tau\circ\phi}(B(x,r)) \geq \triangledown(r)$$

for all open balls with radius $r \geq \sigma$, provided that

$$|\tau \circ \phi(g) - L(g)(\tau)| < \eta \quad \text{for all} \quad g \in \mathcal{H},$$

where $\mu_{\tau\circ\phi}$ is the Borel probability measure induced by the state $\tau \circ \phi$.

We will use the following uniqueness theorem.

Theorem 4.2 ([12, Theorem 5.9]) *Let Y be a compact metric space and let T be a compact Choquet simplex. Suppose that $L: C(Y)_{\text{s.a.}} \to \operatorname{Aff}(T)$ is a unital strictly positive affine map. Let $\epsilon > 0$ and let $\mathcal{F} \subset C(Y)$ be a finite subset. There exists a finite subset $\mathcal{H} \subset C(Y)_{\text{s.a.}}$, a finite subset $\mathcal{P} \subset \underline{K}(C(Y))$, a finite subset $\mathcal{U} \subset U_c(K_1(C(Y)))$, $\delta > 0$, and $\eta > 0$ satisfying the following. Suppose that $\phi_1, \phi_2: C(Y) \to A$ are two*

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unital monomorphisms for some unital simple C^*-algebra A of tracial at most one with $T(A) = T$ such that

$$
\begin{aligned}
&[\phi_1]|_{\mathcal{P}} = [\phi_2]|_{\mathcal{P}}, \\
&|\tau \circ \phi_i(g) - L(g)(\tau)| < \delta \quad \text{for all} \quad g \in \mathcal{H},\ i = 1, 2, \quad \text{and} \\
&\operatorname{dist}(\overline{\phi_1(v)\phi_2(v^*)}, \overline{1_A}) < \eta \quad \text{for all} \quad v \in \mathcal{U}.
\end{aligned} \tag{4.1}
$$

Then there is a unitary $w \in A$ such that

$$\|w^*\phi_1(f)w - \phi_2(f)\| < \epsilon \quad \text{for all} \quad f \in \mathcal{F}.$$

Proof This is a variation of [12, Theorem 5.9]. Let $\triangledown\colon (0,1) \to (0,1)$ be as in Definition 4.1 (depending on L). For any $0 < d < 1$, if δ is small enough and $\mathcal{H}$ is large enough, by the discussion above in Definition 4.1 and by (4.1),

$$\mu_{\tau\circ\phi_i}(B(x,r)) \geq \triangledown(r)$$

for all $1 > r > d$, $i = 1, 2$. Note also that (4.1) implies that

$$|\tau \circ \phi_1(g) - \tau \circ \phi_2(g)| < 2\delta \quad \text{for all} \quad g \in \mathcal{H}.$$

Therefore, Theorem 4.2 follows from [12, Lemma 5.7], and [12, Theorem 5.9]. ∎

The following is a well-known lemma.

Lemma 4.3 *Let Y be a compact metric space such that $U(C(Y)) = U_0(C(Y))$. Then for any $z \in K_1(C(Y))$ there is an integer $m \geq 1$ and a unitary $v \in M_m(C(Y))$ such that*

$$[v] = z \quad \text{and} \quad \det(v)(y) = 1 \quad \text{for all} \quad y \in Y.$$

Proof There exists an integer $m \geq 1$ and a unitary $w \in M_m(C(Y))$ such that $[w] = z$. Put $w_{00}(y) = \det(w)(y)$ for all $y \in Y$. Then $w_{00} \in U(C(Y)) = U_0(C(Y))$. It follows that

$$w_0(y) = \begin{pmatrix} w_{00}^*(y) & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ & & \ddots & \\ & & & 1 \end{pmatrix} \in U_0(M_m(C(Y))).$$

Define $v = w_0 w$. Then $[v] = [w] = z$ and

$$\det(v)(y) = \det(w_0)(y)\det(w)(y) = 1 \quad \text{for all} \quad y \in Y. \qquad ∎$$

Lemma 4.4 *Let (X, α) be a Cantor minimal system, let Ω be a connected finite dimensional compact metric space with $U(C(\Omega)) = U_0(C(\Omega))$ and let*

$$\phi\colon X \to \mathrm{Homeo}(\Omega)$$

be a continuous map. Denote $A = C(X \times \Omega) \rtimes_{\alpha\times\phi} \mathbb{Z}$. Suppose that there is $x \in X$ and an integer $k \geq 1$ such that

$$[\Phi_y] = [\mathrm{id}_{C(\Omega)}] \quad \text{in} \quad KL(C(\Omega), C(\Omega)), \tag{4.2}$$

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where $\Phi_y: C(\Omega) \to C(\Omega)$ *is defined by*

$$\Phi_y(f) = f \circ \phi^{-1}_{\alpha^{-k+1}(y)} \circ \phi^{-1}_{\alpha^{-k+2}(y)} \circ \cdots \circ \phi^{-1}_y \quad \textit{for all} \quad f \in C(\Omega)$$

and for all $y \in \{\alpha^j(x) : j \in \mathbb{Z}\}$ *and* α^k *is minimal. Let* $x \in X$*. Then for any* $N \in \mathbb{N}$, $\epsilon > 0$ *and any finite subset* $\mathcal{F} \subset C(X \times \Omega)$, *there is an integer* $M > N$, *a clopen neighborhood* O *of* x *and partial isometry* $w \in A_x$ *that satisfy the following:*

(i) $\alpha^{-N}(O), \alpha^{-N+1}(O), \ldots, O, \alpha(O), \ldots, \alpha^M(O)$ *are mutually disjoint and* $\mu(O) < \epsilon/M$ *for every* α*-invariant probability measure* μ;
(ii) $w^*w = \chi_O$ *and* $ww^* = \chi_{\alpha^M(O)}$;
(iii) $u^{*i}wu^i \in A_x$ *for* $i = 0, 1, \ldots, N-1$;
(iv) $\|wf - fw\| < \epsilon$ *for all* $f \in \mathcal{F}$.

Proof Since $U(C(\Omega))$ is connected and X is zero-dimensional, one has

$$U(C(X \times \Omega)) = U_0(C(X \times \Omega)).$$

It follows from Lemma 2.6 that

$$\rho_A(K_0(A)) = \rho_A(K^0(X, \alpha, \mathbb{Z})).$$

Therefore, by Proposition 3.3, the embedding $\iota: A_x \to A$ gives

$$\rho_{A_x}(K_0(A_x)) = \rho_A(K_0(A)). \tag{4.3}$$

Note A_x is a unital simple AH-algebra with no dimension growth, by Theorem 3.2. So $TR(A_x) \le 1$. It is generated by $C(X \times \Omega)$ and $uC((X \setminus \{x\}) \times \Omega)$. The $g \to 1_X \otimes g$ gives a unital embedding from $C(\Omega)$ into $C(X \times \Omega)$. Therefore, there is a unital embedding $\iota: C(\Omega) \to A_x$. Let $L: C(\Omega) \to \mathrm{Aff}(T(A_x))$ be the unital strictly positive affine homomorphism induced by ι. Note that $L(g) = \tau(1 \otimes g)$ for all $g \in C(\Omega)_{\mathrm{s.a.}}$ and for all $\tau \in T(A_x)$.

