THE DOUBLE-SIX OF LINES OVER $PG(3, 4)$

JAMES W. P. HIRSCHFELD

(received 14 June, 1963)

1. Introduction

The object of this paper is to consider the existence of the double-six over $GF(2^n)$ and particularly over $GF(4)$.

There is a neat representation in [5] of the double-six over $GF(4)$, of the 27 lines of the cubic surface on which it lies, and of those 36 linear complexes for which the 36 double-sixes on this surface are self-polar.

2. Existence of the double-six

In [3], any 4 lines have 2 transversals. Therefore given 5 skew lines a_1, a_2, a_3, a_4, a_5 with a transversal b_6, there exist lines b_1, b_2, b_3, b_4, b_5 such that b_i is the second transversal besides b_6 of a_j, a_k, a_l, a_m. Then the lines b_1, b_2, b_3, b_4, b_5 have a transversal a_6, and there exists a unique quadric with respect to which the double-six is self-polar. Many proofs have been given of this theorem over the real and complex fields, e.g. Baker [1] p. 159. The proofs of this theorem are valid also over the other infinite fields and the finite fields of more than four elements which are not of characteristic two.

Over any field consider the skew hexagon $a_1 b_3 a_2 b_1 a_3 b_2$ with vertices $A_{ij} = a_i \cap b_j$. Without loss of generality, take the syzygies among the six points as

$$A_{12}+A_{23}+A_{31}+A_{21}+A_{13}+A_{32} = 0$$
$$m_3 A_{12}+m_1 A_{23}+m_2 A_{31}+l_3 A_{21}+l_2 A_{13}+l_1 A_{32} = 0$$

If the hexagon is self-polar with respect to a linear complex then each point A_{ij} lies in the opposite plane $a_j b_i$; so a linear relation is required among sets of 4 points like $A_{12}, A_{23}, A_{21}, A_{31}$. Therefore $l_1 = l_2$. Similarly $l_1 = l_2 = l_3$, $m_1 = m_2 = m_3$. Therefore $A_{12}+A_{23}+A_{31} = 0$, $A_{21}+A_{13}+A_{32} = 0$, i.e. sets of three alternate vertices are collinear. If the reference system is taken so that the vertices of the hexagon $A_{13}, A_{12}, A_{31}, A_{21}$ are X_0, X_1, X_2, X_3 respectively and

$$A_{32} = (1, 0, 0, 1), \quad A_{23} = (0, 1, 1, 0),$$

then the equation of the complex is $p_{01}+p_{23} = 0$. Thus there is a unique linear complex with respect to which the hexagon is self-polar.

https://doi.org/10.1017/S144678870002276X Published online by Cambridge University Press

Over $GF(2^n)$, if two points $X(x_0x_1x_2x_3)$ and $Y(y_0y_1y_2y_3)$ are conjugate with respect to

$$\sum_{i\leq j} a_{ij}x_ix_j = 0$$

then

$$\sum_{i<j} a_{ij}(x_iy_j+x_jy_i) = 0.$$

There is no condition on the a_{ii}; so there can be no unique quadric for self-polarity of the hexagon, but with the conditions stated, there is a unique linear complex.

Over any field, the coordinates of the line through $X(x_0x_1x_2x_3)$ and $Y(y_0y_1y_2y_3)$ are p_{01}, p_{02}, p_{03}, p_{12}, p_{31}, p_{23} where $p_{ij} = x_iy_j-x_jy_i$ and $p_{01}p_{23}+p_{02}p_{31}+p_{03}p_{12} = 0$. Such line coordinates uniquely determine a line and are uniquely determined by the line. If two points are conjugate with respect to a linear complex the line joining them is self-polar and lies in the complex. If two lines a, b or a point A and a plane α are polar with respect to a linear complex C, denote these by $a\mathscr{C}b$, $A\mathscr{C}\alpha$.

If r, s are two lines, their mutual invariant is

$$\varpi(r, s) = r_{01}s_{23}+r_{02}s_{31}+r_{03}s_{12}+r_{12}s_{03}+r_{31}s_{02}+r_{23}s_{01}.$$

r, s intersect if, and only if, $\varpi(r, s) = 0$.

