BULL. AUSTRAL. MATH. SOC. 10F05
VOL. 19 (1978), 457-466.

Linear approximation by primes

Kee-Wai Lau and Ming-Chit Liu

In this present paper we shall prove the following. Suppose that $\lambda_1, \lambda_2, \lambda_3$ are any non-zero real numbers not all of the same sign and that λ_1/λ_2 is irrational. If η is any real number and $0 < \alpha < 1/9$, then there are infinitely many prime triples (p_1, p_2, p_3) for which

$$|\eta+\lambda_1p_1+\lambda_2p_2+\lambda_3p_3| < \left(\max p_j\right)^{-\alpha} .$$

1. Introduction

Suppose that $\lambda_1, \ldots, \lambda_s$ are any non-zero real numbers, not all of the same sign and not all in rational ratio. In 1946, Davenport and Heilbronn [3] proved that if k is a positive integer and $s \geq 2^k + 1$, then for any $\varepsilon > 0$ the inequality

$$\left|\sum_{j=1}^{s} \lambda_j n_j^k\right| < \varepsilon \tag{1.1}$$

has infinitely many solutions in integers $n_j \geq 1$. This result sparked off a series of investigations (for information, see the introductions in [9], [10], [11]). Schwarz [8] was able to replace all the n_j in (1.1) by primes p_j and obtained a better lower bound for s if $k \geq 12$. For the special case $k = 1$, Baker [1] introduced a new kind of approximation by showing that for any number $A > 0$ the inequality

$$|\lambda_1p_1+\lambda_2p_2+\lambda_3p_3| < \left(\log \max p_j\right)^{-A} \tag{1.2}$$

Received 17 October 1978.

has infinitely many solutions in primes p_j. Later, Ramachandra [7] refined matters still more and replaced the $(\log \max p_j)^{-A}$ in (1.2) by $\exp\left\{-(\log p_1p_2p_3)^{\frac{1}{2}}\right\}$. Recently Vaughan [9, p.374] made remarkable progress by proving that the right hand side of (1.2) can be replaced by $(\max p_j)^{-1/10}(\log \max p_j)^{20}$. He also remarked that it is interesting that one can save as much as $1/10$ and on the generalized Riemann hypothesis, only $1/5$ may be saved. The object of this paper is to show that we can save as much as $1/9 - \delta$ for any $\delta > 0$. We have:

THEOREM. *Suppose that* $\lambda_1, \lambda_2, \lambda_3$ *are any non-zero real numbers not all of the same sign and that* λ_1/λ_2 *is irrational. If* η *is any real number and* $0 < \alpha < 1/9$, *then there are infinitely many prime triples* (p_1, p_2, p_3) *for which*

$$|\eta+\lambda_1p_1+\lambda_2p_2+\lambda_3p_3| < (\max p_j)^{-\alpha} .$$

Our proof is a refinement of the elegant argument of Vaughan's [9] which, in general principle, is based on the method of Davenport and Heilbronn [3].

2. Notation and definitions

Throughout, x is a real variable and n, p, with or without suffices, denote any positive integer and any prime respectively. Since λ_1/λ_2 is irrational it is known [4, Theorem 183] that there are infinitely many convergents a/q with $(a, q) = 1$, $1 \le q$, such that

$$|(\lambda_1/\lambda_2)-(a/q)| < 1/2q^2 . \tag{2.1}$$

For the given $\alpha < 1/9$, let A be any constant with

$$\max(1/5, 2\alpha) < A < 2/9 . \tag{2.2}$$

Put

$$X = q^{1/(1-2A)}, \quad L = \log X , \tag{2.3}$$

$$\varepsilon = X^{-A/2}L^{20} , \tag{2.4}$$

https://doi.org/10.1017/S0004972700008984 Published online by Cambridge University Press

$$(2.5) \qquad K_\varepsilon = \begin{cases} \varepsilon^2 & \text{if } x = 0 , \\ (\sin \pi\varepsilon x)^2/(\pi x)^2 & \text{otherwise,} \end{cases}$$

$$e(x) = \exp(i2\pi x) ,$$

$$(2.6) \qquad S(x) = \sum_{p\le X} e(px) \log p , \quad S_j(x) = S(\lambda_j x) ,$$

$$(2.7) \qquad F(x) = \prod_{j=1}^{3} S_j(x) .$$

Throughout, $\delta > 0$ is a small number, and constants implied by the symbols $\ll$ and $\gg$ may depend on λ_j, δ, η, and A only.

Let $\tau_1 = X^{A-1}$, $\tau_2 = \delta X^{1-4A}$, $\tau_3 = X^A$, and $E_1 = \{x : |x| \le \tau_1\}$, $E_2 = \{x : \tau_1 < |x| \le \tau_2\}$, $E_3 = \{x : \tau_2 < |x| \le \tau_3\}$, $E_4 = \{x : \tau_3 < |x|\}$.