Without loss of generality, we may assume that

$$\mathcal{F} = \{f \otimes 1_\Omega, 1_X \otimes g : f \in \mathcal{F}_0 \quad \text{and} \quad g \in \mathcal{F}_1\},$$

where $\mathcal{F}_0 \subset C(X)$ and $\mathcal{F}_1 \subset C(\Omega)$ are finite subsets. There exists a clopen neighborhood B_x of x such that

$$|f(x) - f(y)| < \epsilon/8 \quad \text{for all} \quad y \in B_x \quad \text{and for all } f \in \mathcal{F}_0. \tag{4.4}$$

Since α^k is minimal, we can find $n > N$ such that $\alpha^{kn}(x) \subset B_x$. Choose a sufficiently small clopen neighborhood O_x of x such that (i) holds and

$$\alpha(y) \in B_x \quad \text{for all} \quad y \in O_x.$$

Moreover, we may also require that $O_x \cup \alpha^{kn}(O_x) \subset B_x$. Let $p_1 = \chi_{O_x}$ and $q_1 = \chi_{\alpha^{kn}(O_x)}$. Put $M = kn$. Then

$$u^M g p_1 u^{*M} = u^M g u^{*M} q_1 \quad \text{for all} \quad g \in C(\Omega).$$

Define $\Psi_x = \Phi_{\alpha^{(1-k)(n-1)}(x)} \circ \Phi_{\alpha^{(2-k)(n-2)}(x)} \circ \cdots \circ \Phi_x$. It follows that

$$[\Psi_x] = [\mathrm{id}_{C(\Omega)}] \times [\mathrm{id}_{C(\Omega)}] \times \cdots \times [\mathrm{id}_{C(\Omega)}] = [\mathrm{id}_{C(\Omega)}]. \tag{4.5}$$

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Let $\mathcal{H} \subset C(\Omega)_{\text{s.a.}}$, $\mathcal{P} \subset \underline{K}(C(\Omega))$, and $\mathcal{U} \subset U_c(K_1(C(\Omega)))$ be finite subsets, let $\delta > 0$ and $\eta > 0$ be required by Theorem 4.2 for $\epsilon/4$ (in place of ϵ) and $\mathcal{F}_1$ (in place of $\mathcal{F}$) associated with L given above. Let $\mathcal{V} = \{v_1, v_2, \ldots., v_m\} \subset M_K(C(\Omega))$ such that $\mathcal{U} \subset \{\overline{v_i} : 1 \le i \le m\}$, and by Lemma 4.3,

$$\det(v_i)(y) = 1 \quad \text{for all} \quad y \in \Omega, \quad i = 1, 2, \ldots, m.$$

There is a finite subset $\mathcal{G} \subset C(\Omega)$ and $\delta_1 > 0$ satisfying the following. Suppose that $h_1, h_2 : C(\Omega) \to B$ are two unital homomorphisms (for any unital C^*-algebra B) such that

$$\|h_1(g) - h_2(g)\| < \delta_1 \quad \text{for all} \quad g \in \mathcal{G}.$$

Then $[h_1]|_{\mathcal{P}} = [h_2]|_{\mathcal{P}}$. Let $\mathcal{G}_1 = \mathcal{G} \cup \mathcal{H}$ and let

$$U = \operatorname{diag}(\overbrace{u, u, \ldots, u}^{K}).$$

There is a neighborhood O of x with $O \subset O_x$ such that

$$\begin{aligned} &\|u^M g p u^{*M} - \Psi_x(g) q\| < \min\{\delta, \delta_1, \epsilon/8\} \quad \text{for all} \quad g \in \mathcal{G}_1 \quad \text{and} \\ &\|U^M v_i P U^{*M} - (\Psi_x \otimes \mathrm{id}_{M_K})(v_i) Q\| < \eta, \quad 1 \le i \le m, \end{aligned} \tag{4.6}$$

where

$$p = \chi_O, \quad q = \chi_{\alpha^M(O)}, \quad P = p \otimes \mathrm{id}_{M_K}, \quad \text{and} \quad Q = q \otimes \mathrm{id}_{M_K}.$$

Define $\psi_{1,0}, \psi_{2,0} : C(\Omega) \to C(X \times \Omega)$ by $\psi_{1,0}(f) = f$ (as constant along X) and $\psi_{2,0}(f) = \Psi_x(f)$ for all $f \in C(\Omega)$. It follows from (4.5) that

$$[\psi_{2,0}] = [\psi_{1,0}] \quad KL(C(\Omega), C(X \times \Omega)). \tag{4.7}$$

Define $\psi_1', \psi_2' : C(\Omega) \to q(C(X \times \Omega))q$ by $\psi_1'(g) = \psi_{1,0}(g)|_{\alpha^M(O)}$ and $\psi_2'(g) = \psi_1' \circ \psi_{2,0}(g) = \Psi_x(g) \cdot q$ for all $g \in C(\Omega)$. It follows from (4.7) that

$$[\psi_1'] = [\psi_2'] \quad \text{in} \quad KL(C(\Omega), qC(X \times \Omega)q).$$

Denote by j the embedding $qC(X \times \Omega)q \to qA_x q$, $\psi_i = j \circ \psi_i'$, $i = 0, 1$. Then

$$[\psi_1] = [\psi_2] \quad \text{in} \quad KL(C(\Omega), qA_x q).$$

It follows from (4.6) and Thereom 3.2 that

$$|t \circ \psi_1(g) - t \circ \psi_2(g)| < \delta \quad \text{for all} \quad g \in \mathcal{H} \quad \text{and for all } t \in T(qA_x q).$$

Note

$$\tau \circ \psi_1(g) = \tau(q \otimes g) \quad \text{for all} \quad \tau \in T(A) \quad \text{and for all } g \in C(\Omega)_{\text{s.a.}}.$$

It follows that

$$L(g)(\tau) = \frac{\tau \circ \psi_1(g)}{\tau(q)} \quad \text{for all} \quad g \in C(\Omega)_{\text{s.a.}}$$

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for all $\tau \in T(A)$.

Note also that $\psi_1(v_i)\psi_2(v_i^*) \in M_K(C(X \times \Omega))$ for some integer $m \geq 1$. In virtue of [5, Theorem 10, Chapter VI],

$$\begin{aligned}\Delta\big(\psi_1(v_i)\psi_2(v_i^*)\big) &= \Delta\big(\det(\psi_1(v_i)\psi_2(v_i^*))\big) = \Delta\big(\det(\psi_1(v_i))\det(\psi_2(v_i^*))\big)\\ &= \Delta\big(\psi_1(\det(v_i))\psi_2(\det(v_i^*))\big) = \Delta(1_{qA_xq}) \in \rho_{A_x}(K_0(qA_xq)).\end{aligned}$$

It follows that

$$\operatorname{dist}(\overline{\psi_1(v_i)\psi_2(v_i^*)}, \overline{1_{qA_xq}}) = 0. \tag{4.8}$$

It follows from Theorem 4.2 that there is a unitary $w_1 \in qA_xq$ such that

$$\|w_1\psi_2(g)w_1^* - \psi_1(g)\| < \epsilon/4 \quad \text{for all} \quad g \in \mathcal{F}_1.$$

