Canad. Math. Bull. Vol. **40** (4), 1997 pp. 498–508

MATRIX TRANSFORMATIONS BASED ON DIRICHLET CONVOLUTION

CHIKKANNA SELVARAJ AND SUGUNA SELVARAJ

ABSTRACT. This paper is a study of summability methods that are based on Dirichlet convolution. If $f(n)$ is a function on positive integers and x is a sequence such that $\lim_{n\to\infty}\sum_{k\le n}\frac{1}{k}(f*x)(k)=L$, then x is said to be A_f-*summable* to L. The necessary and sufficient condition for the matrix A_f to preserve bounded variation of sequences is established. Also, the matrix A_f is investigated as $\ell-\ell$ and $G-G$ mappings. The strength of the A_f-matrix is also discussed.

1. **Introduction.** If $f(n)$ and $g(n)$ are real-valued functions defined on positive integers, then the Dirichlet convolution of f and g is given by

$$(f*g)(n)=\sum_{d\mid n}f\Big(\frac{n}{d}\Big)g(d).$$

In 1960, Rubel [6] introduced the sequence to sequence summability method as follows:

A sequence $\{a_n\}$ is A_μ-summable to L if

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k}(\mu*a)(k)=L,$$

where $\mu(n)$ is the *Möbius function*. It can be easily seen that this summability method is indeed a matrix transformation, by considering

$$\begin{aligned}\sum_{k=1}^{n}\frac{1}{k}(\mu*a)(k)&=\sum_{k=1}^{n}\frac{1}{k}\sum_{d\mid n}\mu\Big(\frac{k}{d}\Big)a_d\\&=\sum_{d=1}^{n}\frac{1}{d}\sum_{q\le\frac{n}{d}}\frac{\mu(q)}{q}a_d.\end{aligned}$$

Thus the lower triangular matrix A_μ is given by

$$A_\mu(n,k)=\begin{cases}\frac{1}{k}\sum_{q\le\frac{n}{k}}\frac{\mu(q)}{q}, & \text{if } k\le n,\\ 0, & \text{if } k>n.\end{cases}$$

Rubel [6] proved that the matrix A_μ satisfies the Silverman-Toeplitz conditions and hence is regular. In 1964, Segal [7] introduced the generalization of the matrix A_μ and gave the characterization of classes of functions $f(n)$ satisfying the condition that the matrix A_f given by

$$A_f(n,k)=\begin{cases}\frac{1}{k}\sum_{q\le\frac{n}{k}}\frac{f(q)}{q}, & \text{if } k\le n,\\ 0, & \text{if } k>n\end{cases}$$

Received by the editors May 9, 1996.

AMS subject classification: 11A25, 40A05, 40C05, 40D05.

is regular. In this paper, we investigate the classes of functions $f(n)$ which satisfy the condition that the corresponding matrices A_f preserve bounded variation of sequences. Also, we give necessary and sufficient conditions for the matrix A_f to be a $G-G$, $G-\ell$ and $\ell-\ell$ matrix. In Section 3 we establish two comparison theorems related to the summability method A_f. We have given below the basic notations and definitions used in this paper.

$$B = \Big\{x : \sum_{k=1}^{\infty} |\Delta x_k| < \infty\Big\}.$$

$$\ell = \Big\{x : \sum_{k=1}^{\infty} |x_k| < \infty\Big\}.$$

$$G = \{x : x_k = O(r^k) \text{ for some } r\epsilon(0,1)\}$$

$$\equiv \{x : \limsup_k |x_k|^{\frac{1}{k}} < 1\}.$$

DEFINITION 1.1. We call a matrix A an $X-Y$ matrix if Ax is in the set Y whenever x is in X.