Let C be the non-special linear complex

$$a_{01}p_{01}+a_{02}p_{02}+a_{03}p_{03}+a_{12}p_{12}+a_{31}p_{31}+a_{23}p_{23} = 0$$

with $a_{01}a_{23}+a_{02}a_{31}+a_{03}a_{12} = 1$.

Put $\mathscr{R} = \sum a_{ij}r_{ij}$, $\mathscr{S} = \sum a_{ij}s_{ij}$ where r_{ij}, s_{ij} are coordinates of the lines r, s. If $r\mathscr{C}s$ then $r_{ij}+s_{ij} = \mathscr{R}a_{kl}$ as shown by Baker [1] p. 64 in a proof which is valid over any field.

If r and s are conjugate, i.e. r meets s', the polar of s, then if s'_{ij} are the coordinates of s',

$$r_{01}s'_{23}+r_{02}s'_{31}+r_{03}s'_{12}+r_{12}s'_{03}+r_{31}s'_{02}+r_{23}s'_{01} = 0.$$

But $s_{ij}+s'_{ij} = \mathscr{S}a_{kl}$, so $\varpi(r, s) = \mathscr{R}\mathscr{S}$.

The mutual invariant of the polars of two lines r, s is

$$\sum (\mathscr{R}a_{ij}-r_{kl})(\mathscr{S}a_{kl}-s_{ij}) = \varpi(r, s).$$

Now take a skew hexagon $a_1b_3a_2b_1a_3b_2$ with sets of alternate vertices collinear and find the unique linear complex C such that $a_i\mathscr{C}b_i$, $i = 1, 2, 3$. Let b_6 meet a_1, a_2, a_3 and a_4, a_5 meet b_1, b_2, b_3, b_6. Select b_4, b_5 such that $a_i\mathscr{C}b_i$, $i = 4, 5$, and select a_6 such that $a_6\mathscr{C}b_6$. Then b_4, b_5 meet a_1, a_2, a_3, a_6 and a_6 meets b_1, b_2, b_3, b_4, b_5. Let $\varpi_{ij} = \varpi(a_i, a_j) = \varpi(b_i, b_j)$ and if C is $\sum c_{ij}p_{ij} = 0$ put $\mathscr{A}_i = \sum c_{jk}a^i_{jk}$ where a^i_{jk} are the line coordinates of a_i.

We have then that $\varpi_{ij} = \mathscr{A}_i \mathscr{A}_j$ except for $ij = 45$. To complete the theorem, a_4 must be proved conjugate to a_5 i.e. $\varpi_{45} = \mathscr{A}_4 \mathscr{A}_5$.

Consider $W = |(\varpi_{ij})|$, $i, j = 1, \cdots, 5$, evaluated over the complex field. Since (ϖ_{ij}) is symmetric with diagonal elements zero, $W = 2w$. Over any field the condition that the a_i, $i = 1, \cdots, 5$ have a transversal is $w = 0$: the formula for W is given by Todd [7], p. 145, ex. 41. Substituting in this identity the values for the ϖ_{ij} excluding ϖ_{45}, gives the result that $\varpi_{45} = 3\mathscr{A}_4 \mathscr{A}_5$. Therefore such a double-six exists over $GF(2^n)$ and only over $GF(2^n)$, $n \geqq 2$. I point out that there is no degeneracy of the figure since the condition that four lines, for example a_1, a_2, a_3, a_4, should have two distinct transversals is $|(\varpi_{ij})| \neq 0$, $i, j = 1, \cdots, 4$, as in Todd [7], p. 145, ex. 35. Now $|(\varpi_{ij})| = \mathscr{A}_1^2 \mathscr{A}_2^2 \mathscr{A}_3^2 \mathscr{A}_4^2$ and thus the condition is satisfied. Similarly the other conditions for non-degeneracy are satisfied.

3. Existence of the double-six over $GF(4)$

I will now prove that over $GF(4)$ every skew hexagon which gives rise to a non-degenerate double-six has the property that sets of alternate vertices are collinear.

Consider a general skew hexagon over $GF(4)$.