There is a unitary normalizer $w_2 \in A_x \cap C^*(X,\alpha)$ of $C(X)$ such that $w_2pw_2^* = q$. Note that w_2 has the form $w_2 = \sum_{m\in\mathbb{Z}} u^m \chi_{\Gamma^{-1}(m)}$, where $\Gamma\colon X \to \mathbb{Z}$ is a continuous map. Define $\psi_3, \psi_4, \psi_5\colon C(\Omega) \to pA_xp$ by $\psi_3(g) = gp$ and

$$\psi_4(g) = w_2^*w_1u^Mgpu^{*M}w_1^*w_2 \quad \text{and} \quad \psi_5(g) = w_2^*w_1(g\circ\Psi_x)pw_1^*w_2$$

for all $g \in C(\Omega)$. As above, we compute that

$$[\psi_5] = [\psi_3] \quad \text{in} \quad KL(C(\Omega), pA_xp).$$

By (4.6), the choice of $\mathcal{G}_1$ and δ_1, $[\psi_5]|_{\mathcal{P}} = [\psi_4]|_{\mathcal{P}}$. It follows from Theoerem 3.2 that

$$\tau\circ\psi_3(g) = \tau\circ\psi_4(g) \quad \text{for all} \quad g \in C(\Omega) \quad \text{and for all } \tau \in T(pA_xp).$$

It is clear that $\psi_3(v_i)\psi_4(v_i^*) \in CU(pAp)$. It follows that

$$\widetilde{\Delta}_\tau(\overline{\psi_3(v_i)\psi_4(v_i^*)}) \in \overline{\rho_A(K_0(A))}.$$

Therefore, by (4.3),

$$\operatorname{dist}(\overline{\psi_1(v_i)\psi_2(v^*)}, \overline{1_{qA_xq}}) = 0.$$

By applying Theorem 4.2 again, we obtain a unitary $w_3 \in pA_xp$ such that

$$\|w_3gpw_3^* - \psi_4(g)\| < \epsilon/4 \quad \text{for all} \quad g \in \mathcal{F}_1.$$

Put $w = w_2w_3$. Then $w \in A_x$ and

$$\begin{aligned}w^*w &= pw_3^*w_2^*w_2w_3p = p = \chi_O,\\ ww^* &= w_2w_3w_3^*w_2^* = w_2pw_2^* = q = \chi_{\alpha^M(O)}.\end{aligned}$$

So (ii) holds. Moreover (see also (4.6)),

$$\begin{aligned}\|wgpw^* - gq\| &\leq \|w_2(w_3gpw_3^*)w_2^* - w_2\psi_4(g)w_2^*\| + \|w_2\psi_4(g)w_2^* - gq\|\\ &< \epsilon/4 + \|w_1u^Mgpu^{*M}w_1^* - gq\|\\ &< \epsilon/4 + \epsilon/8 + \|w_1\Psi_x(g)w_1^* - gq\|\\ &< \epsilon/4 + \epsilon/8 + \epsilon/4 = 5\epsilon/8 \quad \text{for all} \quad g \in \mathcal{F}_1.\end{aligned}$$

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It follows that

$$\|wg - gw\| = \|wgp - gqw\| = \|(wgp - gqw)w^*\| = \|wgpw^* - gq\| < 5\epsilon/8$$

for all $g \in \mathcal{F}_1$. Since $O \cup \alpha^M(O) \subset B_x$, by (4.4), for all $f \in \mathcal{F}_0$,

$$\begin{aligned}\|wf - fw\| &\le \|wpf - wpf(x)\| + \|wpf(x) - f(x)qw\| + \|f(x)qw - fw\| \\ &< \epsilon/8 + \|wf(x) - f(x)w\| + \epsilon/8 = \epsilon/4.\end{aligned}$$

Thus (iv) holds. To see (iii), we note that

$$pu^i = pu\chi_{\alpha^{-1}(O)}u\chi_{\alpha^{-2}(O)}\cdots u\chi_{\alpha^{-i}(O)},$$
$$(u^{*i}q)^* = qu\chi_{\alpha^{M-1}(O)}u\chi_{\alpha^{M-2}(O)}\cdots u\chi_{\alpha^{M-i}(O)}$$

for $i = 1, 2, \dots, N-1$. Since $x \in O$, (i) implies that pu^i and $u^{*i}q$ are in A_x. From this one concludes that $u^{*i}wu^i \in A_x$, which proves the lemma. ∎

In Lemma 4.4, if $k = 1$, assumption (4.2) implies that

$$[\widetilde{\phi}_x] = [\mathrm{id}_{C(\Omega)}] \quad \text{in} \quad KL(C(\Omega), C(\Omega)).$$

Corollary 4.5 *In the case where $k = 1$, Lemma 4.4 holds if the condition $U(C(\Omega)) = U_0(C(\Omega))$ is replaced by the following: for each $z \in U(C(\Omega))/U_0(C(\Omega))$, there exists $v \in U(C(\Omega))$ with $[u] = z$ and $h \in C(X)_{\mathrm{s.a.}}$ such that*

$$\widetilde{\phi}_y(v) = v\exp(ih(y)) \quad \textit{for all} \quad y \in X. \tag{4.9}$$

Proof For each $z \in U(C(\Omega))/U_0(C(\Omega))$, choose $v \in U(C(\Omega))$ such that $[u] = z$ and (4.9) holds. Therefore, the rotation map

$$\Delta_t(v(\alpha \times \phi)(v^*)) = -t(h) + \rho_{C(X\times\Omega)}\big(K_0(C(X \times C(\Omega)))\big)$$

for all $t \in T(C(X \times \Omega))$. Since α is minimal, $C(X) \rtimes_\alpha \mathbb{Z}$ is a unital A$\mathbb{T}$-algebra of real rank zero. In particular, $\rho_B(K_0(B))$ is dense in $\mathrm{Aff}(T(B))$, where $B = C(X) \rtimes_\alpha \mathbb{Z}$. Note that

$$\rho_B(K_0(B)) = \rho_A(K^0(X, \alpha, \mathbb{Z})) \quad \text{and} \quad h(y) \in C(X)_{\mathrm{s.a.}}.$$

It follows that for each $\tau \in T(A)$,

$$\Delta_\tau(v(\alpha \times \phi)(v^*)) \in \overline{\rho_A(K^0(X, \alpha, \mathbb{Z}))}. \tag{4.10}$$

It follows from the last part of Proposition 2.5 and (4.10) that

$$\overline{\rho_{A_x}(K_0(A_x))} = \overline{\rho_A(K_0(A))}.$$

In the proof of Lemma 4.4, if $v_j \in U(C(\Omega))$, then

$$\psi_1(v_j)\psi_2(v_j)^* = v_j q \Psi_x(v_j^*) q = v_j \widetilde{\phi}_x(v_j^*) = \exp(-ih_j)$$

for some $h_j \in C(X)_{\mathrm{s.a.}}$. It follows that

$$D_\tau(\psi_1(v_j)\psi_2(v_j)) = -\tau(h_j q) \in \overline{\rho_{A_x}(K_0(qA_xq))},$$

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since $h_j q \in C(X)$. If $v_j \notin U(C(\Omega))$ and $v_j \in M_K(C(\Omega))$ with $K > 1$, let $w_j = \det v_j \in U(C(\Omega))$ and

$$v_j' = \begin{pmatrix} \det v_j^* & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ & & \ddots & \\ & & & 1 \end{pmatrix} v_j.$$

Note that $\det(v_j') = 1$. As in the proof of Lemma 4.4, one has

$$\Delta\big(\psi_1(v_j')\psi_2((v_j')^*)\big) \in \overline{\rho_{A_x}(K_0(qA_xq))}.$$

Put $w_j' = (v_j'(v_j^*))^*$. Then

$$\begin{aligned}\Delta\big(\psi_1(v_j)\psi_2(v_j^*)\big) &= \Delta\big(\psi_1(w_j'v_j')\psi_2((w_j'v_j')^*)\big)\\ &= \Delta\big(\psi_1(v_j')\psi_2((v_j')^*)\psi_2((w_j')^*)\psi_1(w_j')\big)\\ &= \Delta\big(\psi_1(v_j')\psi_2((v_j')^*)\big) + \Delta\big(\psi_2((w_j')^*)\psi_1(w_j')\big)\\ &= \Delta\big(\psi_1(v_j')\psi_2((v_j')^*)\big) - \Delta\big(\psi_1(w_j)\psi_2(w_j^*)\big) \in \overline{\rho_{A_x}(K_0(qA_xq))}.\end{aligned}$$

Thus (4.8) also holds here. The rest of the proof is exactly the same as for Lemma 4.4. ■

The following lemma is taken from the proof of [14, Theorem 5.6].