DEFINITION 1.2. An $X-Y$ matrix A is said to be *sum-preserving* if for each $x \in X$,

$$\sum_{n=1}^{\infty} (Ax)_n = \sum_{k=1}^{\infty} x_k.$$

DEFINITION 1.3. A sequence $\{x_n\}$ is of bounded variation if and only if

$$\sum_{k=1}^{\infty} |\Delta x_k| = \sum_{k=1}^{\infty} |x_k - x_{k+1}| < \infty.$$

Fricke and Fridy [2] introduced the set G as the set of sequences that are dominated by a convergent geometric sequence and gave characterizations of $G-\ell$ and $G-G$ matrices as follows:

THEOREM 1.1. *The matrix A is a $G-\ell$ matrix if and only if*

$$\sum_{n=0}^{\infty} |a_{nk}| = M_k < \infty \quad \textit{for } k = 0, 1, \ldots,$$

and

$$\limsup_k M_k^{\frac{1}{k}} \leq 1.$$

THEOREM 1.2. *The matrix A is a $G-G$ matrix if and only if for each $\epsilon > 0$, there exists a constant M and an $r \in (0,1)$ such that*

$$|a_{nk}| \leq Mr^n(1+\epsilon)^k \quad \textit{for all } n \textit{ and } k.$$

THEOREM 1.3. *The $\ell-\ell$ or $G-\ell$ matrix A is sum-preserving if and only if for each k,*

$$\sum_{n=1}^{\infty} a_{nk} = 1.$$

2. **$\mathbf{A_f}$ Transformations of various sets.** In this section we consider the A_f-transformations of the sets B, ℓ and G.

In [5] Mears gave necessary and sufficient conditions for a matrix $[a_{nk}]$ to preserve the bounded variation of sequences. It can be easily verified that a lower triangular matrix $[a_{nk}]$ maps the set B into B if and only if

$$\sup_k \sum_{n=1}^{\infty} |a_{n,k} - a_{n-1,k}| < \infty.$$

THEOREM 2.1. *The matrix A_f preserves the bounded variation of the sequences if and only if $\frac{f(n)}{n}$ is in ℓ.*

PROOF. Assuming that $\frac{f(n)}{n}$ is in ℓ , we want to show that

$$\sup_k \sum_{n=1}^{\infty} \left| \frac{1}{k} \sum_{q \le \frac{n}{k}} \frac{f(q)}{q} - \frac{1}{k} \sum_{q \le \frac{n-1}{k}} \frac{f(q)}{q} \right| < \infty. \tag{1}$$

For each k, let us evaluate the summation over n by considering the cases where (i) n is not divisible by k and (ii) n is divisible by k.

CASE (i). If n is not divisible by k, then $k \ge 2$. So, $n = kp + m$ where p is an integer ≥ 0 and $1 \le m < k$. Therefore, $[\frac{n}{k}] = p$ and $[\frac{n-1}{k}] = [\frac{kp+m-1}{k}] = p$, since $m - 1 < k$. Hence, for each such n,

$$\sum_{q \le \frac{n}{k}} \frac{f(q)}{q} - \sum_{q \le \frac{n-1}{k}} \frac{f(q)}{q} = 0.$$

Thus for each k in (1), the n's not divisible by k contribute zeros to the summation over n.

CASE (ii). If n is divisible by k, then for $k = 1$,

$$\sum_{n=1}^{\infty} \left| \sum_{q \le n} \frac{f(q)}{q} - \sum_{q \le n-1} \frac{f(q)}{q} \right| = \sum_{n=1}^{\infty} \left| \frac{f(n)}{n} \right|.$$

If n is divisible by k and $k \ge 2$, we have $n = kp$ for some positive integer p. Therefore, $[\frac{n}{k}] = p$ and $[\frac{n-1}{k}] = [p - \frac{1}{k}] = p - 1$. Hence, for each such n,

$$\sum_{q \le \frac{n}{k}} \frac{f(q)}{q} - \sum_{q \le \frac{n-1}{k}} \frac{f(q)}{q} = \frac{f(p)}{p}.$$

Thus for each k,

$$\frac{1}{k} \sum_{n=1}^{\infty} \left| \sum_{q \le \frac{n}{k}} \frac{f(q)}{q} - \sum_{q \le \frac{n-1}{k}} \frac{f(q)}{q} \right| = \frac{1}{k} \sum_{\substack{n=kp \\ p=1,2,\ldots}} \left| \frac{f(p)}{p} \right| = \frac{M}{k},$$

by the assumption. Hence A_f is a $B - B$ matrix.

