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MACKEY BOREL STRUCTURE FOR THE QUASI-DUAL OF A SEPARABLE C^*-ALGEBRA

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1. Introduction. Let A be a separable C^*-algebra. Two representations π and π_1 of A on the Hilbert spaces H and H_1, respectively are said to be *quasi-equivalent* (denoted by $\pi \sim \pi_1$) if projections of $H \oplus H_1$ on the invariant subspaces H and H_1 of $(\pi \oplus \pi_1)(A)$ have the same central support in the commutant $(\pi \oplus \pi_1)(A)'$ of $(\pi \oplus \pi_1)(A)$, or equivalently, if there is an isomorphism ϕ of $\pi(A)''$ onto $\pi_1(A)''$ such that $\phi(\pi(x)) = \pi_1(x)$ for all $x \in A$ (cf. [**5**, § 5]). A representation π of A is said to be a *factor representation* if the center of $\pi(A)''$ consists of scalar multiples of the identity. The relation $\sim$ partitions the set of factor representations of A into quasi-equivalence classes. Let $\tilde{A}$ be the set of all quasi-equivalence classes of factor representations of A. Let $[\pi]$ denote the quasi-equivalence class containing the representation π.

Let H be any Hilbert space and let $\mathrm{Rep}(A, H)$ be the space of all representations of A on H with the topology of pointwise convergence, i.e., $\pi_n \to \pi$ if and only if $\pi_n(x)\zeta \to \pi(x)\zeta$ for all $x \in A$, $\zeta \in H$. For each $\pi \in \mathrm{Rep}(A, H)$, let $H(\pi)$ denote the *essential subspace* of π, i.e. the orthogonal complement of the subspace $\{\zeta \in H | \pi(x)\zeta = 0 \text{ for all } x \in A\}$. The subspace $H(\pi)$ is invariant under $\pi(A)$. For every π let π' denote the restriction of π to $H(\pi)$. Let $\mathrm{Fac}(A, H)$ be the set of all non-zero representations π of A on H such that the restriction π' of π to $H(\pi)$ is a factor representation on $H(\pi)$ and, for each $n = 1, 2, \ldots, \infty$, let

$$\mathrm{Fac}_n(A, H) = \{\pi \in \mathrm{Fac}(A, H) | \dim H(\pi) = n\}.$$

Let $\mathrm{Fac}(A, H)$ and $\mathrm{Fac}_n(A, H)$ have the topology induced by $\mathrm{Rep}(A, H)$.

From now on let H be a fixed separable infinite dimensional Hilbert space and let $\{H_n | n = 1, 2, \ldots, \infty\}$ be an increasing sequence of subspaces of H with $\dim H_n = n$ and $H_\infty = H$. Let $\mathrm{Fac}_n(A, H_n)$ be the space of all nonzero factor representations of A on H_n. The family M of all subsets X of $\tilde{A}$ such that each set $\{\pi \in \mathrm{Fac}_n(A, H_n) | [\pi] \in X\}$ is a Borel set in $\mathrm{Fac}_n(A, H_n)$ is a σ-ring and M is called the *Mackey Borel structure* of $\tilde{A}$ (cf. [**5**, § 7]).

A state f of A is said to be a *factor state* of A if the canonical representation π_f induced by f is a factor representation (cf. [**5**, § 2]). Let $F(A)$ be the space of all factor states of A with the relativized w^*-topology. We say the factor states f and g are quasi-equivalent (denoted by $f \sim g$) if $\pi_f \sim \pi_g$.

Now let $\sim$ be the map of $\mathrm{Fac}(A, H)$ (respectively $\mathrm{Fac}_\infty(A, H_\infty)$, $F(A)$) into

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$\tilde{A}$ given by $\sim(\pi) = [\pi']$ (respectively $\sim(\pi) = [\pi]$, $\sim(f) = [\pi_f]$). In this note we show that $\sim$ is surjective and the Mackey Borel structure of $\tilde{A}$ is the quotient Borel structure, i.e., X is a Mackey Borel set in $\tilde{A}$ if and only if $\sim^{-1}(X)$ is Borel.