Lemma 4.6 *Let (X,α) be a Cantor minimal system, let Ω be a compact connected finite dimensional metric space, and let $\phi\colon X \to \mathrm{Homeo}(\Omega)$ be a continuous map. Suppose that there is $x \in X$ such that, for any $N \in \mathbb{N}$, $\delta > 0$, and any finite subset $\mathcal{F} \subset C(X \times \Omega)$, there is an integer $M > N$, a clopen neighborhood O of x and partial isometry $w \in A_x$ that satisfy the following:*

(i) *$\alpha^{-N}(O), \alpha^{-N+1}(O), \dots, O, \alpha(O), \dots, \alpha^M(O)$ are mutually disjoint and $\mu(O) < \delta/M$ for every α-invariant probability measure μ;*
(ii) *$w^*w = \chi_O$ and $ww^* = \chi_{\alpha^M(O)}$;*
(iii) *$u^{*i}wu^i \in A_x$ for $i = 0, 1, \dots, N-1$;*
(iv) *$\|wf - fw\| < \epsilon$ for all $f \in \mathcal{F}$.*

Then for any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exists a projection $e \in A_x$ satisfies the following:

(a) *$\|ea - ae\| < \epsilon$ for all $a \in \mathcal{F}$,*
(b) *$\mathrm{dist}(pap, eA_xe) < \epsilon$ for all $a \in \mathcal{F}$,*
(c) *$\tau(1-e) < \epsilon$ for all $\tau \in T(A)$.*

Proof It suffices to show that for any $\epsilon > 0$, any finite subset $\mathcal{F} \subset C(X \times \Omega)$, and any nonzero element $a \in A_+ \setminus \{0\}$, there exists a projection $e \in A_x \subset A$ such that the following hold:

(i)′ $\|ef - fe\| < \epsilon$ for all $a \in \mathcal{F} \cup \{u\}$,
(ii)′ $\mathrm{dist}(efe, eA_xe) < \epsilon$ for all $f \in \mathcal{F}$,
(iii)′ $\tau(1-e) < \epsilon$ for all $\tau \in T(A)$.

Without loss of generality, we may assume that $\mathcal{F}^* = \mathcal{F}$. Choose $N \subset \mathbb{N}$ so that $2\pi/N < \epsilon$. Put $\mathcal{G} = \bigcup_{i=0}^{N-1} u^i \mathcal{F} u^{*i}$. We obtain an integer $M > N$, a clopen neighborhood O of x, and a partial isometry $w \in A_x$ satisfying (i)–(iv).

Put $p = \chi_O$ and $q = \chi_{\alpha^M(O)}$. Define

$$P(t) = p\cos^t + w\sin t\cos t + w^* \sin t \cos t + q\sin^2 t \quad t \in [0, \pi/2].$$

Then $P(0) = p$ and $P(\pi/2) = q$. Moreover, one checks that $P(t)$ is a continuous path of projections. By (ii), (iii), and by the choice of $\mathcal{G}$, one has

$$\|u^{i*}P(t)u^i f - f u^{i*}P(t)u^i\| < \epsilon$$

for all $t \in [0, \pi/2]$, $i = 0, 1, \dots, N-1$ and $f \in \mathcal{F}$. Define

$$e = 1 - \Big(\sum_{i=0}^{M-N} u^i p u^{i*} + \sum_{i=1}^{N-1} u^{i*} P(i\pi/2N) u^i \Big).$$

Using (i) and (ii), one verifies that e is a projection. By the assumption that $u^{i*}wu^i \in A_x$, $e \in A_x$. By (ii) and the fact that

$$\{ p, upu^*, u^2pu^{2*}, \dots, u^{M-N}p(u^{M-N})^*, \\ u^*P(\pi/2N)u, u^{2*}P(2\pi/2N)u^2, \dots, (u^*)^{N-1}P((N-1)\pi/2N)u^{N-1} \}$$

is a set of orthogonal projections, we can verify that

$$\|fe - ef\| < \epsilon \quad \text{for all} \quad f \in \mathcal{F}.$$

Since

$$\|P(i\pi/2N) - P((i-1)\pi/2N)\| < \pi/N < \epsilon, \quad i = 1, 2, \dots, N,$$

one can further verify that $\|ue - eu\| < \epsilon$. It is clear that $efe \in A_x$ for all $f \in C(X \times \Omega)$. Note that $eue = eu(1-p)e$. Therefore, $eue \in A_x$. One also has

$$\tau(1-e) < M\tau(p) < \epsilon \quad \text{for all} \quad \tau \in T(A). \qquad \blacksquare$$

Lemma 4.7 *Let A be a unital simple C^*-algebra and let $B \subset A$, where $1_B = 1_A$ and B is a unital simple AH-algebra with no dimension growth such that $T(B) = T(A)$. Suppose that for any $\epsilon > 0$ and any subset $\mathcal{F} \subset A$, there is a projection $e \in B$ such that*

(i) $\|ea - ae\| < \epsilon$ *for all* $a \in \mathcal{F}$,
(ii) $\operatorname{dist}(eae, eBe) < \epsilon$ *for all* $a \in \mathcal{F}$,
(iii) $\tau(1-e) < \epsilon$ *for all* $\tau \in T(A)$.

Then A has tracial rank at most one.

Proof We first show that, with the assumption, for any given $d \in A_+ \setminus \{0\}$, one can require that $1 - e \lesssim d$. We can assume that $0 \le d \le 1$. Put $\sigma = \inf\{\tau(d) : \tau \in T(A)\}$. Since A is simple and $T(A)$ is compact, $\sigma > 0$. Choose $\epsilon_0 = \min\{\sigma/2, \epsilon/2\}$. By the assumption, there is a projection $e_1 \in B$ such that

$$\|e_1 d e_1 - d_1\| < \epsilon_0/32 \quad \text{and} \quad \tau(1-e_1) < \epsilon_0/2 \quad \text{for all} \quad \tau \in T(A)$$

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for some $d_1 \in e_1Be_1$. Since

$$\tau(d) = \tau(e_1de_1) + \tau((1-e_1)d(1-e_1)) \quad \text{for all} \quad \tau \in T(A),$$

$$\|e_1de_1\| \geq \sigma - \epsilon_0/2 \geq \sigma/2.$$

It follows that

$$\|d_1\| \geq \sigma_2 - \epsilon_0/32 \geq 15\sigma/32.$$

Put $\delta = \epsilon_0/32$. Then by [23, Proposition 2.2], $0 \neq f_\delta(d_1) \lesssim e_1de_1 \sim d^{1/2}e_1d^{1/2} \lesssim d$. Since B has tracial rank at most one, e_1Be_1 has property (SP). In particular, there is a non-zero projection $e_2 \in \overline{f_\delta(d_1)Bf_\delta(d_1)}$. Put

$$\epsilon_2 = \min\{\epsilon/2, \inf\{\tau(e_2) : \tau \in T(A)\}\}.$$

Then, by the assumption, there is a projection $e \in B$ such that

(i) $\|ea - ae\| < \epsilon_2 \leq \epsilon/2$ for all $a \in \mathcal{F}$,
(ii) $\mathrm{dist}(eae, eBe) < \epsilon_2 \leq \epsilon/2$ for all $a \in \mathcal{F}$,
(iii) $\tau(1-e) < \epsilon_2 \leq \tau(e_2)$ for all $\tau \in T(A)$.