Conversely, if the matrix A_f maps B into B, then the condition (1) is true. When $k = 1$ we get that

$$\sum_{n=1}^{\infty} \left| \sum_{q \le n} \frac{f(q)}{q} - \sum_{q \le n-1} \frac{f(q)}{q} \right| = \sum_{n=1}^{\infty} \left| \frac{f(n)}{n} \right| \le M.$$

Hence, the theorem is proved.

https://doi.org/10.4153/CMB-1997-059-6 Published online by Cambridge University Press

COROLLARY 2.1. *If A_g is a $B-B$ matrix and $f(n) = O\big(g(n)\big)$, then A_f is also a $B-B$ matrix.*

We give below some examples of arithmetic functions such that the associated matrices A_f preserve bounded variation of sequences.

EXAMPLE 2.1. Let $f(n) = O\big(g(n)\big)$, where $g(n) = \frac{1}{n^m}$ for fixed $m > 0$. Then

$$\sum_{n=1}^{\infty}\left|\frac{g(n)}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n^s}, \quad \text{where } s > 1$$
$$= \zeta(s)$$

where ζ is the *Riemann-Zeta function*.

EXAMPLE 2.2. Let $f(n) = O\big(g(n)\big)$, where $g(n) = \frac{\mu(n)}{n^m}$ for fixed $m > 0$. Since $|\mu(n)| \leq 1$ for all n, we get that

$$\sum_{n=1}^{\infty}\left|\frac{g(n)}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n^s} = \zeta(s).$$

EXAMPLE 2.3. Let $f(n) = \frac{d(n)}{n^m}$ for fixed $m > 0$, where $d(n)$ is the *divisor function* given by $d(n) =$ number of positive divisors of n. Then

$$\sum_{n=1}^{\infty}\left|\frac{f(n)}{n}\right| = \sum_{n=1}^{\infty}\frac{d(n)}{n^s}, \quad \text{where } s > 1,$$
$$= \zeta^2(s)$$

by the formula in [1, p. 24].

EXAMPLE 2.4. Let $f(n) = \frac{\Lambda(n)}{n^m}$ for fixed $m > 0$, where $\Lambda(n)$ is *von Mangoldt function* given by

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^m \text{ where } p \text{ is prime and } m > 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then, by the formula in [1, p. 24], we get that

$$\sum_{n=1}^{\infty}\left|\frac{f(n)}{n}\right| = \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}, \quad \text{where } s > 1,$$
$$= -\frac{\zeta'(s)}{\zeta(s)}.$$

EXAMPLE 2.5. Let $f(n) = \frac{\psi(n)}{n^s}$ for fixed $s > 1$, where $\psi(n)$ is the *Chebyshev's function* given by $\psi(n) = \sum_{k \leq n} \Lambda(k)$. Since $\psi(n) = O(n)$ [1, p. 5], we get that

$$\sum_{n=1}^{\infty}\left|\frac{f(n)}{n}\right| \leq \sum_{n=1}^{\infty}\frac{M}{n^s} = M\zeta(s).$$

EXAMPLE 2.6. Let $f(n) = \frac{\phi(n)}{n^m}$ for fixed $m > 0$, where $\phi(n)$ is the *Euler's function* given by $\phi(n)$ = number of $m \le n$ such that $(m, n) = 1$. Then

$$\sum_{n=1}^{\infty}\left|\frac{f(n)}{n}\right| = \sum_{n=1}^{\infty}\frac{\phi(n)}{n^s}, \quad \text{where } s > 1,$$
$$= \frac{\zeta(s-1)}{\zeta(s)}.$$

Next, we give the necessary and sufficient conditions on the arithmetic function $f(n)$ such that the associated matrix A_f maps G into G and ℓ into ℓ.

THEOREM 2.2. *The matrix A_f is a $G - G$ matrix if and only if $f(n) \in G$.*

PROOF. Suppose that $f(n)$ is in G. Then there exists an $M > 0$ and an $r \in (0, 1)$ such that

$$|f(n)| \le Mr^n, \quad \text{for all } n.$$

Therefore,

$$|A_f(n,k)| = \left|\frac{1}{k}\sum_{q\le\frac{n}{k}}\frac{f(q)}{q}\right|$$
$$\le \frac{M}{k}\sum_{q\le n+1} r^q$$
$$= \frac{Mr}{k}(1-r)^n.$$

Hence, given $\epsilon > 0$, there exists an $H > 0$ and $1 - r = s \in (0, 1)$ such that

$$|A_f(n,k)| \le Hs^n(1+\epsilon)^k,$$

for all n and k. By Theorem 1.2, A_f is a $G - G$ matrix.

