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GENERALIZED RESOLVENT EQUATIONS AND UNSYMMETRIC DIRICHLET SPACES

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0. Introduction. Let $(X, \mathscr{B}, \mu)$ and $(X, \mathscr{B}, \mu')$ be measure spaces with the measures μ and μ' totally finite. Suppose $\{U_\lambda : \lambda > 0\}$ is a family of positive (i.e., $\phi \geqq 0 \Rightarrow U_\lambda \phi \geqq 0$) continuous linear operators from $L^2(X, d\mu')$ to $L^2(X, d\mu)$ with the following additional properties: if $\phi \geqq 0$ then $U_\lambda \phi$ is non-decreasing as λ increases, while $\lambda^{-1} U_\lambda \phi$ is non-increasing.

A family $\{M_\lambda : \lambda > 0\}$ of continuous linear operators from $L^2(X, d\mu)$ to $L^2(X, d\mu')$ satisfies the "generalized resolvent equation" relative to $\{U_\lambda\}$ if

$$(0.1) \qquad M_\lambda - M_\nu = M_\lambda (U_\nu - U_\lambda) M_\nu$$

for positive λ and ν. If $U_\lambda = \lambda I$, then (0.1) is just the well-known resolvent equation. The family $\{M_\lambda\}$ is called *submarkov* if M_λ is a positive operator and

$$(0.2) \qquad M_\lambda U_\lambda 1 \leqq 1 \quad \mu' - \text{a.e.};$$

it is *conservative* if

$$(0.3) \qquad M_\lambda U_\lambda 1 = 1 \quad \mu' - \text{a.e.}$$

Families of operators satisfying (0.1) for choices of U_λ other than λI are found in the theory of boundary value problems associated with stochastic processes. For example, Fukushima [**2**] classified all the conservative, symmetric Brownian processes over an arbitrary bounded domain in $\mathbf{R}^n$. This amounts to finding all the conservative resolvents $\{G_\lambda : \lambda > 0\}$ on $L^2(E, dx)$ (where E is a bounded domain in $\mathbf{R}^n$ and dx is Lebesgue measure), given by symmetric kernels of the form

$$(0.4) \qquad G_\lambda(x, y) = G_\lambda^0(x, y) + H_\lambda(x, y),$$

where G_λ^0 is the minimal resolvent density corresponding to the absorbing barrier Brownian motion on E. The term $H_\lambda(x, y)$ is λ-harmonic (i.e., $\frac{1}{2}\Delta H_\lambda = \lambda H_\lambda$) in x for each fixed $\lambda > 0$ and $y \in E$. This λ-harmonic term has the so-called *Feller representation*:

$$(0.5) \qquad H_\lambda(x, y) = \int_{\partial E} \int_{\partial E} K_\lambda(\eta, x) M_\lambda(\eta, \xi) K_\lambda(\xi, y) \mu_0(d\xi) \mu_0(d\eta),$$

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where ∂E is the Martin boundary, K_λ is the Poisson-Martin kernel for the λ-harmonic functions, and μ_0 is the canonical representing measure for 1, i.e.,

$$(0.6) \qquad 1 = \int_{\partial E} K(\xi, x)\mu_0(d\xi).$$

The kernel $K(\xi, x)$ is the Poisson-Martin kernel for the harmonic functions. The kernels $M_\lambda(\eta, \xi)$ in the Feller representation give rise to mappings which satisfy the conditions of Section 1 with

$$(0.7) \qquad \mathscr{U}_\lambda \phi(\xi) = \int_{\partial E} \mathscr{U}_\lambda(\xi, \eta)\phi(\eta)\mu_0(d\eta),$$

where

$$(0.8) \qquad \mathscr{U}_\lambda(\xi, \eta) = \lambda \int_{\partial E} K_\lambda(\xi, x)K(\eta, x)dx = \mathscr{U}_\lambda(\eta, \xi).$$

If $\{G_\lambda : \lambda > 0\}$ is a conservative resolvent family, then it turns out that $M_\lambda \mathscr{U}_\lambda 1 = 1$ i.e., the family $\{M_\lambda\}$ is conservative. Thus, the classification of all the conservative resolvents of type (0.4) amounts to a study of the conservative families $\{M_\lambda : \lambda > 0\}$ in the Feller representation which satisfy (0.1) with $\{\mathscr{U}_\lambda\}$ given by (0.7) and (0.8).

Each conservative family $\{M_\lambda : \lambda > 0\}$ determines a Dirichlet space on ∂E in which range M_λ (which is independent of λ) is dense. The Dirichlet form for the space has the structure

$$(0.9) \qquad \mathscr{E}(\psi, \phi) = U\langle \psi, \phi \rangle + N(\psi, \phi)$$

where

$$(0.10) \qquad U\langle \psi, \phi \rangle = \lim_{\lambda \to \infty} \frac{1}{2} \int_{\partial E} \int_{\partial E} [\phi(\xi) - \phi(\eta)][\psi(\xi) - \psi(\eta)] \times \mathscr{U}_\lambda(\xi, \eta)\mu_0(d\xi)\mu_0(d\eta),$$

and the form N is a Dirichlet form, i.e., $N(\phi, \phi) \geqq 0$ and

$$(0.11) \qquad N((\phi - c)^+, \phi \wedge c) \geqq 0$$

for all positive constants c, and all ϕ in the Dirichlet space. The conservativity requirement for the resolvent is reflected in the requirement that $N(1, 1) = 0$. Since $\mathscr{E}(1, 1) = 0$, we do not get a norm from $\mathscr{E}(\phi, \phi)^{1/2}$, but we can add a suitable L^2 term to get an actual norm:

$$(0.12) \qquad |||\phi|||^2 = \mathscr{E}(\phi, \phi) + \alpha\|\phi\|_2^2,$$

and for each $\alpha > 0$ the space is complete in this norm.

Conversely, any Dirichlet space with a form of type (0.9) with $N(1, 1) = 0$ determines a conservative family $\{M_\lambda : \lambda > 0\}$ which in turn gives a conservative Brownian motion resolvent, via the Feller representation. We refer to [**2**] for the technical details.

Kunita [**4**] extended Fukushima's results to unsymmetric Brownian motions in which the Laplacian is replaced by an unsymmetric elliptic operator of the form

$$(0.13)\quad L = \sum \frac{\partial}{\partial x_i}(a_{ij}\partial/\partial x_j) + \sum b_i \partial/\partial x_i,$$

where $a_{ij} = a_{ji}$ is bounded, measurable, uniformly positive definite, and $b_i \in L^{p_0}(E, dx)$ for some $p_0 > n$. Kunita did not use Dirichlet spaces on the boundary, but he did make use of the Feller representation to get the generator Q of the Ueno Markovian semigroup on the boundary. We shall go into more of the details of this example in Section 7.

In the present paper, we shall be interested in submarkov (not necessarily conservative) generalized resolvent families $\{M_\lambda : \lambda > 0\}$, whose adjoint family $\{M_\lambda' : \lambda > 0\}$ is also submarkov, and which determine "unsymmetric" Dirichlet spaces in the sense of Definitions 5.1 and 5.2 in Section 5. We make use of the concept of unsymmetric Dirichlet forms developed in [**3**]. The family $\{U_\lambda : \lambda > 0\}$ which we consider is of a rather general type, not necessarily connected with differential operators. The results are therefore applicable to other types of processes with Feller representations.