Since $T(B) = T(A)$ and B has tracial rank at most one, $1 - e \lesssim e_2 \lesssim f_\delta(d_1) \lesssim d$.

Note that B has tracial rank at most one. The same argument used in the proof of [19, Lemma 4.4] shows that A has tracial rank at most one. Another way to reach the conclusion is to apply [9, Lemma 4.3]. ■

Theorem 4.8 *Let (X,α) be a Cantor minimal system, let Ω be a compact connected finite dimensional metric space with $U(C(\Omega)) = U_0(C(\Omega))$, and let $\phi: X \to \mathrm{Homeo}(\Omega)$ be a continuous map. Suppose that there exists $x \in X$ and an integer $k \geq 1$ such that*

$$[\Phi_y] = [\mathrm{id}_{C(\Omega)}] \quad \textit{in} \quad KL\big(C(\Omega), C(\Omega)\big),$$

where

$$\Phi_y(f) = f \circ \phi^{-1}_{\alpha^{-k}(y)} \circ \phi^{-1}_{\alpha^{1-k}(x)} \circ \cdots \circ \phi^{-1}_{\alpha^{-1}(x)} \circ \phi^{-1}_x \quad \textit{for all} \quad f \in C(\Omega)$$

for all $y \in \{\alpha^{j-1}(x) : j \in \mathbb{N}\}$ and suppose that α^k is minimal.

If $\alpha \times \phi$ is minimal, then $A = C(X \times \Omega) \rtimes_{\alpha\times\phi} \mathbb{Z}$ has tracial rank at most one. Consequently, A is isomorphic to a unital simple AH-algebra with no dimension growth.

Proof This follows from Lemmas 4.4, 4.6, and 4.7. ■

Remark 4.9 In this paper, we mainly consider the case where ϕ is a constant map. We state the above in a greater generality for the future usage.

Theorem 4.10 *Let (X,α) be a Cantor minimal system, let Ω be a compact connected finite dimensional metric space, and let $\phi: X \to \mathrm{Homeo}(\Omega)$ be a continuous* *map. Suppose that* $[\phi_y] = [\mathrm{id}_{C(\Omega)}]$ *in* $KL\big(C(\Omega), C(\Omega)\big)$ *for all* $y \in X$, *and for each* $z \in U(C(\Omega))/U_0(C(\Omega))$, *there exists* $v \in U(C(\Omega))$ *and* $h \in C(X)_{\mathrm{s.a.}}$ *such that*

$$\widetilde{\phi}_y(v) = v\exp(ih(y)) \quad \text{for all} \quad y \in X.$$

If $\alpha \times \phi$ *is minimal on* $X \times \Omega$, *then* $A = C(X \times \Omega) \rtimes_{\alpha\times\phi} \mathbb{Z}$ *is isomorphic to a unital simple AH-algebra with no dimension growth.*

Proof This follows from Corollary 4.5 and Lemmas 4.6 and 4.7. ∎

5 Examples

5.1 Notation

Let (X, α) be a Cantor minimal system and let $\phi: X \to \mathbb{T}^n$ be a continuous map. One can write $\mathbb{T}^n = \mathbb{R}/\mathbb{Z} \times R/\mathbb{Z} \times \cdots \mathbb{R}/\mathbb{Z}$. Given $\eta \in \mathbb{T}^n$, one can write $\eta = (t_1, t_2, \dots, t_n)$, where $t_j \in \mathbb{R}/\mathbb{Z}$. Define $\eta(\xi) = (s_1 + t_1, s_2 + t_2, \dots, s_n + t_n)$, where $\xi = (s_1, s_2, \dots, s_n) \in \mathbb{T}^n$. Define $(\alpha\times\phi)(x, \xi) = (\alpha(x), \phi_x(\xi))$ for all $(x, \xi) \in X \times \mathbb{T}^n$. Since X is totally disconnected, we may also write $\phi_x = (\exp(i\theta_1(x)), \exp(i\theta_2(x)), \dots, \exp(i\theta_n(x)))$, where $\theta_j \in C(X)_{\mathrm{s.a.}}$. Note that for each $x \in X$, $[\phi_x] = [\mathrm{id}_{C(\Omega)}]$. Let z be the standard unitary generator of $C(\mathbb{T})$. Denote by $z_j \in C(\mathbb{T}^n)$ the function that maps $(s_1, s_2, \dots, s_n)$ to s_j. Then $\widetilde{\phi}_y(z_j) = z_j\exp(i\theta_j(y))$ for all $y \in X$, $j = 1, 2, \dots, n$. Therefore, if $\alpha \times \phi$ is minimal, then Theorem 4.10 applies. In particular, when $\alpha \times \phi$ is minimal, $C(X \times \mathbb{T}^n) \rtimes_{\alpha\times\phi} \mathbb{Z}$ is a unital simple C^*-algebra with tracial rank at most one. In the case where $n = 1$, [14, Lemma 4.2] provides a necessary and sufficient condition for $\alpha \times \phi$ being minimal (see also [25]).

5.2 Definition

Let $\{m_n\}$ be a sequence of integers with $m_n \geq 2$ and $m_n | m_{n+1}$. Let $\lambda_n: \mathbb{Z}/m_{n+1} \to \mathbb{Z}/m_n$ be the quotient map. The inverse limit $\lim_{\leftarrow} \mathbb{Z}/m_n$ is the Cantor set. The so-called odometer action α is defined by $\alpha(x) = x + 1$ for $x \in \lim_{\leftarrow} \mathbb{Z}/m_n$. Such action is always minimal. Moreover, the family $\{\alpha^k : k \in \mathbb{N}\}$ is equicontinuous on the Cantor set ([3, II.9.6.7]).

Lemma 5.1 *For each integer* $k \geq 2$, *there exists an odometer action* α *on the Cantor set such that* α^k *is minimal.*

Proof Fix $k \geq 2$. Choose a sequence of integers $\{m_n\}$ such that $(k, m_n) = 1$; *i.e.*, k and m_n are relatively prime and $m_n | m_{n+1}$, $n = 1, 2, \dots$. Fix $x \in \lim_{\leftarrow} \mathbb{Z}/m_n\mathbb{Z}$. We will show that $\{\alpha^{mk}(x) : m \in \mathbb{N}\}$ is dense. Let $y \in \lim_{\leftarrow} \mathbb{Z}/m_n\mathbb{Z}$. Fix $\epsilon > 0$. Since $\{\alpha^m : m \in \mathbb{N}\}$ is equicontinuous, there is $\delta > 0$ such that, for any pair of $z_1, z_2 \in \lim_{\leftarrow} \mathbb{Z}/m_n\mathbb{Z}$,

$$\mathrm{dist}(\alpha^m(z_1), \alpha^m(z_2)) < \epsilon/2 \quad \text{for all} \quad m \in \mathbb{N},$$

provided $\mathrm{dist}(z_1, z_2) < \delta$.