Conversely, if A_f is a $G-G$ matrix, then for all n and k, we have a $B > 0$ and $s \in (0, 1)$ such that

$$|A_f(n,k)| \le Bs^n(1+\epsilon)^k.$$

When $k = 1$, for all n,

$$\left|\sum_{q\le n}\frac{f(q)}{q}\right| \le Bs^n.$$

Therefore, for each n,

$$|f(n)| \le \left|\frac{f(n)}{n}\right|$$
$$= \left|\sum_{q\le n}\frac{f(q)}{q} - \sum_{q\le n-1}\frac{f(q)}{q}\right|$$
$$\le \left|\sum_{q\le n}\frac{f(q)}{q}\right| + \left|\sum_{q\le n-1}\frac{f(q)}{q}\right|$$
$$\le Bs^n + Bs^{n-1} = Hs^n.$$

https://doi.org/10.4153/CMB-1997-059-6 Published online by Cambridge University Press

Hence, the theorem is true.

The next two examples demonstrate the following general relationship.

REMARK 2.1. If A_f is a $G - G$ matrix, then $A_f^{-1}[G]$ is larger than G.

EXAMPLE 2.7. Let $a_n = \frac{1}{n^s}$ for fixed $s > 0$. Then $a_n \notin G$, because

$$\limsup_n |a_n|^{\frac{1}{n}} = 1.$$

If A_f is $G - G$ matrix, then by Theorem 2.2, $f(n) \in G$ which yields that

$$|f(n)| \le Mr^n, \quad \text{for some } r \in (0, 1).$$

Therefore, for each n,

$$\begin{aligned} |(A_f a)_n| &\le \sum_{k=1}^{n} \frac{a_k}{k} \sum_{q \le \frac{n}{k}} \frac{|f(q)|}{q} \\ &< \sum_{k=1}^{n} \frac{1}{k^{s+1}} \sum_{q \le n+1} Mr^q \\ &= H(1-r)^n \sum_{k=1}^{n} \frac{1}{k^{s+1}} \\ &< H\zeta(s+1)(1-r)^n. \end{aligned}$$

Since $1 - r \in (0, 1)$, we have that $(A_f a)_n \in G$.

EXAMPLE 2.8. If we choose $a_n = \frac{\mu(n)}{n^s}$ for $s > 0$, then we get $\limsup_n |a_n|^{\frac{1}{n}} = 1$, because $|\mu(n)| = 1$ or 0, for all n. So, $a_n \notin G$ and as in Example 2.7, we can show that $(A_f a)_n \in G$.

THEOREM 2.3. *The following statements are equivalent:*

(i) The matrix A_f is $\ell - \ell$;

(ii) The matrix A_f is $G - \ell$;

(iii) The first column of the matrix A_f is in ℓ.

PROOF. It is obvious that (i) implies (ii), because $G \subset \ell$. To see that (ii) implies (iii), suppose that A_f is $G - \ell$. Therefore, by Theorem 1.1, we get that

$$\sum_{n=1}^{\infty} |A_f(n, k)| = M_k < \infty, \quad \text{for all } k.$$

Hence,

$$\sum_{n=1}^{\infty} |A_f(n, 1)| = M_1 < \infty.$$

In order to prove the remaining implication, we need the following lemmas.

https://doi.org/10.4153/CMB-1997-059-6 Published online by Cambridge University Press

LEMMA 2.1. *In the matrix A_f, all column sums are equal.*

PROOF. For any fixed k,

$$\sum_{n=1}^{\infty} A_f(n,k) = \sum_{n=1}^{\infty} \frac{1}{k} \sum_{q \le \frac{n}{k}} \frac{f(q)}{q}$$
$$= \frac{1}{k}\left[\sum_{n<k}\left(\sum_{q\le \frac{n}{k}} \frac{f(q)}{q}\right) + \sum_{k\le n<2k}\left(\sum_{q\le \frac{n}{k}} \frac{f(q)}{q}\right) + \sum_{2k\le n<3k}\left(\sum_{q\le \frac{n}{k}} \frac{f(q)}{q}\right) + \cdots\right].$$

The first term of the above infinite sum is zero. In each of the remaining terms, if $pk \le n < (p+1)k$, then $[\frac{n}{k}] = p$ and n takes exactly k different values. Thus,

$$\sum_{n=1}^{\infty} A_f(n,k) = \frac{1}{k}\left[k\sum_{q\le 1}\frac{f(q)}{q} + k\sum_{q\le 2}\frac{f(q)}{q} + \cdots\right]$$
$$= \sum_{n=1}^{\infty}\sum_{q\le n}\frac{f(q)}{q}$$
$$= \sum_{n=1}^{\infty} A_f(n,1),$$

which proves the lemma.