2. Borel structures. For every $n = 1, 2, \ldots, \infty$, let J_n be the space of all partial isometries of H with domain support H_n taken with the *-strong topology and let U_n be the subgroup of J_n of all partial isometries of J_n whose range support is also H_n. Then the pair (U_n, J_n) can be made into a transformation group by defining the action of U_n on J_n as $(u, j) \to ju^*$. If both J_n and U_n are *polonais* spaces (i.e. they are separable and metrizable by a complete metric) and if the function $(u, j) \to ju^*$ is continuous (U_n, J_n) is said to be a polonais transformation group. A subset T_n of J_n is said to be a transversal of J_n/U_n if T_n intersects each equivalence class $\{jU_n | j \in J_n\}$ in precisely one point. (cf. [**6**]). The first lemma applies to L. T. Gardner's analysis [**8**, Lemma 1] of polonais transformation groups to a wider class of objects needed in studying quasi-equivalence relations.

LEMMA 1. *For every $n = 1, 2, \ldots, \infty$, the pair (U_n, J_n) is a polonais transformation group and J_n/U_n has a Borel transversal T_n.*

Proof. Since the proof for $n = \infty$ is found in [**8**, Lemma 1], we assume that n is finite. Let $\{\zeta_i\}$, $\{\zeta_i'\}$ and $\{\zeta_i''\}$ be sequences of unit vectors that are dense in the sets $\{\zeta \in H_n | \, ||\zeta|| = 1\}$, $\{\zeta \in H \ominus H_n | \, ||\zeta|| = 1\}$ and $\{\zeta \in H | \, ||\zeta|| = 1\}$ respectively. Let

$$S_{ik} = \{x \in L_1(H) | \operatorname{lub}_j |(x\zeta_i, \zeta_j'')| > 1 - k^{-1}\}$$

for all $i, k = 1, 2, \ldots$, and let

$$S_{ijk} = \{x \in L_1(H) | \, |(x\zeta_i', \zeta_j'')| < k^{-1}\}$$

for all $i, j, k = 1, 2, \ldots$. The sets S_{ik}, S_{ik}^*, S_{ijk}, S_{ijk}^* are open subsets of $L_1(H)$ with the *-strong topology. Thus the sets

$$J_n = (\bigcap S_{ik}) \cap (\bigcap S_{ijk})$$
$$U_n = J_n \cap J_n^*$$

are G_δ sets in $L_1(H)$. Because $L_1(H)$ is a polonais space in the *-strong topology, the spaces J_n and U_n are polonais in the *-strong topology.

The remainder of the proof showing that (U_n, J_n) is a transformation group and that J_n/U_n has a Borel transversal can be proved in the same way as Lemma 1 in [**8**].

Now let θ_n be the map of $\mathrm{Fac}_n(A, H_n) \times J_n$ into $\mathrm{Rep}(A, H)$ given by $\theta_n(\pi, j) = j\pi j^*$. We have the following proposition.

PROPOSITION 2. *For every $n = 1, 2, \ldots, \infty$, the subset $\mathrm{Fac}_n(A, H)$ is a Borel subset of $\mathrm{Rep}(A, H)$ and the restriction θ_n' of θ_n to $\mathrm{Fac}_n(A, H_n) \times T_n$ is a Borel* *isomorphism of* $\mathrm{Fac}_n(A, H_n) \times T_n$ *onto* $\mathrm{Fac}_n(A, H)$ *such that* $\theta_n'(\pi, j)' \sim \pi$ *for all* $(\pi, j) \in \mathrm{Fac}_n(A, H_n) \times T_n$.