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There is an integer $j \geq 1$ and $x', y' \in \mathbb{Z}/m_j\mathbb{Z}$ such that $x_0 = \{x'_n\}, y_0 = \{y'_n\} \in \lim_{\leftarrow} \mathbb{Z}/m_n\mathbb{Z}$ and $x'_n = \gamma_{j,n}(x')$ and $y'_n = \gamma_{j,n}(y')$ for all $n < j$, where $\gamma_{j,n} = \gamma_n \circ \gamma_{n+1} \circ \cdots \circ \gamma_j$ and such that $\operatorname{dist}(x_0, x) < \delta$ and $\operatorname{dist}(y_0, y) < \delta$. We can assume that $\delta < \epsilon/2$. Since $(k, m_j) = 1$, there is $m \in \mathbb{N}$ such that $mk \equiv 1(m_j)$ or $mk \equiv -1(m_j)$. Since $-(m_j - 1) \equiv 1(m_j)$, in fact, in both case, there is an integer $l_1 \geq 1$ such that $l_1 k \equiv 1(m_j)$. We may assume that $y' = x' + m$ in $\mathbb{Z}/m_j\mathbb{Z}$. Then $y' = x' + ml_1k$ in $\mathbb{Z}/m_j\mathbb{Z}$. Then one computes that

$$\alpha^{ml_1k}(x_0) = x_0 + ml_1k = y_0.$$

It follows that

$$\begin{aligned}\operatorname{dist}(\alpha^{ml_1k}(x), y) &\leq \operatorname{dist}(\alpha^{ml_1k}(x), \alpha^{ml_1k}(x_0)) + \operatorname{dist}(\alpha^{ml_1k}(x_0), y)\\ &< \epsilon/2 + \operatorname{dist}(y', y) < \epsilon.\end{aligned}$$

■

The following is a result of K. Strung ([24, Proposition 2.1, Section 5]). We quote here for the convenience. Note that if Ω is connected, β^m is minimal for any non-zero integer m (see, for example, [3, II 9.6.7]).

Proposition 5.2 *Let α be an odometer action on the Cantor set and let Ω be compact metric space. Suppose that $\beta\colon \Omega \to \Omega$ is a minimal homeomorphism such that β^m is minimal for all $m \in \mathbb{N}$. Then $\alpha \times \beta$ is a minimal homeomorphism on $X \times \Omega$.*

Example 5.3 Let $\beta\colon S^{2n+1} \to S^{2n+1}$ $(n = 1, 2, \dots)$ be a minimal homeomorphism. It is known that such β exists. Fathi and Herman ([6]) showed that there exists a unique ergodic and minimal diffeomorphism on S^{2n+1}. The group $\mathbb{R}/\mathbb{Z}$ can act on S^{2n+1} freely as rotations. By a result of A. Windsor, there are minimal homeomorphisms β on S^{2n+1} such that β can have any number of ergodic measures ([28]). It follows from Proposition 5.2 that $\alpha \times \beta$ are minimal homeomorphism on $X \times S^{2n+1}$, where α is an odometer that has many invariant probability measures.

Corollary 5.4 *Let α be an odometer on the Cantor set and let (S^{2n+1}, β) be a minimal dynamical system with $n \geq 1$. Then $\alpha \times \beta$ is minimal and $A = C(X \times S^{2n+1}) \rtimes_{\alpha\times\beta} \mathbb{Z}$ has tracial rank at most one.*

Proof It follows from Example 5.3 that $\alpha \times \beta$ is minimal. Since β is minimal, it does not have a fixed point. Therefore, β has zero degree. If follows that $[\beta] = [\mathrm{id}]$ in $KK(C(S^{2n+1}), C(S^{2n+1}))$. Moreover, $U(C(S^{2n+1})) = U_0(C(S^{2n+1}))$. Thus Theorem 4.8 applies (with $k = 1$). ■

Example 5.5 Consider an $\mathbb{R}/\mathbb{Z}$ action on RP^{2n+1}. We identify PR^{2n+1} as $SO(2n)$. Define $\gamma\colon \mathbb{R}/\mathbb{Z} \to SO(2n)$ by

$$\gamma(t) = \begin{pmatrix} \cos(\pi t/2) & \sin(\pi t/2) & 0 & & \\ -\sin(\pi t/2) & \cos(\pi t/2) & 0 & & \\ 0 & 0 & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}.$$

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Define an action $\mathbb{R}/\mathbb{Z} \times SO(2n) \to SO(2n)$ by

$$\Gamma(t)(x) = \gamma(t)x \quad \text{for all} \quad t \in \mathbb{R}/\mathbb{Z} \quad \text{and} \quad x \in SO(2n).$$

It is clear that Γ is free and a C^∞ -diffeomorphism. For each $1/n^2 > \delta > 0$ and $r \in \mathbb{Q}/\mathbb{Z}$, by [28], there is a minimal diffeomorphism $\beta_r: SO(2n) \to SO(2)$ such that

$$\operatorname{dist}(\beta_r(x), \gamma(r)x) < \delta \quad \text{for all} \quad x \in SO(2n).$$

Corollary 5.6 *Let $n \geq 1$ be an integer. There are odometer actions α on the Cantor set such that for any minimal homeomorphism β on RP^{2n+1}, $A = C(X \times RP^{2n+1}) \rtimes_{\alpha\times\beta} \mathbb{Z}$ is a unital simple C^*-algebra with tracial rank at most one.*

Proof First it is well known that $H^1(C(RP^{2n+1}), \mathbb{Z}) = \{0\}$. In other words,

$$U\big(C(RP^{2n+1})\big) = U_0\big(C(RP^{2n+1})\big).$$

Note that

$$K_0(C(RP^{2n+1})) = \mathbb{Z} \oplus G_0 \quad \text{and} \quad K_1(C(RP^{2n+1})) = \mathbb{Z},$$

where G_0 is a finite group such that $2g = 0$ for all $g \in G_0$. Any automorphism on $K_0(C(RP^{2n+1}))$ induced by an automorphism on $C(RP^{2n+1})$ has the form

$$\begin{pmatrix} \mathrm{id}_{\mathbb{Z}} & 0 \\ \phi_{2,1} & \phi_{2,2} \end{pmatrix}, \tag{5.1}$$

where $\phi_{2,1}: \mathbb{Z} \to G_0$ and $\phi_{2,2}: G_0 \to G_0$ are homomorphisms, since it sends identity of $C(RP^{2n+1})$ to itself and G_0 is finite. Automorphisms of the form of (5.1) form a subgroup. It is a finite group; suppose that its order is k_1. Then for any automorphism $\phi: C(RP^{2n+1}) \to C(RP^{2n+1})$, $\phi_{*0}^{k_1} = \mathrm{id}_{K_0(C(RP^{2n+1}))}$.