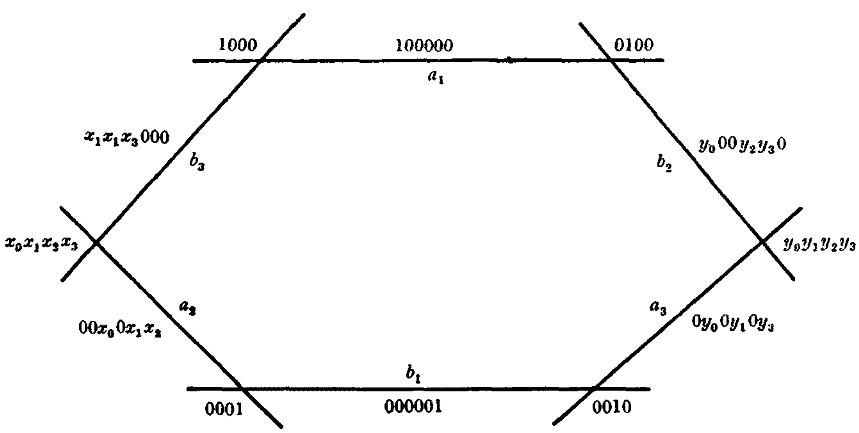

Fig. 1

For the hexagon to be skew, the following pairs of lines must be skew:

$$b_1b_3,\ a_1a_2,\ b_1b_2,\ a_1a_3,\ a_2a_3,\ a_2b_2,\ a_3b_3,\ b_2b_3.$$

The conditions respectively are:

$$x_1, x_2, y_0, y_3, r_{01}, r_{02}, r_{13}, r_{23} \neq 0, \quad \text{where} \quad r_{ij} = x_i y_j + x_j y_i.$$

There are 3 lines b_μ meeting a_1, a_2, a_3 in no point of the hexagon, where

https://doi.org/10.1017/S144678870002276X Published online by Cambridge University Press

$b_\mu: (\mu x_0+x_1)(\mu y_0+y_1), x_2(\mu y_0+y_1), y_3(\mu x_0+x_1), \mu x_2(\mu y_0+y_1), \mu y_3(\mu x_0+x_1), 0.$

The three lines correspond to $\mu = 1, \omega, \omega^2$

Let them be b_4, b_5, b_6. Similarly a_4, a_5, a_6 are defined by

$a_\lambda: 0, \lambda y_0(x_2+\lambda x_3), y_0(x_2+\lambda x_3), \lambda x_1(y_2+\lambda y_3), x_1(y_2+\lambda y_3), (x_2+\lambda x_3)(y_2+\lambda y_3).$

Since b_μ is required not to meet b_1, $(\mu x_0+x_1)(\mu y_0+y_1) \neq 0$ and so $\mu x_0+x_1 \neq 0$, $\mu y_0+y_1 \neq 0$. If m is in $GF(4)$, m^3 is either 0 or 1. Now $\mu x_0 \neq x_1$ ($\mu = 1, \omega, \omega^2$), so that $x_0^3 \neq x_1^3$. Since $x_1 \neq 0$, $x_0 = 0$. Also since $y_0 \neq 0$, $y_1 = 0$.

Similarly since a_λ does not meet a_1, $(x_2+\lambda x_3)(y_2+\lambda y_3) \neq 0$ for $\lambda = 1$, ω, ω^2; but $x_2 \neq 0$, $y_3 \neq 0$, therefore $x_3 = 0$, $y_2 = 0$. So the sets of alternate vertices of the hexagon are

$$\begin{array}{lll} 1000, & 0001, & y_0 0 0 y_3 \\ 0100, & 0x_1x_20, & 0010. \end{array}$$

Therefore the two sets of three alternate vertices are collinear.

The a_λ, b_μ satisfy all the conditions required to form a double-six; in particular

$$\begin{aligned} \varpi(a_\lambda, b_\mu) &= \lambda^2 x_1^2 y_3^2 + \lambda\mu x_1 x_2 y_0 y_3 + \mu^2 x_2^2 y_0^2 \\ &= (\lambda x_1 y_3 + \omega\mu x_2 y_0)(\lambda x_1 y_3 + \omega^2 \mu x_2 y_0). \end{aligned}$$

Therefore $\varpi(a_\lambda, b_\mu) = 0$ gives a (2, 2) correspondence between the a_λ and b_μ.