Proof. The map θ_n is continuous since J_n is taken with its *-strong topology. We show that θ_n' is a one-one function. Let π, π_1 be in $\mathrm{Fac}_n(A, H_n)$ and j, j_1 be in T_n. If $j\pi j^* = j_1\pi_1 j_1^*$, then the ranges of j and j_1 coincide. This follows from the fact that both $\{\pi(x)\zeta | x \in A, \zeta \in H_n\}$ and $\{\pi_1(x)\zeta | x \in A, \zeta \in H_n\}$ are dense in H_n; otherwise, the projection corresponding to the closure would be a nontrivial projection in the center of the von Neumann algebra generated by $\pi(A)$ or $\pi_1(A)$. This proves that $j^*j_1 = u \in U_n$ and thus that $j = j_1$ and $\pi = \pi_1$. Hence the map θ_n' is one-one. Because $\mathrm{Fac}_n(A, H_n) \times T_n$ is a Borel subset of $\mathrm{Fac}_n(A, H_n) \times J_n$ and in particular is a standard Borel space (i.e. is isomorphic to a Borel subspace of a polonais space) [**5**, 7.1.2, 3.7.1, Lemma 1; **1**, p. 5] and because $\mathrm{Rep}(A, H)$ is a standard Borel space [**5**, 3.7.1] and in particular is a countably generated Borel space, the image of $\mathrm{Fac}_n(A, H_n) \times T_n$ under θ_n' is a Borel subset of $\mathrm{Rep}(A, H)$ [**1**, Proposition 2.5]. Now it is clear that $\theta_n'(\pi, j)' \sim \pi$ and that $\dim H(\theta_n'(\pi, j)) = n$. Therefore, the image of θ_n' is contained in $\mathrm{Fac}_n(A, H)$. Thus, it remains to be shown that θ_n' maps onto $\mathrm{Fac}_n(A, H)$. If $\pi \in \mathrm{Fac}_n(A, H)$, there is a $j \in J_n$ and a $\pi_1 \in \mathrm{Fac}_n(A, H_n)$ with $j\pi_1 j^* = \pi$. But j may be written as $j = j_1 u$ for $j_1 \in T_n$ and u in U_n. Because the restriction π_2 of $u\pi_1 u^*$ to its invariant subspace H_n is a nonzero factor representation, we see that π is in the image of θ_n' by expressing π as $\pi = \theta_n'(\pi_2, j_1)$. Hence the image of θ_n' is precisely $\mathrm{Fac}_n(A, H)$.

COROLLARY 3. *The set* $\mathrm{Fac}(A, H)$ *is a Borel subset of* $\mathrm{Rep}(A, H)$.

Proof. The set $\mathrm{Fac}(A, H)$ may be written as $\mathrm{Fac}(A, H) = \bigcup \mathrm{Fac}_n(A, H)$.

We now give the first characterization of the Mackey Borel structure for $\tilde{A}$.

THEOREM 4. *The map* $\sim : \pi \rightarrow [\pi | H(\pi)] = [\pi']$ *of* $\mathrm{Fac}(A, H)$ *into* $\tilde{A}$ *is surjective and the quotient Borel structure of* $\mathrm{Fac}(A, H)/\sim\, = \tilde{A}$ *is the Mackey Borel structure.*

Proof. If π is a nonzero factor representation of A and if ζ is a nonzero vector in $H(\pi)$, then π is quasi-equivalent to the representation π restricted to the invariant subspace $K = \mathrm{clos}\{\pi(x)\zeta | x \in A\}$ (cf. [**5**, 5.3.5]). Since K is separable the representation π is quasi-equivalent to a representation $\pi_1' = \pi_1 | H(\pi_1)$ with π_1 in $\mathrm{Fac}(A, H)$ obtained by identifying K with a subspace of H. Thus the map $\pi \rightarrow [\pi']$ of $\mathrm{Fac}(A, H)$ into $\tilde{A}$ is surjective.

For every $n = 1, 2, \ldots, \infty$, let η_n be the natural injection of $\mathrm{Fac}_n(A, H_n)$ into $\mathrm{Fac}_n(A, H)$. It is clear that the map η_n is continuous and that $\pi \sim \pi_1$ for π, π_1 in $\mathrm{Fac}_n(A, H_n)$ if and only if $\eta_n(\pi)' \sim \eta_n(\pi_1)'$.

Now let X be a subset of $\tilde{A}$ and let $X' = \{\pi \in \mathrm{Fac}(A, H) | [\pi'] \in X\}$. We notice that $\eta_n^{-1}(X') = \{\pi \in \mathrm{Fac}_n(A, H_n) | [\pi] \in X\}$. On the one hand, assuming that X' is a Borel set in $\mathrm{Fac}(A, H)$, we show that X is a Mackey Borel set. In fact, the set $\eta_n^{-1}(X')$ is a Borel set in $\mathrm{Fac}_n(A, H_n)$ due to the continuity of