In Section 1 we give the most basic information about $\{U_\lambda\}$ and $\{M_\lambda\}$. In Section 2 various quadratic forms determined by M_λ and U_λ are defined and studied. In Section 3 we discuss "continuity conditions" for quadratic forms, a type of condition satisfied by Kunita's unsymmetric Dirichlet forms. The complete spaces that result from using norms associated with our quadratic forms are studied in Section 4. The generalized concept of Dirichlet space that we use is defined and applied in Section 5. In Section 6 we show that each of these Dirichlet spaces whose associated Dirichlet form is of specified type determines a submarkov family $\{M_\lambda : \lambda > 0\}$ satisfying the requirements set down in Sections 1 and 3. Section 7 is devoted to illustrative examples.

1. The basic operators. Let $(X, \mathscr{B}, m)$ and $(X, \mathscr{B}, m')$ be σ-finite measure spaces.

We are given a family of linear operators $\{\mathscr{U}_\lambda : \lambda > 0\}$ with the following properties:

U1: $\mathscr{U}_\lambda : L^\infty(X, dm') \to L^1(X, dm)$

$\mathscr{U}_\lambda' : L^\infty(X, dm) \to L^1(X, dm')$.

(Here $\mathscr{U}_\lambda'$ denotes the adjoint of $\mathscr{U}_\lambda$.)

U2: $\phi \geqq 0 \Rightarrow \mathscr{U}_\lambda \phi \geqq 0$

U3: $\phi \geqq 0$ and $\lambda_2 \geqq \lambda_1 \Rightarrow \mathscr{U}_{\lambda_2}\phi \geqq \mathscr{U}_{\lambda_1}\phi$ and

$$\lambda_2^{-1}\mathscr{U}_{\lambda_2}\phi \leqq \lambda_1^{-1}\mathscr{U}_{\lambda_1}\phi$$

U4: $\mathscr{U}_\lambda 1 > 0$ m – a.e. and $\mathscr{U}_\lambda' 1 > 0$ m' – a.e.

U5: $\mathscr{U}_\lambda 1 \uparrow +\infty$ and $\mathscr{U}_\lambda' 1 \uparrow +\infty$ a.e. as $\lambda \uparrow +\infty$.

We form three new measures:

$$\begin{aligned} d\mu &= \mathscr{U}_1 1\, dm \\ (1.1) \qquad d\mu' &= \mathscr{U}_1' 1\, dm' \\ d\tilde{\mu} &= \tfrac{1}{2}(d\mu + d\mu'). \end{aligned}$$

Clearly, the measures $d\mu$ and $d\mu'$ are absolutely continuous with respect to $d\tilde{\mu}$.

Define

$$(1.2) \qquad U_\lambda \phi(\xi) = \mathscr{U}_\lambda \phi(\xi)[\mathscr{U}_1 1(\xi)]^{-1}$$

if $\mathscr{U}_1 1(\xi) < \infty$ and zero otherwise. Similarly,

$$(1.3) \qquad U_\lambda' \phi(\xi) = \mathscr{U}_\lambda' \phi(\xi)[\mathscr{U}_\lambda' 1(\xi)]^{-1}$$

if $\mathscr{U}_1' 1(\xi) < \infty$ and zero otherwise. Then

$$\begin{aligned} (1.4) \qquad & 1 \leqq U_\lambda 1 \leqq \lambda \quad m \text{ – a.e.} \\ & 1 \leqq U_\lambda' 1 \leqq \lambda \quad m' \text{ – a.e.} \end{aligned}$$

Therefore, both U_λ and U_λ' map bounded functions into bounded functions. We next extend the mappings U_λ and U_λ' so that

$$\begin{aligned} (1.5) \qquad & U_\lambda : L^2(X, d\mu') \rightarrow L^2(X, d\mu) \\ & U_\lambda' : L^2(X, d\mu) \rightarrow L^2(X, d\mu') \end{aligned}$$

with U_λ' now denoting the L^2 adjoint of U_λ. In fact, if $\phi \in L^\infty(X, d\mu')$, then

$$\begin{aligned} (1.6) \qquad \int (U_\lambda \phi)^2 d\mu &\leqq \int (U_\lambda \phi^2)(U_\lambda 1) d\mu \leqq \lambda \int U_\lambda \phi^2 d\mu \\ &= \lambda \int \phi^2 U_\lambda' 1 d\mu' \leqq \lambda^2 \int \phi^2 d\mu'. \end{aligned}$$

Thus U_λ extends as claimed with

$$(1.7) \qquad \|U_\lambda\| \leqq \lambda.$$

Similar considerations apply to U_λ' and show that the extension of U_λ' is actually the L^2-adjoint of U_λ.

Associated with the family $\{U_\lambda : \lambda > 0\}$, we shall consider families $\{M_\lambda : \lambda > 0\}$ of operators from $L^2(X, d\mu)$ to $L^2(X, d\mu')$ satisfying the "generalized resolvent equation" (0.1). Note that (0.1) is equivalent to

$$(1.8) \quad M_\lambda - M_\nu = M_\nu(U_\nu - U_\lambda)M_\lambda,$$

for positive λ and ν. The adjoint operators M_λ' map $L^2(X, d\mu')$ into $L^2(X, d\mu)$ and satisfy

$$(1.9) \quad M_\lambda' - M_\nu' = M_\nu'(U_\nu' - U_\lambda')M_\lambda' = M_\lambda'(U_\nu' - U_\lambda')M_\nu'.$$

Note that both Range (M_λ) and Range (M_λ') are independent of λ. We denote them by $R(M)$ and $R(M')$, respectively.

In this paper we shall be interested in the case that the following conditions are satisfied:

$$\begin{aligned} &\text{M1: } \phi \geqq 0 \Rightarrow M_\lambda \phi \geqq 0 && (\forall \lambda > 0) \\ &\text{M2: } M_\lambda U_\lambda 1 \leqq 1 && (\mu' - \text{a.e.}) \\ &\qquad M_\lambda' U_\lambda' 1 \leqq 1 && (\mu - \text{a.e.}) \end{aligned}$$

(the "submarkov" conditions). It is easily seen that if $\phi \geqq 0$, then both $M_\lambda \phi$ and $M_\lambda' \phi$ decrease as λ increases.

Let us define a new family of operators by

$$(1.10) \quad \begin{aligned} S_\lambda &= M_\lambda U_\lambda \\ \bar{S}_\lambda &= M_\lambda' U_\lambda'. \end{aligned}$$

We then have, for bounded ϕ

$$(1.11) \quad \int (S_\lambda \phi)^2 U_\lambda' 1 d\mu' \leqq \int (S_\lambda \phi^2) U_\lambda' 1 d\mu' \leqq \int \phi^2 U_\lambda' \bar{S}_\lambda 1 d\mu' \leqq \int \phi^2 U_\lambda' 1 d\mu' \leqq (1 \vee \lambda) \int \phi^2 d\mu'$$

and similarly

$$(1.12) \quad \int (\bar{S}_\lambda \phi)^2 U_\lambda 1 d\mu \leqq \int \phi^2 U_\lambda 1 \leqq (1 \vee \lambda) \int \phi^2 d\mu.$$

Thus, since $(\lambda \wedge 1) \leqq U_\lambda 1 \leqq (\lambda \vee 1)$ we see that, as a mapping from $L^2(d\mu')$ into itself, we have

$$\|S_\lambda\| \leqq [(\lambda \vee 1)/(\lambda \wedge 1)]^{1/2}$$

and similarly for $\bar{S}_\lambda$ as a mapping from $L^2(d\mu)$ to itself. If we knew that $U_\lambda 1 \geqq c\lambda$ for some positive constant, then we could conclude that $\sup \{\|S_\lambda\| : \lambda > 0\} < \infty$, but we shall not make that assumption.