Repeating the above proof for $\sum_{n=1}^{\infty} |A_f(n,k)|$, we get the following lemma.

LEMMA 2.2. *In the matrix A_f, absolute column sums are equal.*

Now, let us prove that (iii) implies (i) of Theorem 2.3. Consider

$$\sup_k \sum_{n=1}^{\infty} |A_f(n,k)| = \sum_{n=1}^{\infty} |A_f(n,1)| < \infty,$$

by (iii). Hence by the Knopp-Lorentz [4] Theorem, A_f is an $\ell - \ell$ matrix.

EXAMPLE 2.9. For fixed $s > 1$, let

$$f(n) = \begin{cases} 1 - \zeta(s+1), & \text{if } n = 1, \\ \frac{1}{n^s}, & \text{if } n > 1. \end{cases}$$

Then in the corresponding matrix A_f,

$$A_f(n,1) = \sum_{q=1}^{n} \frac{f(q)}{q}$$
$$= -\zeta(s+1) + \sum_{q=1}^{n} \frac{1}{q^{s+1}}.$$

Using the known result that

$$\sum_{k\le n} \frac{1}{k^s} = \frac{n^{1-s}}{1-s} + \zeta(s) + O\left(\frac{1}{n^s}\right),$$

https://doi.org/10.4153/CMB-1997-059-6 Published online by Cambridge University Press

we get that

$$\sum_{n=1}^{\infty} |A_f(n,1)| = \sum_{n=1}^{\infty} \left| \frac{1}{sn^s} + O\Big(\frac{1}{n^{s+1}}\Big) \right|,$$

which converges since $s > 1$. Thus, A_f is $\ell - \ell$ and $G - \ell$ matrix.

EXAMPLE 2.10. If $f(n)$ is given by

$$f(n) = \begin{cases} 1, & \text{if } n = 1, \\ \frac{n+1}{(1-n)2^{n+1}}, & \text{if } n > 1, \end{cases}$$

then the matrix A_f is $\ell - \ell$.

Since

$$\sum_{q=1}^{n} \frac{f(q)}{q} = 1 - \sum_{q=2}^{n} \frac{(q+1)}{q(q-1)2^{q-1}} = \frac{1}{n2^{n-1}},$$

we have that

$$\sum_{n=1}^{\infty} |A_f(n,1)| = \sum_{n=1}^{\infty} \frac{1}{n2^{n-1}},$$

which is convergent.

EXAMPLE 2.11. By defining $f(n)$ as

$$f(n) = \begin{cases} \frac{1}{2}, & \text{if } n = 1, \\ \frac{(n+1)-n^2}{(n+1)!}, & \text{if } n > 1, \end{cases}$$

we get that

$$\sum_{q=1}^{n} \frac{f(q)}{q} = \frac{1}{2} - \sum_{q=2}^{n} \frac{q^2-q-1}{(q+1)!} = \frac{1}{2} - \sum_{q=2}^{n} \frac{(q+1)(q-1)-q}{(q+1)!} = \frac{n}{(n+1)!},$$

which yields that

$$\sum_{n=1}^{\infty} |A_f(n,1)| = \sum_{n=1}^{\infty} \frac{n}{(n+1)!}.$$

This sum converges to 1, because the partial sum

$$S_N = 1 - \frac{1}{(N+1)!}$$

approaches 1 as $N \longrightarrow \infty$. Thus, the matrix A_f is $\ell - \ell$ and $G - l$. Also, A_f is sum-preserving over ℓ and G by Theorem 1.3 and Lemma 2.1.

https://doi.org/10.4153/CMB-1997-059-6 Published online by Cambridge University Press