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η_n. Hence, by the definition of the Mackey Borel structure, the set X is a Mackey Borel set. On the other hand, assuming that X is a Mackey Borel set, we show that X' is a Borel set in $\mathrm{Fac}(A, H)$. Because $\eta_n^{-1}(X')$ is a Borel subset of $\mathrm{Fac}_n(A, H_n)$, we have that $\eta_n^{-1}(X') \times T_n$ is a Borel subset of $\mathrm{Fac}_n(A, H_n) \times T_n$. This means that the set

$$X_n = \theta_n'(\eta_n^{-1}(X') \times T_n)$$

is a Borel subset of $\mathrm{Fac}_n(A, H)$ and hence a Borel subset of $\mathrm{Fac}(A, H)$ (Proposition 2). We show X' is a Borel set in $\mathrm{Fac}(A, H)$ by showing that $X' = \bigcup X_n$. If $\pi \in X_n$, then π' is quasi-equivalent to a representation π_1 in $\eta_n^{-1}(X')$. We see that $\pi' \sim \pi_1 \sim \eta_n(\pi_1)'$ and consequently that $\pi \in X'$ since $\eta_n(\pi_1) \in X'$. Hence we get that $\bigcup X_n \subset X'$. Conversely, if $\pi \in X'$ and if $\dim H(\pi) = n$, there is a $(\pi_1, j) \in \mathrm{Fac}_n(A, H_n) \times T_n$ such that $\theta_n(\pi_1, j) = \pi$ (Proposition 2). Since $\pi_1 \sim \pi'$, we get that $[\pi_1] \in X$ and consequently that $\pi_1 \in \eta_n^{-1}(X')$. Hence, $\pi \in X_n$. This proves that $X' \subset \bigcup X_n$. Therefore, we obtain that $X' = \bigcup X_n$.

Before giving the next characterization of the Mackey Borel structure we need another lemma.

LEMMA 5. *For each* $n = 1, 2, \ldots,$ *there is a continuous map* $\phi = \phi_n$ *of* $\mathrm{Fac}_n(A, H_n)$ *into* $\mathrm{Fac}_\infty(A, H_\infty)$ *such that* $\pi \sim \phi(\pi)$ *for every* $\pi \in \mathrm{Fac}_n(A, H_n)$.

Proof. Let K be the direct sum of countably infinitely many copies of H_n. For each $\pi \in \mathrm{Fac}_n(A, H_n)$, let π_0 be the representation in $\mathrm{Fac}_\infty(A, K_\infty)$ given by the direct sum of countably infinitely many copies of π (cf. [**5**, 5.3.1 (v)]). Let u be an isometric isomorphism of the separable infinite dimensional space K onto the separable infinite dimensional space H. Then the map $\phi(\pi) = u\pi_0 u^{-1}$ maps $\mathrm{Fac}_n(A, H_n)$ into $\mathrm{Fac}_\infty(A, H_\infty)$. Since $\pi_0 \sim \pi$ [**5**, 5.3.5], we get that $\phi(\pi) \sim \pi$. Now all that remains is the demonstration that ϕ is continuous. Let $\pi_i \to \pi$ in $\mathrm{Fac}_n(A, H_n)$ and let $\zeta, \xi \in H$. There are sequences $\{\zeta_m\}$ and $\{\xi_m\}$ in H_n such that $\sum ||\zeta_m||^2 < +\infty$ and $\sum ||\xi_m||^2 < +\infty$ so that $u^{-1}\zeta = (\zeta_n) \in K$ and $u^{-1}\xi = (\xi_m) \in K$. Now given $x \in A$ and $\epsilon > 0$, there is a natural number m_0 and an index i_0 such that

$$\sum\{||\zeta_m||^2 | m \geqq m_0\} < \epsilon/2, \quad \sum\{||\xi_m||^2 | m \geqq m_0\} < \epsilon/2,$$

and

$$|(\pi_i(x)\zeta_m, \xi_m) - (\pi(x)\zeta_m, \xi_m)| \leqq \epsilon/m_0$$

for $m \leqq m_0$ and $i \geqq i_0$. Thus we have that

$$\begin{aligned} &|(\phi(\pi)(x)\zeta, \xi) - (\phi(\pi)(x)\zeta, \xi)| \\ &\quad \leqq \sum |(\pi_i(x)\zeta_m, \xi_m) - (\pi(x)\zeta_m, \xi_m)| \\ &\quad \leqq \epsilon + \epsilon||x|| \end{aligned}$$

whenever $i \geqq i_0$. This means that ϕ is continuous.

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We now show that it is not at all necessary to consider $\mathrm{Fac}_n(A, H_n)$ for n finite when working with the Mackey Borel structure.