We shall also need the operators

$$(1.13) \quad \begin{aligned} T_\lambda \phi &= M_\lambda(\phi \cdot U_\lambda 1) \\ \bar{T}_\lambda \phi &= M_\lambda'(\phi \cdot U_\lambda' 1). \end{aligned}$$

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It is easily seen (since $T_\lambda 1 \leqq 1$ and $\bar{T}_\lambda 1 \leqq 1$) that

$$(1.14) \quad \int (T_\lambda \phi)^2 U_\lambda' 1 d\mu' \leqq \int \phi^2 U_\lambda 1 d\mu,$$

and

$$(1.15) \quad \int (\bar{T}_\lambda \phi)^2 U_\lambda 1 d\mu \leqq \int \phi^2 U_\lambda' 1 d\mu'.$$

From this we conclude that if $\phi \in L^2(d\mu)$,

$$(1.16) \quad \int (M_\lambda \phi)^2 U_\lambda' 1 d\mu' \leqq \int [U_\lambda 1]^{-1} \cdot \phi^2 d\mu.$$

Note that the right side is finite since $U_\lambda 1 \geqq 1 \wedge \lambda$. Similarly,

$$(1.17) \quad \int (M_\lambda' \phi)^2 \cdot U_\lambda 1 d\mu \leqq \int [U_\lambda' 1]^{-1} \cdot \phi^2 d\mu'.$$

The right sides of (1.16) and (1.17) have integrands decreasing to zero, and we can conclude that

$$(1.18) \quad \lim_{\lambda\to\infty} \int (M_\lambda \phi)^2 U_\lambda' 1 d\mu' = \lim_{\lambda\to\infty} \int (M_\lambda' \phi)^2 U_\lambda 1 d\mu = 0.$$

If $\phi = M_\nu \eta$, then from (1.8)

$$(1.19) \quad \phi - S_\lambda \phi = M_\lambda(\eta - U_\nu M_\nu \eta)$$

so that

$$(1.20) \quad \int (\phi - S_\lambda \phi)^2 d\mu' \leqq \int [U_\lambda 1]^{-1} (\eta - U_\nu M_\nu \eta)^2 d\mu.$$

Thus

$$(1.21) \quad \lim_{\lambda\to\infty} \int (\phi - S_\lambda \phi)^2 d\mu' = 0.$$

Using similar arguments,

$$(1.22) \quad \lim_{\lambda\to\infty} \int (\phi - \bar{S}_\lambda \phi)^2 d\mu = 0$$

if $\phi \in R(M')$.

If S_λ and $\bar{S}_\lambda$ have operator norms which are bounded in λ, then (1.21) and (1.22) will hold for the closures $\overline{R(M)}$ and $\overline{R(M')}$ in $L^2(d\mu')$ and $L^2(d\mu)$, respectively. However, we do not know that this is true, and shall assume only the following condition:

M3: $\bar{S}_\lambda \psi \to \psi$ (weakly in $L^2(d\mu)$) if $\psi \in R(M) \cap L^2(d\mu)$.

(Note that $R(M) \cap L^2(d\mu)$ is nonempty, since M takes bounded

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functions into bounded functions. This follows from M2 plus the fact that $U_\lambda 1 \geqq 1 \wedge \lambda$.)

2. The associated quadratic forms. Define the measures

$$(2.1)\qquad \begin{aligned} d\mu_\lambda &= U_\lambda 1 d\mu \\ d\mu_\lambda' &= U_\lambda' 1 d\mu' \\ d\tilde{\mu}_\lambda &= \tfrac{1}{2}(d\mu_\lambda + d\mu_\lambda') \end{aligned}$$

for $\lambda > 0$. Note that when $\lambda = 1$, we get the measures defined in (1.1). If ϕ and ψ belong to $L^2(d\tilde{\mu})$, then they also belong to each $L^2(d\tilde{\mu}_\lambda)$, and we can define the symmetric quadratic form

$$(2.2)\qquad \tilde{U}_\lambda\langle \psi, \phi\rangle = \int \psi\phi d\tilde{\mu}_\lambda - \frac{1}{2}\int \psi U_\lambda \phi d\mu - \frac{1}{2}\int \psi U_\lambda' \phi d\mu'.$$

This form has the properties:

$$(2.3)\qquad \tilde{U}_\lambda\langle \phi, \phi\rangle \geqq 0,$$

$$(2.4)\qquad \tilde{U}_\lambda\langle \phi, \phi\rangle \geqq \tilde{U}_\nu\langle \phi, \phi\rangle$$

if $\lambda \geqq \nu \geqq 0$.

From now on we shall consider $\mathscr{U}_\lambda$'s of the form

$$(2.5)\qquad \mathscr{U}_\lambda \phi(\xi) = f_\lambda(\xi)\phi(\xi) + \int \mathscr{U}_\lambda(\xi, \eta)\phi(\eta)m(d\eta),$$

where $f_\lambda \in L^1(dm)$ is non-negative with f_λ non-decreasing in λ and $\lambda^{-1}f_\lambda$ non-increasing. The positive kernel $\mathscr{U}_\lambda$ is integrable on the product space and has the property that $\mathscr{U}_\lambda(\xi, \eta)$ is non-decreasing as λ increases, while $\lambda^{-1}\mathscr{U}_\lambda(\xi, \eta)$ is non-increasing. Then

$$(2.6)\qquad U_\lambda \phi(\xi) = b_\lambda(\xi)\phi(\xi) + \int U_\lambda(\xi, \eta)\phi(\eta)\mu'(d\eta)$$

and

$$(2.7)\qquad U_\lambda'\psi(\xi) = b_\lambda'(\xi)\psi(\xi) + \int U_\lambda(\eta, \xi)\psi(\xi)\mu(d\eta)$$

where

$$(2.8)\qquad \begin{aligned} b_\lambda &= f_\lambda \cdot [\mathscr{U}_1 1]^{-1} \\ b_\lambda' &= f_\lambda \cdot [\mathscr{U}_1' 1]^{-1} \\ U_\lambda(\xi, \eta) &= \mathscr{U}_\lambda(\xi, \eta)\cdot[\mathscr{U}_1 1(\xi)\cdot \mathscr{U}_1'(\eta)]^{-1}. \end{aligned}$$

We use the following notation

$$(2.9)\qquad c_\lambda(\xi) = \int U_\lambda(\xi, \eta)\mu'(d\eta)$$

$$c_\lambda'(\eta) = \int U_\lambda(\xi, \eta)\mu(d\xi).$$

Then

$$(2.10)\qquad b_\lambda(\xi) + c_\lambda(\xi) = U_\lambda 1(\xi)$$
$$b_\lambda'(\eta) + c_\lambda'(\eta) = U_\lambda' 1(\eta).$$

From this representation of U it follows that

$$(2.11)\qquad U_\lambda\phi(\xi) - \phi(\xi)\cdot U_\lambda 1(\xi) = \int U_\lambda(\xi, \eta)[\phi(\eta) - \phi(\xi)]\mu'(d\eta)$$

and

$$(2.12)\qquad U_\lambda'\psi(\xi) - \psi(\xi)\cdot U_\lambda' 1(\xi) = \int U_\lambda(\eta, \xi)[\psi(\eta) - \psi(\xi)]\mu(d\eta).$$