We note that $H_0(RP^{2n+1}, \mathbb{Q}) = \mathbb{Q}$, $H_{2n+1}(RP^{2n+1}, \mathbb{Q}) = \mathbb{Q}$ and $H_i(RP^{2n+1}) = \{0\}$ for all other i. Also, $H_0(RP^{2n+1}, \mathbb{Z}) = \mathbb{Z}$ and $H_{2n+1}(RP^{2n+1}, \mathbb{Z}) = \mathbb{Z}$. Let

$$\widetilde{\beta}: C(RP^{2n+1}) \to C(RP^{2n+1})$$

be the isomorphism induced by β. Note that $\beta_* = \mathrm{id}$ on $H_0(RP^{2n+1}, \mathbb{Z}) = \mathbb{Z}$ and $\widetilde{\beta}_*(1) = \pm 1$ on $H_{2n+1}(RP^{2n+1}, \mathbb{Z}) = \mathbb{Z}$. Let

$$\begin{aligned} L_\beta &= \sum_{k\geq 0} (-1)^k \operatorname{Tr}\big(\beta_*|(H_k(RP^{2n+1}, \mathbb{Q}))\big) \\ &= \operatorname{Tr}\big(\mathrm{id}\,|(H_0(RP^{2n+1}, \mathbb{Q})\big) + (-1)^{2n+1} \operatorname{Tr}\big(\beta_*|(H_{2n+1}(RP^{2n+1}, \mathbb{Q}))\big) \end{aligned}$$

be the Lefschetz number. If β is minimal, it does not have a fixed point. So $L_\beta = 0$. It follows that $\beta_*(1) = 1$ on $H_{2n+1}(PR^{2n+1}, \mathbb{Z})$. It follows that, for any minimal homeomorphism β on RP^{2n+1}, $(\phi_\beta)_{*1} = \mathrm{id}_{K_1(C(RP^{2n+1}))}$. We compute that

$$K_0(C(RP^{2n+1}, \mathbb{Z}/2\mathbb{Z})) = \mathbb{Z}/2\mathbb{Z} \oplus G_0 \quad \text{and}$$

$$0 \to \mathbb{Z}/2\mathbb{Z} \to K_1(C(RP^{2n+1}, \mathbb{Z}/2\mathbb{Z}) \to G_0 \to 0.$$

Let k_2 be the order of $\operatorname{Aut}(\mathbb{Z}/2\mathbb{Z} \oplus G_0)$ (which is finite). One also checks, from the above, $K_1(C(RP^{2n+1}), \mathbb{Z}/2\mathbb{Z})$ is a finite abelian group such that $4x = 0$ for all $x \in K_1(C(RP^{2n+1}), \mathbb{Z}/2\mathbb{Z})$. Let k_3 be the order of $\operatorname{Aut}(K_1(C(RP^{2n+1})))$.

Define $k = k_1 \cdot k_2 \cdot k_3$, which depends on n only. Choose a sequence of integers $\{m_j\}$ such that $m_j | m_{j+1}$ for all j and each m_j is prime relative to K. Then by Lemma 5.1, there are odometer actions α on the Cantor set such that α^k is also minimal.

Now let β be a minimal homeomorphism on RP^{2n+1}. Then by the above,

$$
\begin{aligned}
&[(\widetilde{\beta})^k]|_{K_i(C(RP^{2n+1}))} = \mathrm{id}, \quad i = 0, 1 \quad \text{and} \\
&[\widetilde{\beta})^k]|_{K_i(C(RP^{2n+1}), \mathbb{Z}/2\mathbb{Z})} = \mathrm{id}, \quad i = 0, 1.
\end{aligned}
\tag{5.2}
$$

Note that

$$
KL\big(C(RP^{2n+1}), C(RP^{2n+1})\big) = \mathrm{Hom}_\Lambda \underline{K}\big(C(RP^{2n+1}), C(RP^{2n+1})\big)
$$

It follows from [2, 2.11] that, to check $[\widetilde{\beta}^k] = [\mathrm{id}]$, it suffices to show that (5.2) holds, since $2g = 0$ for all $g \in G_0$. Therefore,

$$
[\widetilde{\beta}^k] = [\mathrm{id}] \quad \text{in} \quad KL(C(RP^{2n+1}), C(RP^{2n+1})).
$$

By the assumption we also have that α^k is minimal. Hence Theorem 4.8 applies to $\alpha \times \beta$. ∎

6 Applications

In this section we consider $A = C(\Omega) \rtimes_\beta \mathbb{Z}$, where Ω is a connected compact metric space and β is a minimal homeomorphism on Ω. Specific examples are the cases where $\Omega = S^{2n+1}$ or $\Omega = RP^{2n+1}$, where $n \geq 1$. It should be noted that there are no minimal homeomorphisms on even spheres or even dimensional real projective spaces. Our results can also apply to other connected spaces.

Theorem 6.1 *Let Ω be a connected, compact, metric space with finite covering dimension such that $U(C(\Omega)) = U_0(C(\Omega))$ and let $\beta\colon \Omega \to \Omega$ be a minimal homeomorphism. Suppose that $[\widetilde{\beta}^k] = [\mathrm{id}]$ in $KL(C(\Omega), C(\Omega))$ for some integer $k \geq 1$, where $\widetilde{\beta}(f) = f \circ \beta^{-1}$ for all $f \in C(\Omega)$. Then $A = C(\Omega) \rtimes_\beta \mathbb{Z}$ has rationally tracial rank at most one;* i.e., *$A \otimes U$ has tracial rank at most one for any infinite dimensional UHF-algebra U. In particular, A is in $\mathcal{A}$.*

Proof First we note that since Ω has finite covering dimension, it follows from [27] that A has finite nuclear dimension. Let α be an odometer action on the Cantor set such that α^k is also minimal. It follows from Proposition 5.2 that $\alpha \times \beta$ is a minimal action. Let $B = C(X \times \Omega) \rtimes_{\alpha \times \beta} \mathbb{Z}$ and $C = C(X) \rtimes_\alpha \mathbb{Z}$.

It follows from Theorem 4.8 that B has tracial rank at most one. Consider the embedding $\imath\colon A \to B$ by sending $C(\Omega) \to C(X \times \Omega)$ and sending the implementing unitary to the implementing unitary in a natural way. Any tracial state τ of B is given by $\alpha \times \beta$-invariant Borel probability measure. Let τ_0 be the unique tracial state on $C = C(X) \rtimes_\alpha \mathbb{Z}$ that is given by the α-invariant Borel probability measure. Therefore, $K_0(C) = K^0(X, \alpha)$ has a unique state. Then each $\alpha \times \beta$-invariant tracial state on $C(X \times \Omega) = C(X) \otimes C(\Omega)$ has the form $\tau_0 \otimes \tau_1$, where τ_1 is a β-invariant tracial state on $C(\Omega)$. It follows that the map $\imath_\sharp: T(B) \to T(A)$ induced by $\imath$ is a homeomorphism. It follows from Lemma 2.6, since Ω is connected and $H^1(\Omega, \mathbb{Z}) = \{0\}$, that $\rho_B(K_0(B)) = \rho_C(K^0(X, \alpha))$ and that $\rho_A(K_0(A)) = \mathbb{Z}$. Therefore, if $\tau_0 \otimes \tau_1$ and $\tau_0 \otimes \tau_1'$ are two tracial states then they induce the same state on $K_0(A)$ as well as the same state on $K_0(B)$. It follows from [30, Theorem 4.2] that $A \otimes U$ has tracial rank at most one. It follows from [18] that $A \otimes U$ has tracial rank at most one for all infinite dimensional UHF-algebras U. Since $A = C(\Omega) \rtimes_\beta \mathbb{Z}$, it satisfies the Universal Coefficient Theorem. Furthermore, by [27], A is $\mathcal{Z}$-stable. Therefore, A is in the class of unital separable amenable simple C^*-algebras that are in $\mathcal{A}$. ∎

Theorem 6.2 *Let Ω be a connected compact metric space with finite covering dimension such that $H^1(\Omega, \mathbb{Z}) = \{0\}$ and $K_i(C(\Omega)) = \mathbb{Z} \oplus G_i$, where G_i is a finite group. Suppose that $\beta: \Omega \to \Omega$ is a minimal homeomorphism. Then $A = C(\Omega) \rtimes_\beta \mathbb{Z}$ has rational tracial rank at most one and is in $\mathcal{A}$.*

Proof This is a corollary of Theorem 6.1. We note that $U(C(\Omega)) = U_0(C(\Omega))$. Therefore, it suffices to show that $[\widetilde{\beta}^k] = [\mathrm{id}]$ in $KL(C(\Omega))$. Similar to the proof of Corollary 5.6, it is easy to see that there exists an integer $k_i \geq 1$ such that $((\widetilde{\beta})^{k_i})_{*i} = (\widetilde{\beta})_{*i}^{k_i} = \mathrm{id}_{K_i(C(\Omega))}$, $i = 0, 1$.