Here we partition the real line into four regions instead of the usual three regions such as in [9]. Our new region, E_3, is similar to the range (20) in [2]. By introducing such a new region we are able to obtain our result, $\alpha < 1/9$. In §3 we shall give a proof for the estimation of a certain integral over E_3 and in §4 known integral estimations over the remaining regions, E_1, E_2, E_4, will be used.

3. The integral over E_3

LEMMA 1. *If integers* b, r *satisfy* $(b, r) = 1$, $1 \le r$, *and if* $2 \le Y$, *then*

$$\sum_{p\le Y} e(bp/r) \log p \ll \left(r^{1/2}Y^{1/2}+Y^{5/7}r^{3/14}+Yr^{-1/2}\right)(\log Y)^{17} .$$

Proof. This follows immediately from Theorem 16.1 in [6].

Put

$$(3.1) \qquad \theta_0 = 0 , \quad \theta_m = m(2-9A)/2 , \quad x_m = \delta X^{1-4A+\theta_m} ,$$

$$E_{3m} = \{x : |x| \in (x_{m-1}, x_m]\} ,$$

where $m = 1, 2, \ldots$, $N = [2(5A-1)/(2-9A)] + 1$. We see that

$$1 - 4A + \theta_N > A .$$

So

(3.2) $$\bigcup_{m=1}^{N} E_{3m} \supset E_3 .$$

LEMMA 2. *If* $x \in E_{3m}$, *then*

$$\min(|S_1(x)|, |S_2(x)|) \ll X^{1-A/2+3\theta_m/4} L^{17} .$$

Proof. Put

(3.3) $$Q = X^{1-A} .$$

For each $x \in E_{3m}$ and $j = 1, 2$, by Theorem 36 in [4] there are $(a_j, q_j) = 1$, $1 \le q_j \le Q$, such that

(3.4) $$\left|\lambda_j x - a_j q_j^{-1}\right| \le q_j^{-1} Q^{-1} .$$

We see that

(3.5) $$a_1 a_2 \ne 0 ;$$

for if $a_1 = 0$, then by (3.4), (3.3), and (2.2),

$$|x| \le |\lambda_1|^{-1} Q^{-1} < \delta X^{1-4A} .$$

This is impossible as $x \in E_{3m}$.

Next suppose that

(3.6) $$\max(q_1, q_2) \le X^{A-m(2-9A)/4} .$$

Write

$$a_2 q_1 (\lambda_1/\lambda_2) - a_1 q_2 = (a_2/q_2)(q_1 q_2/\lambda_2 x)(\lambda_1 x - (a_1/q_1)) - (a_1/q_1)(q_1 q_2/\lambda_2 x)(\lambda_2 x - (a_2/q_2)) = T_1 - T_2 ,$$

say. By (3.4), (3.6), and (3.3), we have

$$|T_1| \le \left(|\lambda_2 x| + q_2^{-1}Q^{-1}\right)\left(q_1 q_2/|\lambda_2 x|\right)q_1^{-1}Q^{-1} = \left(q_2 + Q^{-1}|\lambda_2 x|^{-1}\right)Q^{-1}$$
$$\ll X^{2A-1-m(2-9A)/4} .$$

Similarly we have $|T_2| \ll X^{2A-1-m(2-9A)/4}$.

Hence, in view of (2.3), we have

(3.7) $$|a_2 q_1(\lambda_1/\lambda_2) - a_1 q_2| \ll X^{2A-1-m(2-9A)/4} < 1/2q .$$

Now for any integers a', q' satisfying $1 \le q' < q$, by (2.1) we have

(3.8) $$|q'(\lambda_1/\lambda_2) - a'| \ge q'\left((|a'q - aq'|/qq') - |(a/q) - (\lambda_1/\lambda_2)|\right)$$
$$> q'\left((1/qq') - (1/2q^2)\right) > 1/2q .$$

Put $q' = |a_2 q_1|$ and $a' = \pm a_1 q_2$. By (3.5) we see that $1 \le q'$. It follows from (3.7), (3.8), and (2.3) that

(3.9) $$|a_2 q_1| \ge q = X^{1-2A} .$$

But by (3.4), (3.6), (3.1), and $x \in E_{3m}$, we have

(3.10) $$|a_2 q_1| = |(a_2/q_2)|q_1 q_2 \le \left(|\lambda_2 x| + Q^{-1}q_2^{-1}\right)q_1 q_2 \ll x_m X^{2A-m(2-9A)/2}$$
$$\ll \delta X^{1-2A} < q .$$

Since (3.10) contradicts (3.9), (3.6) must be false. Therefore we may assume that