Let

$$0 \leqq U(\xi, \eta) = \lim_{\lambda\uparrow\infty} U_\lambda(\xi, \eta) \leqq +\infty$$

and

$$N_n = \{(\xi, \eta): U(\xi, \eta) \geqq n\}.$$

For each fixed n, we define

$$U_\lambda^{(n)}(\xi, \eta) = U_\lambda(\xi, \eta) \qquad (\xi, \eta) \notin N_n$$
$$= 0 \qquad (\xi, \eta) \in N_n$$

and

$$\Phi_n(\xi) = \int U_\lambda^{(n)}(\xi, \eta)[\phi(\xi) - \phi(\eta)]\mu'(d\eta).$$

Then

$$(2.13)\qquad \lim_{\lambda\to\infty} \int \Phi_n^{2}(\xi)\cdot [U_\lambda 1]^{-1}\mu(d\xi) = 0$$

because of condition U5 (Section 1). Similarly, if

$$\Psi_n(\xi) = \int U_\lambda^{(n)}(\eta, \xi)[\psi(\xi) - \psi(\eta)]\mu(d\eta),$$

then

$$(2.14)\qquad \lim_{\lambda\to\infty} \int \Psi_n^{2}\cdot [U_\lambda' 1]^{-1}\mu'(d\xi) = 0.$$

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In terms of the kernel $U_\lambda(\xi, \eta)$ we have

$$(2.15)\quad \tilde{U}_\lambda\langle\psi, \phi\rangle = \frac{1}{2}\int\int U_\lambda(\xi, \eta)[\phi(\xi) - \phi(\eta)] \times [\psi(\xi) - \psi(\eta)]\mu'(d\eta)\mu(d\xi)$$

for all ϕ and ψ in $L^2(d\tilde{\mu})$.

Let

$$(2.16)\quad \mathscr{D} = \{\phi \in L^2(d\tilde{\mu}): \sup_\lambda \tilde{U}_\lambda\langle\phi, \phi\rangle < \infty\}.$$

If $\phi \in \mathscr{D}$ and $\psi \in \mathscr{D}$, we write

$$(2.17)\quad \tilde{U}\langle\psi, \phi\rangle = \lim_{\lambda\to\infty} \tilde{U}_\lambda\langle\psi, \phi\rangle,$$

the limit being finite in this case. If $U(\xi, \eta) < \infty$ almost everywhere in the product space, then we have

$$(2.18)\quad \tilde{U}\langle\psi, \phi\rangle = \frac{1}{2}\int\int U(\xi, \eta)[\psi(\xi) - \psi(\eta)][\phi(\xi) - \phi(\eta)]\mu(d\xi)\mu'(d\eta).$$

If $U(\xi, \eta) = +\infty$ on set S of positive $\mu \times \mu'$, then each $\phi \in \mathscr{D}$ would have to be constant on that set. Thus (2.18) would hold if the integration is extended just over the set where $U(\xi, \eta)$ is finite. Or, alternatively, we can simply define the integrand in (2.18) to be zero on the set S. With this understanding, we shall use (2.18) in both cases.

It is easily seen that $\mathscr{D}$ is a Hilbert space with inner product and norm

$$(2.19)\quad (\psi, \phi)_{\mathscr{D}} = \tilde{U}\langle\psi, \phi\rangle + \int \psi\phi d\tilde{\mu}$$

$$\|\phi\|_{\mathscr{D}}^2 = \tilde{U}\langle\phi, \phi\rangle + \int \phi^2 d\tilde{\mu}.$$

The semi-norm $\tilde{U}\langle\phi, \phi\rangle$ is not a norm, since $\tilde{U}\langle 1, 1\rangle = 0$.

LEMMA 2.1. *If* $\phi \in \mathscr{D}$, *then*

$$(2.20)\quad \lim_{\lambda\to\infty} \int [U_\lambda\phi - \phi U_\lambda 1]^2 \cdot [U_\lambda 1]^{-1} d\mu = 0$$

and

$$(2.21)\quad \lim_{\lambda\to\infty} \int [U_\lambda'\phi - \phi U_\lambda' 1]^2 \cdot [U_\lambda' 1]^{-1} d\mu' = 0.$$

Proof. To prove (2.20), we use (2.13) plus

$$(2.22)\quad \int [U_\lambda\phi - \phi U_\lambda 1]^2 \cdot [U_\lambda 1]^{-1} d\mu - \int \Phi_n^2 \cdot [U_\lambda 1]^{-1} d\mu \leqq \int_{N_n} \int U(\xi, \eta)[\phi(\xi) - \phi(\eta)]^2 \mu(d\xi)\mu(d\eta).$$

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The right side can be made as small as we please by taking n sufficiently large, provided $\phi \in \mathscr{D}$. A similar argument proves (2.21).

We next turn to various quadratic forms associated with the particular family $\{M_\lambda : \lambda > 0\}$ under consideration. These involve the mappings S_λ, $\bar{S}_\lambda$, T_λ and $\bar{T}_\lambda$ introduced in (1.10) and (1.13).

We shall use the following notation for inner products:

$$\int \psi\phi d\mu = (\psi, \phi)$$

$$(2.23)\quad \int \psi\phi d\mu' = (\psi, \phi)'$$

$$\int \psi\phi d\tilde{\mu} = (\psi, \phi)_2.$$

Definition 2.1. We define

$$(2.24)\quad \mathscr{E}_\lambda(\psi, \phi) = (\psi, U_\lambda(\phi - S_\lambda\phi)) = (U_\lambda'(\psi - \bar{S}_\lambda\psi), \phi)'$$

for $\psi \in L^2(d\mu)$ and $\phi \in L^2(d\mu')$;

$$(2.25)\quad \mathscr{E}_\lambda^{(1)}(\psi, \phi) = (\psi, U_\lambda'1(\phi - S_\lambda\phi))'$$

for ϕ and ψ in $L^2(d\mu')$;

$$(2.26)\quad \bar{\mathscr{E}}_\lambda^{(1)}(\psi, \phi) = (U_\lambda 1 \cdot (\psi - \bar{S}_\lambda\psi), \phi)$$

and ϕ and ψ in $L^2(d\mu')$;

$$(2.27)\quad V_\lambda(\psi, \phi) = \int \psi\phi d\tilde{\mu}_\lambda - \int \psi T_\lambda \phi d\mu_\lambda' = \int \psi\phi d\tilde{\mu}_\lambda - \int \bar{T}_\lambda\psi \cdot \phi d\mu_\lambda,$$

for ψ and $\phi \in L^2(d\tilde{\mu})$.