THEOREM 6. *The map* $\sim : \pi \rightarrow [\pi]$ *of* $\mathrm{Fac}_\infty(A, H_\infty)$ *into* $\tilde{A}$ *is surjective and the quotient Borel structure of* $\mathrm{Fac}_\infty(A, H_\infty)/\sim\ = \tilde{A}$ *is the Mackey Borel structure on* $\tilde{A}$.

Proof. Let π be a nonzero factor representation of A on a separable Hilbert space $H(\pi)$. As in Lemma 5, the direct sum of countably infinitely many copies of π gives a factor representation that is quasi-equivalent to π and is unitarily equivalent to a factor representation in $\mathrm{Fac}_\infty(A, H_\infty)$. This proves that the map $\sim$ is surjective.

Now let X be a subset of $\tilde{A}$ and let $X' = \{\pi \in \mathrm{Fac}_\infty(A, H_\infty) | [\pi] \in X\}$. First let X' be a Borel set of $\mathrm{Fac}_\infty(A, H_\infty)$. We show that $X \in M$ by showing that $X_n' = \{\pi \in \mathrm{Fac}_n(A, H_n) | [\pi] \in X\}$ is Borel in $\mathrm{Fac}_n(A, H_n)$. Because the map ϕ_n of $\mathrm{Fac}_n(A, H_n)$ into $\mathrm{Fac}_\infty(A, H_\infty)$ constructed in Lemma 5 is continuous, it is sufficient to show that $X_n' = \phi_n^{-1}(X')$. However, this relation is clear since $\phi_n(\pi) \sim \pi$ for all $\pi \in \mathrm{Fac}_n(A, H_n)$. Thus we get that $X \in M$. Conversely, let $X \in M$. By definition of the Mackey Borel structure, the set X' is a Borel subset of $\mathrm{Fac}_\infty(A, H_\infty)$. This proves that the quotient Borel structure of $\mathrm{Fac}_\infty(A, H_\infty)/\sim\ = \tilde{A}$ is the Mackey Borel structure.

The next theorem characterizes the Mackey Borel structure in terms of the factor states. We first clarify the relation between the factor states of a C^*-algebra A without identity and the factor states of the C^*-algebra A_e with identity adjointed (cf. [**5**, 1.3.7]).

LEMMA 7. *Let A be a C^*-algebra without identity. Let f_0 be the unique element of $F(A_e)$ that vanishes on A. Then the map e of $F(A)$ into $F(A_e)$ mapping each element into its unique extension to A_e is a Borel isomorphism onto the Borel set $F(A_e) - \{f_0\}$ of $F(A_e)$.*

Proof. Let $e(f)$ be defined on A_e by setting $e(f)(1) = 1$. Then e is a one-one continuous open map of $F(A)$ into $F(A_e)$ whose range is $F(A_e) - \{f_0\}$ (cf. [**9**, § 3]). Because $F(A_e)$ and consequently $F(A)$ are standard Borel spaces (cf. [**10**, 3.4.5]), the map e is a Borel isomorphism of $F(A)$ onto the Borel set (actually the open set) $F(A_e) - \{f_0\}$ [**1**, Lemma 2.5].

THEOREM 8. *The map* $\sim : f \rightarrow [\pi_f]$ *of* $F(A)$ *into* $\tilde{A}$ *is surjective and the quotient Borel structure of* $F(A)/\sim\ = \tilde{A}$ *is the Mackey Borel structure.*

Proof. First assume A has an identity element. Let ζ be a unit vector in H and let ω be the functional on $L(H)$ given by $\omega(x) = (x\zeta, \zeta)$. For $\pi \in \mathrm{Fac}_\infty(A, H_\infty)$, the canonical representation induced by the state $\omega \circ \pi$ is equivalent to a subrepresentation of π [**3**, Proposition 3] and thus is quasi-equivalent to π [**5**, 5.3.5]. Thus the map ψ of $\mathrm{Fac}_\infty(A, H_\infty)$ defined by $\psi(\pi) = \omega \circ \pi$ maps $\mathrm{Fac}_\infty(A, H_\infty)$ into $F(A)$. Now it is clear that ψ is continuous. We show that ψ is surjective. If $f \in F(A)$, let π be the direct sum of countably infinitely many copies of π_f. The representation π is a factor representation of A on the separable infinite dimensional space K equal to countably infinitely many copies of the representation space $H(f)$ of f. There is an isometric isomorphism u of K onto H that takes $(\zeta_f, 0, 0, \ldots)$ into ζ. Here ζ_f is a cyclic vector for $H(f)$ under $\pi_f(A)$ such that $(\pi_f(x)\zeta_f, \zeta_f) = f(x)$ for all $x \in A$. This means that the image under ψ of the factor representation $u\pi u^{-1}$ on H is precisely f. Hence ψ is surjective. In particular, the map $\sim$ of $F(A)$ into $\tilde{A}$ is surjective since the map $\pi \rightarrow [\pi]$ of $\mathrm{Fac}_\infty(A, H_\infty)$ into $\tilde{A}$ is surjective (Theorem 6) and since $\pi \sim \pi_{\psi(\pi)}$ for every $\pi \in \mathrm{Fac}_\infty(A, H_\infty)$.