We have shown that there exist double-sixes over $GF(4)$ and every one is self-polar with respect to a unique linear complex. Also, over $GF(4)$ the double-six is uniquely determined by the skew hexagon. This is not true over any other field.

It is easily seen from the above exposition that over $GF(2^n)$, $n > 2$, there exist double-sixes which are not self-polar.

4. The geometry of the double-six and its associated cubic surface in $PG(3, 4)$

$PG(3, 4)$ contains 85 points, 357 lines, 85 planes. We have established the existence of the double-six in the smallest field in which it could be defined. There are 12 lines a_i, b_i, $i = 1, \cdots, 6$, such that a_i does not meet a_j, b_i does not meet b_j, a_i does meet b_j, $i \neq j$; and $a_i \mathscr{C} b_i$ for some unique linear complex C.

Denote the plane containing a_i, b_j by $[a_i, b_j]$ and the intersection of a_i, b_j by (a_i, b_j). Let c_{ij} be the intersection of $[a_i, b_j]$, $[a_j, b_i]$, and c'_{ij} the join of (a_i, b_j), (a_j, b_i). c'_{12} is the join of a pair of opposite vertices of the hexagon $a_1 b_3 a_2 b_1 a_3 b_2$. But $[a_1, b_2]$, $[a_2, b_1]$ both contain this pair of vertices in consequence of the collinearities of alternate vertices. Therefore $c_{12} = c'_{12}$ and $c_{ij} = c'_{ij}$ from any hexagon ijk.

https://doi.org/10.1017/S144678870002276X Published online by Cambridge University Press

Hence the lines c_{ij} lie in C.

As in complex space c_{ij} meets a_i, b_j and c_{ij} meets c_{kl} for $k, l \neq i, j$. So we have 6 lines a_i, 6 lines b_i, 15 lines c_{ij}. Each line contains 5 points. Each point is on 3 lines. Each line meets 10 others. Thus the 27 lines contain $27 \times 5 \div 3 = 45$ points.

The unique cubic surface determined by the double-six consists only of the 45 points lying on the 27 lines.* As in complex space, the 27 lines form 36 double-sixes of the types

D	a_1	a_2	a_3	a_4	a_5	a_6	1
	b_1	b_2	b_3	b_4	b_5	b_6	
D_{12}	a_1	b_1	c_{23}	c_{24}	c_{25}	c_{26}	${}^6C_2 = 15$
	a_2	b_2	c_{13}	c_{14}		c_{16}	
D_{123}	a_1	a_2	a_3	c_{56}	c_{46}	c_{45}	${}^6C_3 = 20$
	c_{23}	c_{13}	c_{12}	b_4	b_5	b_6.	

Each of the 36 double-sixes has a unique non-special linear complex with respect to which it is self-polar. Let the linear complexes corresponding to the double-sixes D, D_{ij}, D_{ijk} be δ, δ_{ij}, δ_{ijk}.

Each linear complex C consists of 85 lines, which are the self-polar lines of C. 20 lines of C meet any line of C. Thus there are 5 lines of C through every point of every line of C. Thus the lines of C contain all 85 points of the space. The structure of C can be completed from the fact that if r is a line of C and p, q are polar lines not in C and r meets p, then r meets q.

A special linear complex consists of the 100 lines meeting a fixed line.

The surface is cut by 45 tritangent planes:

$30[a_i, b_j, c_{ij}]$ containing the point (a_i, b_j, c_{ij})

$15[c_{ij}, c_{kl}, c_{mn}]$ containing the point (c_{ij}, c_{kl}, c_{mn}).

The 45 points are all Eckardt points (points of concurrence of 3 coplanar lines, described by Segre [6] 3) of the surface.