Note that all of these forms are defined for ϕ and ψ in $L^2(d\tilde{\mu})$. Whenever the following limits exist, we write

$$\mathscr{E}(\psi, \phi) = \lim_{\lambda\to\infty} \mathscr{E}_\lambda(\psi, \phi)$$

$$(2.28)\quad \mathscr{E}^{(1)}(\psi, \phi) = \lim_{\lambda\to\infty} \mathscr{E}_\lambda^{(1)}(\psi, \phi)$$

$$\bar{\mathscr{E}}^{(1)}(\psi, \phi) = \lim_{\lambda\to\infty} \mathscr{E}_\lambda^{(1)}(\psi, \phi)$$

$$V(\psi, \phi) = \lim_{\lambda\to\infty} V_\lambda(\psi, \phi).$$

LEMMA 2.2. *If* $\phi \in L^2(d\tilde{\mu})$, *then*

$$\mathscr{E}_\lambda^{(1)}(\phi, \phi) \geqq 0$$

$$(2.29)\quad \bar{\mathscr{E}}_\lambda^{(1)}(\phi, \phi) \geqq 0$$

$$V_\lambda(\phi, \phi) \geqq 0.$$

Proof. The statement for $\mathscr{E}_\lambda^{(1)}$ and $\bar{\mathscr{E}}_\lambda^{(1)}$ follows from (1.11) and (1.12).

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For V_λ we make use of (1.14). Thus

$$(2.30)\quad \int \phi T_\lambda \phi d\mu_\lambda' \leqq \left(\int \phi^2 d\mu_\lambda' \right)^{1/2} \left(\int (T_\lambda \phi)^2 d\mu_\lambda' \right)^{1/2}$$
$$\leqq \left(\int \phi^2 d\mu_\lambda' \right)^{1/2} \left(\int \phi^2 d\mu_\lambda \right)^{1/2} \leqq \left(\int \phi^2 d\tilde{\mu}_\lambda \right)^{1/2}.$$

LEMMA 2.3. *For each* $\phi \in L^2(d\tilde{\mu})$ *and each integer* $n \geqq 0$ *we have*

$$(2.31)\quad \mathscr{E}_\lambda^{(1)}(\phi, \phi) \geqq \frac{1}{2} \int_{N_n} \int U_\lambda(\xi, \eta)[\phi(\xi) - \phi(\eta)]^2 \mu(d\xi)\mu'(d\eta)$$
$$- \frac{1}{2} \int_X \Phi_n^2(\xi)[U_\lambda 1]^{-1} \mu(d\xi)$$

where Φ_n *is defined as in* (2.13).

Proof. We have

$$(2.32)\quad \mathscr{E}_\lambda^{(1)}(\phi, \phi) = V_\lambda(\phi, \phi) + \tilde{U}_\lambda\langle \phi, \phi \rangle + \int [\phi U_\lambda 1 - U_\lambda \phi]$$
$$\times [\bar{T}_\lambda \phi - \phi] d\mu \geqq \left\{ V_\lambda(\phi, \phi) - \frac{1}{2} \int (\phi - \bar{T}_\lambda \phi)^2 d\mu_\lambda \right\}$$
$$+ \left\{ \tilde{U}_\lambda \langle \phi, \phi \rangle - \frac{1}{2} \int [\phi U_\lambda 1 - U_\lambda \phi]^2 \cdot [U_\lambda 1]^{-1} d\mu \right\}.$$

But

$$(2.33)\quad \frac{1}{2} \int (\phi - \bar{T}_\lambda \phi)^2 d\mu_\lambda = \frac{1}{2} \int \phi^2 d\mu_\lambda + \frac{1}{2} \int (\bar{T}_\lambda \phi)^2 d\mu_\lambda - \int \phi \bar{T}_\lambda \phi d\mu_\lambda$$
$$\leqq \int \phi^2 d\tilde{\mu}_\lambda - \int \phi \bar{T}_\lambda \phi d\mu_\lambda = V_\lambda(\phi, \phi).$$

In addition, using (2.11), we have

$$(2.34)\quad \tilde{U}_\lambda \langle \phi, \phi \rangle - \frac{1}{2} \int [\phi U_\lambda 1 - U_\lambda \phi]^2 [U_\lambda 1]^{-1} d\mu$$
$$\geqq \int_{N_n^c} \int [\phi(\xi) - \phi(\eta)]^2 \cdot U_\lambda(\xi, \eta) \mu'(d\eta) \mu(d\xi)$$
$$- \frac{1}{2} \int_X \Phi_n^2(\xi)[U_\lambda 1]^{-1} \mu(d\xi),$$

which completes the proof.

COROLLARY 2.1. (i) *If*

$$\sup_\lambda \mathscr{E}_\lambda^{(1)}(\phi, \phi) < \infty,$$

then $\phi \in \mathscr{D}$, *and*

$$(2.35)\quad \sup_\lambda \mathscr{E}_\lambda^{(1)}(\phi, \phi) \geqq \tilde{U}\langle \phi, \phi \rangle;$$

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(ii) *If* $\mathscr{E}^{(1)}(\phi, \phi)$ *exists, then* $\phi \in \mathscr{D}$ *and*

$$(2.36) \quad \mathscr{E}^{(1)}(\phi, \phi) \geqq \tilde{U}\langle \phi, \phi \rangle.$$

Proof. Use (2.31) and (2.13).

COROLLARY 2.2. *If* $\mathscr{E}^{(1)}(\phi, \phi)$ *exists, then so does* $V(\phi, \phi)$ *and*

$$(2.37) \quad \mathscr{E}^{(1)}(\phi, \phi) = \tilde{U}\langle \phi, \phi \rangle + V(\phi, \phi).$$

Proof. If $\mathscr{E}^1(\phi, \phi)$ exists, then $\phi \in \mathscr{D}$ from Corollary 2.1 (ii). From (2.32) we conclude that

$$(2.38) \quad \mathscr{E}_\lambda^{(1)}(\phi, \phi) \geqq V_\lambda(\phi, \phi) + \tilde{U}_\lambda\langle \phi, \phi \rangle - \{L_\lambda(\phi)\}^{1/2} \cdot \left\{ \int (\phi - \bar{T}_\lambda \phi)^2 d\mu_\lambda \right\}^{1/2}$$

where

$$(2.39) \quad L_\lambda(\phi) = \int [U_\lambda \phi - \phi \cdot U_\lambda 1]^2 [U_\lambda 1]^{-1} d\mu.$$

But

$$(2.40) \quad L_\lambda(\phi) \leqq 2\tilde{U}_\lambda\langle \phi, \phi \rangle.$$

Using this and (2.33) we have

$$(2.41) \quad \mathscr{E}_\lambda^{(1)}(\phi, \phi) \geqq [V_\lambda(\phi, \phi)^{1/2} - \tilde{U}_\lambda\langle \phi, \phi \rangle^{1/2}]^2.$$

Since both $\mathscr{E}^{(1)}(\phi, \phi)$ and $\tilde{U}\langle \phi, \phi \rangle$ exist, we must then have

$$\sup_\lambda V_\lambda(\phi, \phi) < \infty.$$

But this implies

$$(2.42) \quad \lim_{\lambda \to \infty} \int [\phi \cdot U_\lambda 1 - U_\lambda \phi][\bar{T}_\lambda \phi - \phi] d\mu = 0,$$

since $\lim_{\lambda \to \infty} L_\lambda(\phi) = 0$ by (2.20). The conclusion of the theorem then follows from (2.32).