(3.11) $$X^{A-m(2-9A)/4} < q_1 \ (\le X^{1-A}) .$$

Put

$$c_n = \begin{cases} e(a_1 p/q_1) \log p & \text{if } n \text{ is a prime } p , \\ 0 & \text{otherwise,} \end{cases}$$

and $z = \lambda_1 x - (a_1/q_1)$. By Theorem 421 in [4], Lemma 1 (with $b = a_1$, $r = q_1$), and (3.4), we have

$$S(\lambda_1 x) = \sum_{n\le X} c_n e(nz) = e(Xz) \sum_{p\le X} e(a_1 p/q_1) \log p$$

$$- \int_1^X 2\pi i z e(Yz)\Big\{\sum_{p\le Y} e(a_1 p/q_1) \log p\Big\} dY$$

$$<< \Big\{q_1^{1/2}X^{1/2}+X^{5/7}q_1^{3/14}+Xq_1^{-1/2}\Big\}L^{17}\Big\{1+Xq_1^{-1}Q^{-1}\Big\} .$$

It follows from (3.11), (3.3), and (3.1) that

$$S(\lambda_1 x) << X^{1-A/2+m(2-9A)/8}L^{17}X^{m(2-9A)/4}$$

$$<< X^{1-A/2+3\theta_m/4}L^{17} .$$

This proves Lemma 2.

LEMMA 3. *We have*

$$\int_{x_{m-1}}^{x_m} |F(x)|K_\varepsilon dx << X^2\varepsilon^2L^{-1} . \tag{3.12}$$

Then

$$\int_{E_3} |F(x)|K_\varepsilon dx << X^2\varepsilon^2L^{-1} . \tag{3.13}$$

Proof. By (2.6), Parseval's identity, and $\sum_{p\le X} 1 << X/L$, we have

$$\int_0^1 |S(y)|^2dy = \sum_{p\le X} (\log p)^2 << XL .$$

So, by (2.5) and (3.1),

$$\int_{x_{m-1}}^{x_m} |S_j(x)|^2K_\varepsilon dx \tag{3.14}$$

$$<< \int_{y>|\lambda_j|x_{m-1}}^{\infty} |S(y)|^2y^{-2}dy << \sum_{n>|\lambda_j|x_{m-1}} n^{-2}\int_{n-1}^{n} |S(y)|^2dy$$

$$<< XL/x_{m-1} << X^{4A-\theta_{m-1}}L .$$

On the other hand, note that

https://doi.org/10.1017/S0004972700008984 Published online by Cambridge University Press

$$|F(x)| \ll \min\left(|S_1(x)|, |S_2(x)|\right) \sum_{j=1}^{3} |S_j(x)|^2 .$$

Then by Lemma 2, (3.14), and (3.1), we have

$$\int_{x_{m-1}}^{x_m} |F(x)| K_\varepsilon dx \ll X^{1-A/2+3\theta_m/4} L^{17} \left(X^{4A-\theta_{m-1}} L\right)$$

$$\ll X^{1+7A/2+(2-9A)(4-m)/8} L^{18} \ll X^2 \varepsilon^2 L^{-1} .$$

The last inequality follows from (2.4) and

$$1 + 7A/2 + (2-9A)(4-m)/8 < 2 - A .$$

So (3.13) follows from (3.12) and (3.2).

4. Completion of the proof

LEMMA 4. *If* $x \in E_2$ *then*

$$\min\left(|S_1(x)|, |S_2(x)|\right) \ll X^{1-A/2} L^{17} .$$

Proof. The proof is similar to that of Lemma 2. But here we put $Q = \delta^{-1} X^{1-A}$ instead. Then following the same argument as that of Lemma 2 we must have

$$\max\left(q_1, q_2\right) > X^A ,$$

since $X = q^{1/(1-2A)}$. Then apply Lemma 1.

LEMMA 5. *We have*

$$\int_{E_2} |F(x)| K_\varepsilon dx \ll X^2 \varepsilon^2 L^{-1} .$$

Proof. This follows from Lemma 4 and the same argument as that of Lemma 12 in [9].

LEMMA 6. *We have*

$$\int_{E_1} e(x\eta) F(x) K_\varepsilon dx \gg X^2 \varepsilon^2 .$$

Proof. Note that we may apply Lemmas 2 to 8 in [9] directly without

https://doi.org/10.1017/S0004972700008984 Published online by Cambridge University Press

any changes since these results do not depend on E_1 .