The cubic surface is in fact a diagonal surface first discovered by Clebsch. In complex space, the diagonal surface has only 10 Eckardt points. Furthermore each of the double-sixes is of the type described by Burnside [2]. This can be seen from their description by Baker [1], p. 168. Burnside showed that such a double-six is invariant under a group of 120 collineations and that all such double-sixes are projectively equivalent. In $PG(3, 4)$ the non-singular cubic surface has a projective group $A(4, 3)$ of order 25,920 as shown by Frame [4]; this can also be calculated directly without much difficulty. Thus the set of 36 double-sixes is transitive under this

* To see that the surface contains no extra points, consider the 5 planes through a_1. They cover the space and meet the surface in the cubic curves $a_1 b_j c_{1j}$, $j = 2, \cdots, 6$. Thus there can be no points on the surface besides the 45 lying on the 27 lines.

https://doi.org/10.1017/S144678870002276X Published online by Cambridge University Press

group of the surface. Therefore each double-six has group of order 720. This group is S_6.

The double-sixes of $PG(3, 4)$ are hence all of the Burnside type. The group of $PG(3, 4)$ has order $\frac{1}{3}(4^4-1)(4^4-4)(4^4-4^2)(4^4-4^3) = 2^{12}\cdot 3^4\cdot 5^2\cdot 7\cdot 17$. The group of any double-six has order $720 = 2^4\cdot 3^2\cdot 5$. Therefore the total number of double-sixes in $PG(3, 4)$ is $2^8\cdot 3^3\cdot 5\cdot 7\cdot 17 = 1,370,880$.

5. A representation in [5]

A line $p(p_{01}p_{02}p_{03}p_{12}p_{13}p_{23})$ has dual line coordinates $\pi_{ij} = \rho p_{i'j'}$. Consider the representation in [5] where lines of [3] become tangent primes to a quadric Ω in [5] and non-special linear complexes become points of [5] not on Ω as follows: The line $(p_{01}p_{02}p_{03}p_{12}p_{13}p_{23})$ becomes the prime

$$\pi_{01}x_0+\pi_{02}x_1+\pi_{03}x_2+\pi_{12}x_3+\pi_{13}x_4+\pi_{23}x_5 = 0.$$

This is a tangent prime to $\Omega : x_0x_5+x_1x_4+x_2x_3 = 0$ with point of contact $(p_{01}p_{02}p_{03}p_{12}p_{13}p_{23})$. The linear complex

$$C : a_{01}\pi_{01}+a_{02}\pi_{02}+a_{03}\pi_{03}+a_{12}\pi_{12}+a_{13}\pi_{13}+a_{23}\pi_{23} = 0$$

becomes the point $(a_{01}a_{02}a_{03}a_{12}a_{13}a_{23})$.

By an earlier result, if two lines p, q in [3] are polar with respect to C

$$p_{ij}+q_{ij} = \lambda a_{ij}.$$

If p, q are two intersecting lines of [3] then the point of contact of the tangent prime representing q lies in the tangent prime representing p.

Let the primes representing a_i, b_i, c_{ij} be $\mathfrak{A}_i$, $\mathfrak{B}_i$, $\mathfrak{C}_{ij}$ with points of contact $\mathfrak{a}_i$, $\mathfrak{b}_i$, $\mathfrak{c}_{ij}$ and the points representing δ, δ_{ij}, δ_{ijk} be $\mathfrak{d}$, $\mathfrak{d}_{ij}$, $\mathfrak{d}_{ijk}$. Consider the double-six D. The row $a_1a_2a_3a_4a_5a_6$ is represented by the simplex with faces $\mathfrak{A}_i$ and vertices $\mathfrak{b}_i$. The row $b_1b_2b_3b_4b_5b_6$ is represented by the simplex with faces $\mathfrak{B}_i$ and vertices $\mathfrak{a}_i$. Thus the two rows of the double-six are represented by two simplexes inscribed and circumscribed to each other and to the quadric primal, and in perspective from the point $\mathfrak{d}$. Similarly for the other 35 double-sixes.

This figure of 63 points $\mathfrak{a}_i$, $\mathfrak{b}_i$, $\mathfrak{c}_{ij}$, $\mathfrak{d}$, $\mathfrak{d}_{ij}$, $\mathfrak{d}_{ijk}$ is isomorphic with the entire space $PG(5, 2)$ as described by Edge [3]. All the 651 linear relations can be obtained from the polar properties like $\mathfrak{a}_i+\mathfrak{b}_i = \mathfrak{d}$ and the relations dependent upon the Eckardt points like $\mathfrak{a}_i+\mathfrak{b}_j = \mathfrak{c}_{ij}$.