LEMMA 2.4. *If* $\mathscr{E}^{(1)}(\phi, \phi)$ *exists, then* $\mathscr{E}(\phi, \phi)$ *exists and*

$$(2.43) \quad \mathscr{E}(\phi, \phi) = \mathscr{E}^{(1)}(\phi, \phi).$$

Proof. We have

$$(2.44) \quad |\mathscr{E}_\lambda(\phi, \phi) - \mathscr{E}_\lambda^{(1)}(\phi, \phi)| = \left| \int [U_\lambda' \phi - \phi U_\lambda' 1][\phi - S_\lambda \phi] d\mu \right| \leqq \{\bar{L}_\lambda(\phi)\}^{1/2} \{2\mathscr{E}_\lambda^{(1)}(\phi, \phi)\}^{1/2},$$

where

$$(2.45) \quad \bar{L}_\lambda(\phi) = \int [U_\lambda' \phi - \phi U_\lambda 1]^2 [U_\lambda' 1]^{-1} d\mu'.$$

We have also used the easily verified inequality

$$(2.46)\quad \int [\phi - S_\lambda\phi]^2 d\mu_\lambda' \leqq 2\mathscr{E}_\lambda^{(1)}(\phi, \phi).$$

The right side of (2.44) goes to zero as $\lambda \to \infty$ by Corollary 2.1 (ii) and (2.21).

THEOREM 2.1. *If* $\phi = M_\nu\eta$ *for some* $\eta \in L^\infty(d\tilde{\mu})$, *then* $\mathscr{E}(\phi, \phi)$ *exists* $\phi \in \mathscr{D}$, *and*

$$(2.47)\quad \mathscr{E}(\phi, \phi) = U\langle\phi, \phi\rangle + V(\phi, \phi).$$

Proof. First, note that

$$(2.48)\quad \phi \in L^\infty(d\tilde{\mu}) \subset L^2(d\tilde{\mu}).$$

Using

$$(2.49)\quad \mathscr{E}_\lambda(\phi, \phi) = (M_\nu\eta, U_\lambda M_\lambda(\eta - U_\nu M_\nu\eta)) = (\bar{S}_\lambda\phi, \eta - U_\nu\phi).$$

If we now let $\lambda \to \infty$ and use condition M3 at the end of Section 1, we have

$$(2.50)\quad \mathscr{E}(\phi, \phi) = (\phi, \eta - U_\nu\phi).$$

We next show that $\phi \in \mathscr{D}$. First, we have the identity

$$(2.51)\quad \mathscr{E}_\lambda(\phi, \phi) - \mathscr{E}_\lambda^{(1)}(\phi, \phi) = \int [U_\lambda'\phi - \phi U_\lambda'1][\phi - S_\lambda\phi]d\mu'$$

$$= \int [\bar{S}_\lambda\phi - \phi][\eta - U_\nu\phi]d\mu.$$

Therefore,

$$(2.52)\quad \sup_\lambda |\mathscr{E}_\lambda(\phi, \phi) - \mathscr{E}_\lambda^{(1)}(\phi, \phi)| \leqq C_\nu\|\phi\|_\infty \cdot \|\eta\|_\infty$$

since $\bar{S}_\lambda 1 \leqq 1$ and $\bar{T}_\lambda 1 \leqq 1$.

But by Corollary 2.1, if $\tilde{U}_\lambda\langle\phi, \phi\rangle \uparrow \infty$, then $\lim_{\lambda\to\infty} \mathscr{E}_\lambda^{(1)}(\phi, \phi) = +\infty$ also, which is false. Therefore $\phi \in \mathscr{D}$. Next we note that

$$(2.53)\quad \left|\int (U_\lambda'\phi - \phi U_\lambda'1)(\phi - S_\lambda\phi)d\mu'\right| \leqq \{\bar{L}_\lambda(\phi)\}^{1/2}\{2\mathscr{E}_\lambda^{(1)}(\phi, \phi)\}^{1/2}.$$

Since $\phi \in \mathscr{D}$ and $\sup \mathscr{E}_\lambda^{(1)}(\phi, \phi) < \infty$, the right side of (2.53) goes to zero as $\lambda \to \infty$ by (2.21). Applying this to (2.51) we see that $\mathscr{E}^{(1)}(\phi, \phi)$ exists and is equal to $\mathscr{E}(\phi, \phi)$. We then get (2.47) from Corollary 2.2.

COROLLARY 2.3. *If* $\phi \in L^\infty(d\tilde{\mu})$, *then for* $\lambda > 0$

$$(2.54)\quad (M_\lambda\phi, \phi) \geqq \int (M_\lambda\phi)^2 d\tilde{\mu}_\lambda,$$

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and consequently

$$(2.55)\quad \int (M_\lambda\phi)^2 d\tilde{\mu}_\lambda \leqq \sqrt{2}\int \phi^2[U_\lambda 1]^{-1}d\mu \leqq \sqrt{2}\int \phi^2 d\mu \leqq 2\int \phi^2 d\tilde{\mu}.$$

Proof. We have

$$(2.56)\quad \mathscr{E}(M_\lambda\phi, M_\lambda\phi) = (M_\lambda\phi, \phi) - (M_\lambda\phi, U_\lambda M_\lambda\phi).$$

But by (2.47)

$$(2.57)\quad \mathscr{E}(M_\lambda\phi, M_\lambda\phi) \geqq \tilde{U}\langle M_\lambda\phi, M_\lambda\phi\rangle \geqq \tilde{U}_\lambda\langle M_\lambda\phi, M_\lambda\phi\rangle.$$

Substituting (2.56) into (2.57) gives (2.54). To get (2.55) we use (2.54) and

$$(2.58)\quad (M_\lambda\phi, \phi) \leqq \left(\int (M_\lambda\phi)^2 d\mu_\lambda\right)^{1/2}\left(\int \phi^2\{U_\lambda 1\}^{-1}d\mu\right)^{1/2}$$
$$\leqq \sqrt{2}\left(\int (M_\lambda\phi)^2 d\tilde{\mu}_\lambda\right)^{1/2}\left(\int \phi^2\{U_\lambda 1\}^{-1}d\mu\right)^{1/2}.$$

COROLLARY 2.4. *The mapping M_λ is a continuous mapping from $L^2(d\tilde{\mu}) \to L^2(d\tilde{\mu})$ and also from $L^2(d\mu) \to L^2(d\tilde{\mu})$.*

Proof. This is an immediate consequence of (2.55) plus the fact that $d\tilde{\mu} \leqq d\tilde{\mu}_\lambda$.

From Corollary 2.4 we see that

$$(2.59)\quad R(M) \subset L^2(d\tilde{\mu}).$$

COROLLARY 2.5. *If $\phi = M_\nu\eta$ with $\eta \in L^2(d\mu)$ then $\mathscr{E}(\phi, \phi)$ exists, $\phi \in \mathscr{D}$, and*

$$(2.60)\quad \mathscr{E}(\phi, \phi) \geqq \tilde{U}\langle\phi, \phi\rangle.$$

Proof. Since we have now shown that $R(M) \subset L^2(d\tilde{\mu})$ the proof that $\mathscr{E}(\phi, \phi)$ exists is the same as that in Theorem 2.1 for the case $\eta \in L^\infty$. Then (2.56) and (2.57) are valid when the ϕ in those formulas is replaced by $\eta \in L^2(d\mu)$. Thus

$$\mathscr{E}(\phi, \phi) \geqq \tilde{U}_\lambda\langle\phi, \phi\rangle \qquad \forall\, \phi \in R(M).$$

Letting $\lambda \uparrow \infty$, we see that $\phi \in \mathscr{D}$ and (2.60) holds.

COROLLARY 2.6. *If $\phi = S_\lambda\eta$ with $\eta \in L^2(d\mu')$, then $\mathscr{E}(\phi, \phi)$ exists and* (2.60) *holds.*

Proof. We need only note that

$$\eta \in L^2(d\mu') \Rightarrow U_\lambda\eta \in L^2(d\mu)$$

to apply the previous corollary.