Let r_i be the order of G_i. For each $1 \leq j \leq (r_i)!$, there exists a short exact sequence

$$0 \to \mathbb{Z}/j\mathbb{Z} \oplus G_i/jG_i \to K_i(C(\Omega), \mathbb{Z}/j\mathbb{Z}) \to G_i^{(j)} \to 0,$$

where $G_i^{(j)} = \{g \in K_i(C(\Omega)) : jg = 0\}$, $i = 0, 1$. Therefore, $K_i(C(\Omega), \mathbb{Z}/j\mathbb{Z})$ is a finite group, $i = 0, 1$. Note that $[\widetilde{\beta}]|_{K_i(C(\Omega), \mathbb{Z}/j\mathbb{Z})} \in \mathrm{Aut}(K_i(C(\Omega), \mathbb{Z}/j\mathbb{Z}))$. However, $\mathrm{Aut}(K_i(C(\Omega), \mathbb{Z}/j\mathbb{Z}))$ is a finite group. Therefore, for some $m_{i,j} \geq 1$ with $i = 0, 1$, $[\widetilde{\beta}^{m_{i,j}}]|_{K_i(C(\Omega), \mathbb{Z}/j\mathbb{Z})} = \mathrm{id}_{K_i(C(\Omega), \mathbb{Z}/j\mathbb{Z})}$. Put

$$k = k_1 \cdot k_2 \cdot \prod_{\substack{1 \leq j \leq (r_i)! \\ i = 0, 1}} m_{i,j}.$$

One checks that $(\widetilde{\beta}^k)_{*i} = \mathrm{id}_{K_i(C(\Omega))}$ and $[\widetilde{\beta}^k]|_{K_i(C(\Omega), \mathbb{Z}/j\mathbb{Z})} = \mathrm{id}_{K_i(C(\Omega), \mathbb{Z}/j\mathbb{Z})}$, for $j = 1, 2, \ldots, (r_i)!$, $i = 0, 1$. Since r_i is the order of G_i, by [2, 2.11],

$$[\widetilde{\beta}^k] = [\mathrm{id}_{C(\Omega)}].$$ ∎

Corollary 6.3 *Let β be a minimal homeomorphism on S^{2n+1}. Then $A = C(S^{2n+1}) \rtimes_\beta \mathbb{Z}$ has rationally tracial rank at most one and is in $\mathcal{A}$.*

Proof As in Corollary 5.4, we note that $U(C(S^{2n+1})) = U_0(C(S^{2n+1}))$ and any minimal homeomorphism has the property $[\beta] = [\mathrm{id}]$. So Theorem 6.1 applies. ∎

Corollary 6.4 *Let $\beta_1, \beta_2: S^{2n+1} \to S^{2n+1}$ be two minimal homeomorphisms and let $A_i = C(S^{2n+1}) \rtimes_{\beta_i} \mathbb{Z}$, $i = 1, 2$. Then $A_1 \cong A_2$ if and only if $T(A_1) \cong T(A_2)$.*

Proof One computes, using the Pimsner–Voiculescu exact sequence ([21]), that $K_i(A_j) = \mathbb{Z} \oplus \mathbb{Z}$, $i = 0, 1$, and $j = 1, 2$. One also computes that the order of $K_0(A_j)$

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is determined by one copy of $\mathbb{Z}$ from the rank of projections of $M_k(C(S^{2n+1}))$ for all k and $K_0(A_1)$ and $K_0(A_2)$ are unital order isomorphic. Furthermore, all traces agree on $K_0(A_1) = K_0(A_2)$. Therefore their Elliott invariant is determined by $T(A_i)$, $i = 1, 2$. Now, by Corollary 6.3, the classification theorem in [10] applies. ∎

Corollary 6.5 *Let β be a minimal homeomorphism on RP^{2n+1} (for $n \geq 1$). Then $A = C(RP^{2n+1}) \rtimes \mathbb{Z}$ has rational tracial rank at most one and is in $\mathcal{A}$.*

Proof As in Corollary 5.6, we note that

$$U(C(RP^{2n+1})) = U_0(C(RP^{2n+1})) \quad \text{and} \quad K_0(C(RP^{2n+1})) = \mathbb{Z} \oplus G,$$

where G is a finite group and $K_1(C(RP^{2n+1})) = \mathbb{Z}$. Thus, Theorem 6.2 applies. ∎

Corollary 6.6 *Let β_1 and β_2 be two minimal homeomorphisms on RP^{2n+1} (for $n \geq 1$) and let $A_i = C(RP^{2n+1}) \rtimes_{\beta_i} \mathbb{Z}$, $i = 1, 2$. Then $A_1 \cong A_2$ if and only if*

$$K_1(A_1) \cong K_1(A_2), \quad (\beta_1)_* = (\beta_2)_* \quad \textit{on} \quad K_0(C(RP^{2n+1})) \quad \textit{and} \quad T(A_1) = T(A_2).$$

Proof By Corollary 6.5, it suffices to show that A_1 and A_2 have the same Elliott invariant. The assumption shows that

$$K_0(C(RP^{2n+1}))/\{z - z \circ (\beta_1)_* : z \in K_0(C(RP^{2n+1}))\} \cong \\ K_0(C(RP^{2n+1}))/\{z - z \circ (\beta_2)_* : z \in K_0(C(RP^{2n+1}))\} \cong \mathbb{Z} \oplus G_0',$$

where G_0' is a quotient of $\mathrm{Tor}(K_0(C(RP^{2n+1})))$. Moreover, they are order isomorphic. By the Pimsner–Voiculescu exact sequence, we may write

$$K_0(A_1) = (\mathbb{Z} \oplus G_0') \oplus \mathbb{Z} \cong K_0(A_2).$$

Since $H^1(RP^{2n+1}, \mathbb{Z}) = \{0\}$, it follows that

$$\rho_{A_1}(K_0(A_1)) = \rho_{A_1}(\mathbb{Z} \oplus G_0') = \mathbb{Z} \quad \text{and} \quad \rho_{A_2}(K_0(A_2)) = \mathbb{Z}.$$

It follows that $K_0(A_1)$ and $K_0(A_2)$ are unital order isomorphic. Since all traces of A_i agree on $K_0(A_i)$, $i = 1, 2$. It follows that A_1 and A_2 have isomorphic Elliott invariant. Thus, [10] applies. ∎

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Department of Mathematics, East China Normal University, Shanghai, China 20062,
and
Department of Mathematics, University of Oregon, Eugene, Oregon 97402, USA
e-mail: hlin@uoregon.edu