Now we show that the quotient Borel structure of $F(A)/\sim\; = \tilde{A}$ is the Mackey Borel structure. Let X be a subset of $\tilde{A}$ and let

$$X' = \{\pi \in \mathrm{Fac}_\infty(A, H_\infty) | [\pi] \in X\} \quad \text{and} \quad X'' = \{f \in F(A) | [\pi_f] \in X\}.$$

Since $\pi \sim \pi_{\psi(\pi)}$ for every $\pi \in \mathrm{Fac}_\infty(A, H_\infty)$, we have that $\psi^{-1}(X'') = X'$. Because ψ is surjective, we also have that $\psi(X') = X''$. If X'' is a Borel set in $F(A)$, the set X' is Borel in $\mathrm{Fac}_\infty(A, H_\infty)$ because ψ is continuous. From Theorem 6 we obtain that $X \in M$ and consequently the quotient Borel structure is contained in the Mackey Borel structure. Conversely, if $X \in M$, then X' and its complement $\mathrm{Fac}_\infty(A, H_\infty) - X'$ are Borel subsets of $\mathrm{Fac}_\infty(A, H_\infty)$. Because X' is saturated (i.e., $\pi \sim \pi_1 \in X'$ for $\pi \in \mathrm{Fac}_\infty(A, H_\infty)$ implies $\pi \in X'$), its complement is also saturated. Due to the continuity of ψ, the analytic sets $\psi(X')$ and $\psi(\mathrm{Fac}_\infty(A, H_\infty) - X')$ form a partition of $F(A)$. This proves that $\psi(X') = X''$ is a Borel set in $F(A)$ [**2**, § 6, Theorem 2, Corollary]. Thus the Mackey Borel structure is contained in the quotient Borel structure. Therefore, the quotient and the Mackey Borel structures coincide in $F(A)/\sim$.

Now assume A does not have an identity. Let A_e be the C^*-algebra A with identity adjoined. As before let X be a subset of $\tilde{A}$ and let $X'' = \{f \in F(A) | [\pi_f] \in X\}$. The set X is Mackey Borel if and only if $\{[\pi] \in \tilde{A}_e | [\pi|A] \in X\} = Y$ is Mackey Borel in $\tilde{A}_e$ since A is a closed two-sided ideal in A_e [**4**, Proposition 2]. The set Y is Mackey Borel if and only if

$$Y'' = \{f \in F(A_e) | [\pi_f] \in Y\} = \{f \in F(A_e) | [\pi_f|A] \in X\}$$

is a Borel set in $F(A_e)$ due to the first part of this theorem. However, we have that $e(X'') = Y''$. In fact, if $f \in X''$, then $\pi = \pi_{e(f)}|A \sim \pi_f$ since the canonical representation induced by $\omega_{\zeta_f} \circ \pi = f$ is quasi-equivalent to a subrepresentation of π [**3**, Proposition 3] and thus is equivalent to π [**5**, 5.3.5]. So $f \in X''$ implies $e(f) \in Y''$. Conversely, if $f \in Y''$, then $g = f|A \in F(A)$ and $\pi_g \sim \pi_f|A$ Thus $f \in Y''$ implies $f|A \in X''$ and $e(f|A) = f$; so $e(X'') = Y''$. Now, by Lemma 7, we have that X'' is a Borel set in $F(A)$ if and only if Y'' is a Borel set in $F(A_e)$. Thus Theorem 8 for C^*-algebras without identity follows from the theorem for algebras with identity.

COROLLARY 9. *The saturation of every point in $F(A)$ is a Borel set in $F(A)$.*

Proof. Every point of $\tilde{A}$ is a Mackey Borel set (cf. [**5**, 7.2.4]).