3. Continuity conditions. We say that a bilinear, positive semi-definite form satisfies condition C, the "continuity condition", on a set $S \times S$ in $L^2(d\tilde{\mu}) \times L^2(d\tilde{\mu})$ if there exists a constant $C > 0$ such that for each ϕ and ψ in S

$$(3.1) \qquad |\mathscr{E}(\psi, \phi)| \leqq C\{\mathscr{E}(\phi, \phi) + \|\phi\|_2{}^2\}^{1/2}\{\mathscr{E}(\psi, \psi) + \|\psi\|_2{}^2\}$$

where $\|\cdot\|_2$ denotes the $L^2(d\tilde{\mu})$ norm.

THEOREM 3.1. *The form $\mathscr{E}$ defined in* (2.28) *satisfies* (3.1) *on* $R(M) \times R(M)$ *if there exists a* $\nu > 0$ *and a constant* $K_\nu > 0$ *such that*

$$(3.2) \qquad |(M_\nu\psi, \phi)| \leqq K_\nu \cdot (M_\nu\psi, \psi)^{1/2}(M_\nu\phi, \phi)^{1/2}$$

for all ϕ and ψ in $L^2(d\mu)$.

(Note that the terms $(M_\nu\psi, \psi)$ and $(M_\nu\phi, \phi)$ make sense since $R(M) \subset L^2(d\tilde{\mu})$. Furthermore, by (2.54) these terms are positive.)

Proof. If (3.2) holds, then

$$\begin{aligned}(3.3) \qquad |\mathscr{E}(M_\nu\psi, M_\nu\phi)| &= |(M_\nu\psi, \phi) - (M_\nu\psi, U_\nu M_\nu\phi)| \leqq K_\nu(M_\nu\psi, \psi)^{1/2} \\ &\quad \times (M_\nu\phi, \phi)^{1/2} + \left(\int (M_\nu\psi)^2 d\mu\right)^{1/2}\left(\int (M_\nu\phi)^2 d\mu'\right)^{1/2} \\ &\leqq 2K_\nu\left[(M_\nu\psi, \psi) + \int (M_\nu\psi)^2 d\tilde{\mu}\right]^{1/2} \\ &\qquad \times \left[(M_\nu\phi, \phi) + \int (M_\nu\phi)^2 d\tilde{\mu}\right]^{1/2}.\end{aligned}$$

But

$$\begin{aligned}(3.4) \qquad (M_\nu\phi, \phi) &= \mathscr{E}(M_\nu\phi, M_\nu\phi) + \int (M_\nu\phi)^2 d\tilde{\mu}_\nu - \tilde{U}_\nu\langle M_\nu\phi, M_\nu\phi\rangle \\ &\leqq 2(\nu \vee 1)\left\{\mathscr{E}(M_\nu\phi, M_\nu\phi) + \int (M_\nu\phi)^2 d\tilde{\mu}\right\}.\end{aligned}$$

Therefore,

$$\begin{aligned}(3.5) \qquad |\mathscr{E}(M_\nu\psi, M_\nu\phi)| &\leqq K_\nu'\left\{\mathscr{E}(M_\nu\psi, M_\nu\psi) + \int (M_\nu\psi)^2 d\tilde{\mu}\right\}^{1/2} \\ &\qquad \times \left\{\mathscr{E}(M_\nu\phi, M_\nu\phi) + \int (M_\nu\phi)^2 d\tilde{\mu}\right\}^{1/2},\end{aligned}$$

which is the continuity condition for $\mathscr{E}$ on $R(M) \times R(M)$.

THEOREM 3.2. *If the form $\mathscr{E}$ defined in* (2.28) *satisfies* (3.1) *on* $R(M) \times R(M)$, *then* (3.2) *holds for all* $\nu > 0$.

Proof. Referring to (3.4) we have

$$(3.6)\qquad \mathscr{E}(M_\nu\phi, M_\nu\phi) + \int (M_\nu\phi)^2 d\tilde{\mu}_\nu = (M_\nu\phi, \phi) + \tilde{U}_\nu\langle M_\nu\phi, M_\nu\phi\rangle$$
$$\leqq (M_\nu\phi, \phi) + 2(\nu \vee 1)\int (M_\nu\phi)^2 d\tilde{\mu} \leqq C_\nu(M_\nu\phi, \phi),$$

by (2.54). Thus, using condition (C) on $R(M) \times R(M)$,

$$(3.7)\qquad |(M_\nu\psi, \phi)| \leqq |\mathscr{E}(M_\nu\psi, M_\nu\phi)| + |(M_\nu\psi, U_\nu M_\nu\phi)|$$
$$\leqq |\mathscr{E}(M_\nu\psi, M_\nu\phi)| + \left(\int (M_\nu\psi)^2 d\tilde{\mu}_\nu\right)^{1/2} \cdot \left(\int (M_\nu\phi)^2 d\tilde{\mu}_\nu\right)^{1/2}$$
$$\leqq A_\nu\left\{\mathscr{E}(M_\nu\psi, M_\nu\psi) + \int (M_\nu\psi)^2 d\tilde{\mu}\right\}^{1/2}$$
$$\cdot\left\{\mathscr{E}(M_\nu\phi, M_\nu\phi) + \int (M_\nu\phi)^2 d\tilde{\mu}\right\}^{1/2}.$$

Combining (3.6) and (3.7) gives (3.2).

Since we shall be interested in forms $\mathscr{E}$ that satisfy the condition (C), we shall assume from now on that $\{M_\lambda : \lambda > 0\}$ satisfies the condition:

M4: Condition (3.2) is satisfied for some value of $\nu > 0$.

LEMMA 3.1. *If* (3.2) *holds for all* ϕ *and* ψ *in* $L^2(d\mu)$ *then*

$$(3.8)\qquad \textit{closure}\ \{R(M')\}\ \textit{in}\ L^2(d\mu)\ \supset R(M).$$

Proof. Let the closure on the left be denoted by $\overline{R(M')}$. We prove (3.8) by showing

$$(3.9)\qquad \overline{\{R(M')\}} \subset R(M).$$

Since $\{R(M')\} = N(M)$, the common null space of the M_λ's, we need only show that

$$(3.10)\qquad N(M) \subset R(M)^\perp.$$

If $\phi \in N(M)$, then (3.2) shows that $\phi \in R(M)^\perp$, which completes the proof.

From this lemma we see that in cases where

$$(3.11)\qquad \sup\{\|S_\lambda\phi\| : \lambda > 0 \text{ and } \phi \in R(\bar{M})\} < \infty,$$

the assumption (3.2) will imply the condition M3 at the end of Section 1. However, we are not assuming (3.11).

4. The complete space determined by the form $\mathscr{E}$. Let $\mathscr{F}$ be the completion of $R(M)$ in the norm

$$(4.1)\qquad \|\phi\|_{\mathscr{F}}^2 = \mathscr{E}(\phi, \phi) + \int \phi^2 d\tilde{\mu} = \mathscr{E}(\phi, \phi) + \|\phi\|_2^2.$$

Clearly,

(4.2) $R(M) \subset \mathscr{F} \subset \mathscr{D} \subset L^2(d\tilde{\mu})$.

