CENTRALIZING AUTOMORPHISMS OF PRIME RINGS

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ABSTRACT. Let R be a prime ring and T be a nontrivial automorphism of R. If \( xx^T - x^T x \) is in the center of the ring for every \( x \) in \( R \), then \( R \) is a commutative integral domain.

An additive mapping \( L \) of a ring \( R \) to itself is called centralizing if \( x(L) - (L)x \) is in the center of \( R \) for every \( x \) in \( R \). In [4] Posner showed that a prime ring must be commutative if it has a nontrivial centralizing derivation (see [1] for another proof). In this note the analogous result for a centralizing automorphism is proved.

THEOREM. If \( R \) is a prime ring with a nontrivial centralizing automorphism, then \( R \) is a commutative integral domain.

This generalizes the results of Divinsky [2] and Luh [3]. Divinsky showed that a simple ring is commutative if it has a nontrivial automorphism \( T \) such that \( xx^T = x^T x \) for all \( x \) in the ring and Luh extended this result to prime rings.

Let \( [x, y] = xy - yx \) and note that \( [x, yz] = y[x, z] + [x, y]z \). Assume that \( R \) is a prime ring and let \( Z \) be the center of \( R \). The next two lemmas will be used in the proof of the theorem.

**LEMMA 1.** [3] Let \( T \) be a nontrivial automorphism of \( R \). If \( [x, x^T] = 0 \) for all \( x \) in \( R \), then \( R \) is commutative.

**Proof.** Linearizing \( [x, x^T] = 0 \) gives \( [x, y^T] = [x^T, y] \) and thus \( [x, (xy)^T] = [x^T, xy] \). But \( [x, (xy)^T] = x^T [x, y^T] \) and \( [x^T, xy] = x[x^T, y] = x[x, y^T] \). Thus \( (x - x^T)[x, y^T] = 0 \) and since \( T \) is an automorphism \( (x - x^T)[x, z] = 0 \) for all \( x \) and \( z \) in \( R \). Since \( y[x, z] = [x, yz] - [x, y]z \), \( (x - x^T)R[x, z] = 0 \). If \( x \neq x^T \), then \( x \) is in the center since \( R \) is prime. Since \( T \) is nontrivial, there must be at least one \( x \) such that \( x \neq x^T \). Suppose \( y \) is not in the center of \( R \). Then \( x + y \) is not in the center and \( y^T = y \), \( (x + y)^T = x + y \). But then \( x = x^T \) which is a contradiction. Hence \( R \) is commutative.

**LEMMA 2.** If \( xy = 0 \) and \( x \) is a nonzero element in \( Z \), then \( y = 0 \).

**Proof.** If \( xy = 0 \), then \( zxy = xzy = 0 \) for all \( z \) in \( R \). Since \( R \) is prime, and \( x \neq 0 \), \( y \) must be 0.

**Proof of the theorem.** Let \( T \) be a nontrivial automorphism of \( R \) such that \( [x, x^T] \) is in \( Z \) for all \( x \) in \( R \). The proof will consist of showing that \( [x, x^T] = 0 \) for
all \( x \) in \( R \) and then using Lemma 1 to conclude that \( R \) is commutative. Linearization of \([x, x^T]\) in \( Z \) gives

(1) \[ [x, y^T] + [y, x^T] \text{ is in } Z \text{ for all } x \text{ and } y \text{ in } R, \]

and thus

(2) \[ [x, [x, y^T] + [y, x^T]] = 0 \text{ for all } x \text{ and } y \text{ in } R. \]

Now \( R \) is a prime ring so \( R \) is either of characteristic two or \( 2x = 0 \) implies \( x = 0 \) for \( x \) in \( R \).

Suppose \( R \) is not of characteristic two and let \( y = x^2 \) in (2). Then \( 0 = [x, x^T] + [x^2, x^T] = [x, 2x^T] + [x, 2x] - 2[x, x^T]^2 \). Hence \( [x, x^T]^2 = 0 \). By Lemma 2 \([x, x^T] = 0 \) for all \( x \) in \( R \) and thus \( R \) is commutative.