3. **Additional theorems.** In this section, we attempt to assess the strength of A_f-matrix. When $f(n) = \mu(n)$, the matrix A_μ is regular [6]. We notice that A_μ is stronger than the identity matrix, because for the non-convergent sequence a_n when $a_n = \frac{\sigma(n)}{n}$, the n-th term of the transformed sequence is given by

$$(A_\mu a)_n = \sum_{k\le n} \frac{1}{k} \sum_{d|k} \mu\left(\frac{k}{d}\right)\frac{\sigma(d)}{d}.$$

Since $\frac{\sigma(k)}{k} = \frac{1}{k}\sum_{d|k} d = \sum_{d|k}\frac{d}{k} = \sum_{d|k}\frac{1}{d}$, by Möbius Inversion formula [1, p. 2] we get that

$$\frac{1}{k} = \sum_{d|k} \mu\left(\frac{k}{d}\right)\frac{\sigma(d)}{d}.$$

Thus,

$$(A_\mu a)_n = \sum_{k\le n}\frac{1}{k^2},$$

which converges, as $n \longrightarrow \infty$.

Segal [8] gave a necessary condition on the function $f(n)$ for the regularity of A_f. We note that if $f(n)$ is a non-negative arithmetic function such that $f(1) \ne 0$ and $f \in c_0$, then A_f is not regular.

Since $f \in c_0$, there exists an N such that $f(n) < \frac{1}{2}f(1)$ for all $n > N$. Hence, for a sequence $a_n \equiv 1$, we have that

$$(A_f a)_n = \sum_{k\le n}\frac{1}{k}\sum_{d|k} f\left(\frac{k}{d}\right).$$

As $n \longrightarrow \infty$, $(A_f a)_n$ approaches

$$\begin{aligned}
\sum_{k=1}^{\infty}\frac{1}{k}\sum_{d|k} f(d) &> \sum_{\substack{p=1\\ p,\ \text{prime}}}^{\infty}\frac{1}{p}[f(1)+f(p)]\\
&= \sum_{p\le N}\frac{1}{p}[f(1)+f(p)] + \sum_{p>N}\frac{1}{p}[f(1)+f(p)]\\
&> M + \sum_{p>N}\frac{1}{p}[f(1) - \frac{1}{2}f(1)]\\
&= M + \frac{f(1)}{2}\sum_{p>N}\frac{1}{p},
\end{aligned}$$

which diverges.

Next, we compare the A_f-matrix with the Divisor matrix [3] which is given by

$$D(n,k) = \begin{cases} \frac{k}{\sigma(n)}, & \text{if } k|n,\\ 0, & \text{otherwise,}\end{cases}$$

where $\sigma(n)$ = sum of divisors of n. Fridy [4] showed that D is a regular matrix and for any triangular matrix M, if M^{-1} exists and $\lim_{n\to\infty} M(n,n) = 0$, then D does not include M.

THEOREM 3.1. *Divisor matrix D does not include A_f-matrix for any f with $f(1) \neq 0$.*

PROOF. A_f is a lower triangular matrix and

$$\begin{aligned} A_f(n,n) &= \frac{1}{n} \sum_{q \leq 1} \frac{f(q)}{q} \\ &= \frac{f(1)}{n} \neq 0 \end{aligned}$$

So, A_f^{-1} exists and $\lim_{n\to\infty} A_f(n,n) = 0$.

THEOREM 3.2. *If a sequence $\{a_n\}$ is Divisor summable to L such that $(Da)_n = L + o(1/n \log\log n)$ as $n \longrightarrow \infty$, then $\{a_n\}$ is A_f-summable to L for any regular A_f.*

PROOF. Since $(Da)_k = \sum_{m|k} \frac{m}{\sigma(k)} a_m$, we get that

$$(Da)_k \sigma(k) = \sum_{m|k} m a_m.$$

Möbius inversion formula yields that

$$k a_k = \sum_{m|k} \mu\left(\frac{k}{m}\right)(Da)_m \sigma(m).$$

Therefore,

$$\begin{aligned} (A_f a)_n &= \sum_{k\leq n} \frac{1}{k} \sum_{q \leq \frac{n}{k}} \frac{f(q)}{q} a_k = \sum_{k \leq n} \frac{a_k}{k} B_k \qquad \text{(say)} \\ &= \sum_{k\leq n} \frac{1}{k^2} \sum_{m|k} \mu\left(\frac{k}{m}\right)(Da)_m \sigma(m) B_k \\ &= \sum_{k\leq n} \frac{L}{k^2} \sum_{m|k} \mu\left(\frac{k}{m}\right)\sigma(m) B_k \\ &+ \sum_{k\leq n} \frac{1}{k^2} \sum_{m|k} \mu\left(\frac{k}{m}\right) o\left(\frac{1}{m \log\log m}\right)\sigma(m) B_k, \quad \text{as } m \longrightarrow \infty \\ &= S_1 + S_2 \quad \text{(say)} \end{aligned}$$