Define $x \circ y = \sum_{i=0}^{i=5} x_iy_{5-i}$ where

$$x = (x_0x_1x_2x_3x_4x_5), \quad y = (y_0y_1y_2y_3y_4y_5).$$

Then the intersection properties of the lines of [3] can be expressed by

https://doi.org/10.1017/S144678870002276X Published online by Cambridge University Press

$\mathfrak{a}_i \circ \mathfrak{b}_j = 0$, etc. With the linear properties these give relations like $\mathfrak{d} \circ \mathfrak{d}_{ij} = 0$, which implies that exactly 5 out of the 16 linear complexes δ, δ_{ij} are independent.

A linear property of the $\mathfrak{d}$'s is also apparent:

$$\mathfrak{d} = \mathfrak{a}_1 + \mathfrak{b}_1, \quad \mathfrak{d}_{123} = \mathfrak{a}_1 + \mathfrak{c}_{23}, \quad \mathfrak{d}_{456} = \mathfrak{b}_1 + \mathfrak{c}_{23},$$

whence

$$\mathfrak{d} + \mathfrak{d}_{123} + \mathfrak{d}_{456} = 0.$$

The existence of this line is equivalent to the existence in [3] of a Steiner trihedral pair. This line with the lines $\mathfrak{d}_{12} + \mathfrak{d}_{13} + \mathfrak{d}_{23} = 0$, $\mathfrak{d}_{45} + \mathfrak{d}_{46} + \mathfrak{d}_{56} = 0$ generate the [5]; this fact is equivalent to one of forty triads of Steiner trihedral pairs providing a trichotomy of the 27 lines, viz.

$$\begin{matrix} c_{14} & c_{25} & c_{36} \\ c_{26} & c_{34} & c_{15} \\ c_{35} & c_{16} & c_{24} \end{matrix} \qquad \begin{matrix} c_{56} & b_6 & a_5 \\ a_6 & c_{46} & b_4 \\ b_5 & a_4 & c_{45} \end{matrix} \qquad \begin{matrix} c_{23} & b_3 & a_2 \\ a_3 & c_{13} & b_1 \\ b_2 & a_1 & c_{12}. \end{matrix}$$

These results may be elaborated, and others formulated by further exposition of these linear and multiplicative properties of the coordinates. In fact all the properties related to the cubic surface in $PG(3, 4)$ can be simply expressed in $PG(5, 4)$ by the multiplication $x \circ y = 0$ and the collinearity $x+y+z = 0$ as previously defined.

These properties are very similar to the results in complex space as shown in Room [5].

The figure of 63 points is symmetric and by view of its isomorphism with $PG(5, 2)$ has an automorphism group isomorphic to the collineation group of $PG(5, 2)$ which has order $2^{15} \cdot 3^4 \cdot 5 \cdot 7^2 \cdot 31$.

References

[1] Baker, H. F., *Principles of Geometry*, Vol. III (Cambridge, 1923).

[2] Burnside, W., On the double-six which admits a group of 120 collineations into itself, *Proc. Camb. Phil. Soc.* 16 (1911), p. 418.

[3] Edge, W. L., Quadrics over $GF(2)$ and their relevance for the cubic surface group, *Canadian J. Maths.* 11 (1959), p. 625.

[4] Frame, J. S., A symmetric representation of the 27 lines on a cubic surface by lines of a finite geometry, *Bull. Amer. Math. Soc.*, 44 (1938), p. 658.

[5] Room, T. G., The Schur quadrics of a cubic surface, *J. London Math. Soc.*, 7 (1932), p. 147.

[6] Segre, B., *The non-singular cubic surfaces* (Oxford, 1942).

[7] Todd, J. A., *Projective and Analytical Geometry*, (Pitman, London, 1958).

University of Sydney.

https://doi.org/10.1017/S144678870002276X Published online by Cambridge University Press