Now suppose that \( R \) is of characteristic two. Then \( [x^2, x^T] = 2x[x, x^T] = 0 \) and \([x^2, x^T] = 2x^T[x, x^T] = 0. \) Let \( y = x^T \) in (1), then \([x, x^T] + [x^T, x^T] = [x, x^T]^2 \) is in \( Z \). Using the Jacobi identity (2) can be rewritten as

(3) \[ [x, [y^T, x]] + [x^T, [x, y]] = 0. \]

Letting \( y = x^3 x^T \) in (3) gives

(4) \[ [x, [(x^3 x^T)^T, x]] + [x^T, [x, x^3 x^T]] = 0. \]

Now \([x, [(x^3 x^T)^T, x]] = [x, (x^3 x^T)^T x + x(x^3 x^T)^T] = [x^2, (x^3 x^T)^T]. \) But expanding the last commutator gives

\[
x[x, (x^3 x^T)^T] + [x, (x^3 x^T)^T]x
\]

\[= x(x^3 x^T)^2[x, x^T] + x[x, (x^T)^3][x, x^T] + x(x^T)^3[x, x^T] x + [x, (x^T)^3][x, x^T] x
\]

since

\[[x, (x^T)^2] = 0.\]

Hence

\[[x, [(x^3 x^T)^T, x]] = [x, (x^T)^3][x, x^T] + [x^T, [x, x^T]] = 2[x, (x^T)^3][x, x^T] = 0.\]

Thus (4) reduces to

(5) \[ [x^T, [x, x^3 x^T]] = 0. \]

But then \( 0 = [x^T, x^3[x, x^T]] = [x^T, x^3][x, x^T] \) and using \([x^T, x^2] = 0 \) results in

(6) \[ x^2[x, x^T]^2 = 0 \text{ for all } x \text{ in } R. \]

By Lemma 2, if \([x, x^T] \neq 0, \) then \( x^2 = 0. \) So assume \( x^2 = 0, \) then \((x^T)^3 = 0 \) and \((x^T)^2 = 0. \) Now \((x^T)^2 = 0^T = 0 \) and \([x, x^T] = xx^T + x^T x = z \) for some \( z \) in \( Z. \)

Therefore \((xx^T + z)(xx^T) = 0 \) and thus \((xx^T)^3 = z(xx^T). \) If \((xx^T)^2 = 0, \) then \( z = 0 \) or \( xx^T = 0. \) But if \( xx^T = 0, \) then \([x, x^T] = 0 \) and hence \([x, x^T] = 0 \) or \( x = 0. \) So from now on, assume that \( x^2 = 0 \) and \((x^T)^2 \neq 0. \)
Now (6) with $xx^T$ replacing $x$ implies that $[xx^T, (xx^T)^T]=0$. Expanding gives $x[x^T, x^T x TT] + [x, x^T x TT] x^T = 0$. If this equation is left multiplied by $x$, then $x[x, x^T x TT] x^T = 0$ and so $xx^T [x, x^T x TT] x^T + x[x, x^T] x^T = 0$. But $xx^T [x, x^T x TT] x^T = x(x^T)^2 [x, x^T x TT] = 0$. Thus $x[x, x^T] x^T x TT = [x, x^T] x x^T x TT = 0$. If $[x, x^T] = 0$, then $xx^T x TT = 0$.

Thus $[x, x^T x TT] x^T = x TT x x^T$, and so $xx^T (x^T)^2 [x, x^T] x^T = 0$. Hence if $[x, x^T] = 0$, then $x^T x = 0$. But this forces $x^T x = 0$ and so $x = 0$ or $[x, x^T] = 0$.

Suppose then that $[x, x^T] = 0$. Letting $y = xx^T$ in (2) results in $[x, [x^T, xx^T]] + [x, (xx^T)^T] = 0$. Thus $x TT [x, x^T] x^T + [x, x^T] x x^T x TT] = 0$. But then $[x, x^T]^2 + 2[x, x^T] [x, x^T] = [x, x^T]^2 = 0$. Therefore $[x, x^T] = 0$ for all $x$ in $R$ and by Lemma 1, $R$ is commutative.

REFERENCES