Since $\sigma(k) = \sum_{m|k} m$ implies that

$$k = \sum_{m|k} \mu\left(\frac{k}{m}\right)\sigma(m),$$

we get that

$$\begin{aligned} S_1 &= \sum_{k \leq n} \frac{L}{k} \sum_{q \leq \frac{n}{k}} \frac{f(q)}{q} \\ &= (A_f L)_n, \end{aligned}$$

where L is a constant sequence. The regularity of A_f yields that $S_1 \longrightarrow L$ as $n \longrightarrow \infty$.

https://doi.org/10.4153/CMB-1997-059-6 Published online by Cambridge University Press

We know that $\sigma(n) = O(n \log\log n)$. Hence,

$$\begin{aligned}
S_2 &= \sum_{k\le n} \frac{1}{k^2} \sum_{m|k} \mu\Big(\frac{k}{m}\Big) o(1) B_k, \quad \text{as } m \longrightarrow \infty \\
&= \sum_{k\le n} \frac{1}{k^2} O\Big(\sum_{m|k} \Big|\mu\Big(\frac{k}{m}\Big)\Big|\Big) B_k, \quad \text{as } k \longrightarrow \infty \\
&= \sum_{k\le n} \frac{1}{k^2} O\Big(\sum_{m|k} 1\Big) B_k \\
&= \sum_{k\le n} \frac{1}{k^2} O\big(d(k)\big) B_k \\
&= \sum_{k\le n} \frac{1}{k^2} O(k^{\frac{1}{2}}) B_k \quad [1, \text{p. } 19] \\
&= \sum_{k\le n} O\Big(\frac{1}{k^{\frac{1}{2}}}\Big) \Big[\frac{1}{k} \sum_{q\le \frac{n}{k}} \frac{f(q)}{q}\Big], \quad \text{as } k \longrightarrow \infty \\
&= \sum_{k\le n} u_k A_f(n,k),
\end{aligned}$$

where $\{u_k\}$ is a sequence such that $|u_k| \le M/k^{\frac{1}{2}}$ and hence $u_k \longrightarrow 0$ as $k \longrightarrow \infty$. Thus, the regularity of A_f yields that $S_2 = (A_f u)_n \longrightarrow 0$ as $n \longrightarrow \infty$. Hence, the theorem is proved.

REFERENCES

1. K. Chandrasekharan, *Arithmetical functions*. Chapter I, Springer-Verlag, New York, 1970.
2. G. H. Fricke and J. A. Fridy, *Matrix summability of geometrically dominated series*. Canad. J. Math. (3) **39**(1987), 568–582.
3. J. A. Fridy, *Divisor summability methods*. J. Math. Anal. Appl. (2) **12**(1965), 235–243.
4. K. Knopp and G. G. Lorentz, *Beiträge zur absoluten limitierung*. Arch. Math. **2**(1949), 10–16.
5. F. M. Mears, *Absolute regularity and the Nořlund mean*. Ann. of Math. (3) **38**(1937), 594–601.
6. L. A. Rubel, *An abelian theorem for number-theoretic sums*. Acta Arith. **6**(1960), 175–177; Correction, Acta Arith. **6**(1961), 523.
7. S. L. Segal, *Dirichlet convolutions and the Silverman-Toeplitz conditions*, Acta Arith. **X**(1964), 287–291.
8. ______, *A note on Dirichlet convolutions*, Canad. Math. Bull. (4) **9**(1966), 457–462.

Penn State University—Shenango Campus
147, Shenango Avenue
Sharon, Pennsylvania 16146
U.S.A.
e-mail: ulf@psuvm.psu.edu
e-mail: sxs32@psuvm.psu.edu