Remark. In [**9**] we showed that the saturation of every open subset of $F(A)$ is open. This means that the hull-kernel Borel structure of $\tilde{A}$ is weaker than the Mackey Borel structure of $\tilde{A}$.

3. Application to central decomposition. Using the definition of the Mackey Borel structure given in Theorem 8, we can identify the central decomposition of states (cf. [**10**, 3.5 ff.]) with the central decomposition of separable representations (cf. [**5**, 8.4]). Then, directly from the result of E. Effros [**7**] characterizing the measures that arise in the central decomposition of separable representations, we obtain a characterization of the measures that arise in the central decomposition of states. This gives an answer to a question raised by S. Sakai [**10**, p. 151].

In the sequel let $\tilde{A}$ have the Mackey Borel structure and let $F(A)$ have the Borel structure induced by the w^*-topology. The following result is patterned after [**5**, 7.3.2].

PROPOSITION 10. *If X is a Borel set in $F(A)$ such that X intersects each quasi-equivalence class of $F(A)$ in at most one point, then the map $f \to [\pi_f]$ is a Borel isomorphism of X onto a Borel subset of $\tilde{A}$.*

Proof. Let $Y = \{(f_1, f_2) \in F(A) \times F(A) | f_1 \sim f_2\}$. The map $\psi \times \psi$ of $\mathrm{Fac}_\infty(A, H_\infty) \times \mathrm{Fac}_\infty(A, H_\infty)$ onto $F(A) \times F(A)$ obtained from the map ψ defined in Theorem 8 is continuous. Since $\psi(\pi_1) \sim \psi(\pi_2)$ if and only if $\pi_1 \sim \pi_2$, the set Y is the image under $\psi \times \psi$ of the set $Y' = \{(\pi_1, \pi_2) \in \mathrm{Fac}_\infty(A, H_\infty) \times \mathrm{Fac}_\infty(A, H_\infty) | \pi_1 \sim \pi_2\}$ and the complement of Y is the image of the complement of Y'. Because Y' is a Borel set of $\mathrm{Fac}_\infty(A, H_\infty) \times \mathrm{Fac}_\infty(A, H_\infty)$ [**5**, 7.3.2], the set Y is a Borel set of $F(A) \times F(A)$ [**2**, § 6, Theorem 2, Corollary]. Therefore, the set $(X \times F(A)) \cap Y$ is a Borel set in $F(A) \times F(A)$. The map that projects a pair of $F(A) \times F(A)$ onto the second coordinate is continuous and is one-one when confined to the set $(X \times F(A)) \cap Y$. This means that the image of $(X \times F(A)) \cap Y$ under the projection is a Borel subset of $F(A)$ [**1**, Proposition 2.5]. But this image is simply the saturation of X. Thus, the set $\{[\pi_f] | f \in X\}$ is a Mackey Borel set in $\tilde{A}$ (Theorem 8). If X' is any Borel subset of X, then the same proof shows that $\{[\pi_f] | f \in X'\}$ is a Borel subset of $\tilde{A}$. Because $f \to [\pi_f]$ is a Borel map of $F(A)$, it is now clear that the map $f \to [\pi_f]$ is a Borel isomorphism of X onto a Borel subset of $\tilde{A}$.

The following theorem can be obtained almost directly from [**7**] by the use of Proposition 10. For this assume A has an identity.

THEOREM 11. *A probability Radon measure μ on $F(A)$ arises from the central decomposition of a state of A if and only if there is a Borel set X in $F(A)$ with $\mu(X) = 1$ such that the weakest Borel structure on X induced by the family of* *maps $f \to f(c)$, where c runs through the center of the sequential weak-operator closure of A in its enveloping von Neumann algebra, coincides with the Borel structure on X induced by $F(A)$. Here each f in $F(A)$ is identified with its unique extension to a σ-weakly continuous functional on its enveloping von Neumann algebra.*

Proof. If the two Borel structures on X coincide, then X intersects each quasi-equivalence class in at most one point because the family of maps fails to distinguish quasi-equivalent functionals. Then X may be identified with a Borel set in $\tilde{A}$ (Proposition 10) and the result of Effros [**7**] gives the theorem.

The converse is to be found in the proof of [**10**, 3.5.7].

Added April 9, 1974. If A is a *GCR* algebra, I have proved that $F(A)$ has a Borel transversal for the relation of quasi-equivalence.

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