Lemma 9 in [9] still holds if we replace τ there by our $\tau_1 = X^{A-1}$. But we need to make a slight modification to (33) in [9] as follows. By Lemma 5 in [9] the integral in (33) is

$$\begin{aligned} &\ll X^{4/3}L^C \int_{-X^{A-1}}^{X^{A-1}} \left(1+X|\lambda_j x|\right)^2 dx \\ &\ll X^{4/3}L^C X^2 X^{3(A-1)} |\lambda_j|^2 \\ &\ll X^{3A+1/3}L^C \ll XL^{-2} . \end{aligned}$$

The last inequality follows from (2.2); that is, $A < 2/9$. Then we continue the proof exactly as in [9, p. 379].

Finally, Lemma 10 in [9] still holds if we replace τ there by our τ_1 . No modification is necessary in the proof. Then Lemma 6 follows.

LEMMA 7. *We have*

$$\int_{E_4} |F(x)| K_\varepsilon dx \ll X^2 \varepsilon^2 L^{-1} .$$

Proof. This is Lemma 13 in [9].

LEMMA 8. *For any real y we have*

$$\int_{-\infty}^{\infty} e(xy) K_\varepsilon dx = \max(0,\ \varepsilon - |y|) .$$

Proof. This is Lemma 1 in [10].

We come now to prove our theorem. By Lemma 8 and (2.7) we have

$$\begin{aligned} J &= \int_{-\infty}^{\infty} e(x\eta) F(x) K_\varepsilon dx \\ &= \sum_{\substack{p_j \le X \\ j=1,2,3}} \left(\prod_{j=1}^{3} \log p_j\right) \max\left(0,\ \varepsilon - \left|\eta + \sum_{j=1}^{3} \lambda_j p_j\right|\right) \\ &\ll L^3 \varepsilon N , \end{aligned}$$

where N is the number of solutions in primes p_j of

$$\left|\eta + \sum_{j=1}^{3} \lambda_j p_j\right| < \varepsilon \le \left(\max p_j\right)^{-A/2}\left(\max \log p_j\right)^{20}$$

with $p_j \le X$ $(j = 1, 2, 3)$. So, by (2.2), that is $\alpha < A/2$, our theorem follows if $JL^{-3}\varepsilon^{-1} \to \infty$ as $X \to \infty$. Now

$$J = \sum_{\nu=1}^{4} \int_{E_\nu} e(x\eta)F(x)K_\varepsilon dx .$$

By Lemmas 5, 3, 7, we have

$$\sum_{\nu=2}^{4} \int_{E_\nu} |F(x)|K_\varepsilon dx \ll X^2\varepsilon^2 L^{-1} . \tag{4.1}$$

So Lemma 6, together with (4.1), shows that $JL^{-3}\varepsilon^{-1} \gg X^2\varepsilon L^{-3}$ as desired. This completes the proof of our theorem.

5. Remark

In §3 and in the proof of Lemma 6 we need $A < 2/9$, which leads to our result $\alpha < 1/9$ (see (2.2)). In fact, in the proof of Lemma 6 we can replace $A < 2/9$ by a better one, namely $A < (\sqrt{21}-1)/15 = 2/(8.37 \ldots)$ if we modify the argument as in [5, §4]. So it seems that the first difficulty encountered in any further improvement lies in §3.

References

[1] A. Baker, "On some diophantine inequalities involving primes", *J. reine angew. Math.* 228 (1967), 166-181.

[2] R.J. Cook, "Diophantine inequalities with mixed powers", *J. Number Theory* 9 (1977), 142-152.

[3] H. Davenport and H. Heilbronn, "On indefinite quadratic forms in five variables", *J. London Math. Soc.* 21 (1946), 185-193.

[4] G.H. Hardy and E.M. Wright, *An introduction to the theory of numbers*, 4th edition (Clarendon Press, Oxford, 1960).

[5] Ming-Chit Liu, "Approximation by a sum of polynomials involving primes", *J. Math. Soc. Japan* 30 (1978), 395-412.

[6] Hugh L. Montgomery, *Topics in multiplicative number theory* (Lecture Notes in Mathematics, **227**. Springer-Verlag, Berlin, Heidelberg, New York, 1971).

[7] K. Ramachandra, "On the sums $\sum_{j=1}^{K} \lambda_j f_j(p_j)$", *J. reine angew. Math.* **262/263** (1973), 158-165.

[8] Wolfgang Schwarz, "Über die Lösbarkeit gewisser Ungleichungen durch Primzahlen", *J. reine angew. Math.* **212** (1963), 150-157.

[9] R.C. Vaughan, "Diophantine approximation by prime numbers, I", *Proc. London Math. Soc.* (3) **28** (1974), 373-384.

[10] R.C. Vaughan, "Diophantine approximation by prime numbers, II", *Proc. London Math. Soc.* (3) **28** (1974), 385-401.

[11] R.C. Vaughan, "Diophantine approximation by prime numbers, III", *Proc. London Math. Soc.* (3) **33** (1976), 177-192.

Department of Mathematics,
University of Hong Kong,
Hong Kong.