If $\{\phi_n\}$ and $\{\psi_n\}$ are Cauchy sequences in $R(M)$ converging to ϕ and ψ in $\|\cdot\|_{\mathscr{F}}$, then $\mathscr{E}(\psi_n, \phi_n)$ is a Cauchy sequence of reals because of the continuity condition (3.1) which holds on $R(M) \times R(M)$. We then define

$$\mathscr{E}(\psi, \phi) = \lim_{n\to\infty} \mathscr{E}(\psi_n, \phi_n),$$

which is independent of the particular pair of sequences chosen. The continuity condition carries over to $\mathscr{F} \times \mathscr{F}$ and (2.60) holds for $\phi \in \mathscr{F}$. We have also

(4.3) $\mathscr{E}(\psi, S_\lambda\phi) = \mathscr{E}_\lambda(\psi, \phi)$

for all ψ and ϕ in $\mathscr{F}$.

For each $\lambda > 0$ we introduce the form

(4.4) $N_\lambda(\psi, \phi) = \mathscr{E}(\psi, \phi) + (\psi, U_\lambda\phi)$

defined on $\mathscr{F} \times \mathscr{F}$.

LEMMA 4.1. *For each $\lambda > 0$ and $\phi \in \mathscr{F}$, we have*

(4.5) $(\lambda \wedge 1)\|\phi\|_{\mathscr{F}}^2 \leqq 4N_\lambda(\phi, \phi) \leqq 4(\lambda \vee 1)\|\phi\|_{\mathscr{F}}^2$.

Proof. We first note that

(4.6) $$N_\lambda(\phi, \phi) = \mathscr{E}(\phi, \phi) - \tilde{U}_\lambda\langle\phi, \phi\rangle + \int \phi^2 d\tilde{\mu}_\lambda \leqq \mathscr{E}(\phi, \phi) + (\lambda \vee 1)\|\phi\|_2^2 \leqq (\lambda \vee 1)\|\phi\|_{\mathscr{F}}^2.$$

On the other hand

(4.7) $$N_\lambda(\phi, \phi) \geqq \int \phi^2 d\tilde{\mu}_\lambda \geqq (\lambda \wedge 1)\|\phi\|_2^2$$

and therefore

(4.8) $$N_\lambda(\phi, \phi) \geqq \tfrac{1}{4}\mathscr{E}(\phi, \phi) + \tfrac{1}{4}(\phi, U_\lambda\phi) + \tfrac{3}{4}(\lambda \wedge 1)\|\phi\|_2^2 \geqq \tfrac{1}{4}\mathscr{E}(\phi, \phi) + \tfrac{1}{4}(\lambda \wedge 1)\|\phi\|_2^2 \geqq \tfrac{1}{4}(\lambda \wedge 1)\|\phi\|_{\mathscr{F}}^2.$$

LEMMA 4.2. *There exists a constant $M > 0$ such that*

(4.9) $\|\phi - S_\lambda\phi\|_{\mathscr{F}} \leqq M\|\phi\|_{\mathscr{F}}$

for all $\phi \in \mathscr{F}$, and $\lambda \geqq 1$.

Proof. Using (4.8), (4.3), and (3.1) we have

(4.10) $$\|\phi - S_\lambda\phi\|_{\mathscr{F}}^2 \leqq 4N_\lambda(\phi - S_\lambda\phi, \phi - S_\lambda\phi) = 4\mathscr{E}(\phi - S_\lambda\phi, \phi) \leqq 4C\|\phi - S_\lambda\phi\|_{\mathscr{F}}\|\phi\|_{\mathscr{F}}$$

which proves the lemma with $M = 4C$.

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COROLLARY 4.1. *If* $\phi \in \mathscr{F}$, *then*

(4.11) $\lim_{\lambda\to\infty} \|\phi - S_\lambda\phi\|_{\mathscr{F}} = 0.$

Proof. This is true for $\phi \in R(M)$, since if $\phi = M_\nu\eta$, then $\phi - S_\lambda\phi = M_\lambda\eta_0$ where $\eta_0 = \eta - U_\nu M_\nu\eta$, and if $\lambda \geqq 1$

$$\|M_\lambda\eta_0\|_{\mathscr{F}}^2 \leqq 4N_\lambda(M_\lambda\eta_0, M_\lambda\eta_0) = 4(M_\lambda\eta_0, \eta_0).$$

The right side approaches zero as $\lambda \to \infty$ by (2.55). We then use (4.9) to conclude that (4.11) holds for all $\phi \in \mathscr{F}$.

COROLLARY 4.2. *If* ϕ *and* ψ *belong to* $\mathscr{F}$, *then*

(4.12) $\mathscr{E}(\psi, \phi) = \lim_{\lambda\to\infty} \mathscr{E}_\lambda(\psi, \phi).$

Proof. We have

$$|E(\psi, \phi) - \mathscr{E}_\lambda(\psi, \phi)| = |\mathscr{E}(\psi, \phi - S_\lambda\phi)| \leqq C\|\psi\|_{\mathscr{F}}\|\phi - S_\lambda\phi\|_{\mathscr{F}}.$$

An application of Corollary 4.1 completes the proof.

COROLLARY 4.3. *If* $\phi \in \mathscr{F}$, *then*

(4.13) $\sup_{\lambda\geqq 1} |\mathscr{E}_\lambda(\phi, \phi)| \leqq A\|\phi\|_{\mathscr{F}}^2$

and

(4.14) $\sup_{\lambda>0} \mathscr{E}_\lambda^{(1)}(\phi, \phi) \leqq B\|\phi\|_{\mathscr{F}}^2$

where A *and* B *are positive constants.*

Proof. Inequality (4.13) follows from (4.3), (3.1), and (4.9).

To prove (4.14), we use (2.51), (2.46), and (4.13) plus the easily verified inequality

(4.15) $\bar{L}_\lambda(\phi) \leqq 2\tilde{U}_\lambda\langle\phi, \phi\rangle \leqq 2\tilde{U}\langle\phi, \phi\rangle$

to conclude that

(4.16)
$$\begin{aligned}\mathscr{E}_\lambda^{(1)}(\phi, \phi) &\leqq |\mathscr{E}_\lambda(\phi, \phi)| + \sqrt{2}\,L_\lambda(\phi)^{1/2}\{\mathscr{E}_\lambda^{(1)}(\phi, \phi)\}^{1/2}\\ &\leqq |\mathscr{E}_\lambda(\phi, \phi)| + 2\tilde{U}\langle\phi, \phi\rangle^{1/2}\{\mathscr{E}_\lambda^{(1)}(\phi, \phi)\}^{1/2}\\ &\leqq |\mathscr{E}_\lambda(\phi, \phi)| + 2\mathscr{E}(\phi, \phi)^{1/2}\{\mathscr{E}_\lambda^{(1)}(\phi, \phi)\}^{1/2}\\ &\leqq B\|\phi\|_{\mathscr{F}}^2.\end{aligned}$$

We next look at some other characterizations of $\mathscr{F}$.

THEOREM 4.1. *If*

(4.17) $\mathscr{G} = \{\phi \in L^2(d\tilde{\mu}) : \sup_{\lambda\geqq 1} \mathscr{E}_\lambda^{(1)}(\phi, \phi) < \infty\},$

then $\mathscr{F} = \mathscr{G}$.

Proof. If $\phi \in \mathscr{F}$, then clearly (4.13) implies that $\phi \in \mathscr{G}$. To prove $\mathscr{G} \subset \mathscr{F}$, suppose $\phi \in \mathscr{G}$. Then $S_\lambda\phi \in \mathscr{F}$ and

$$(4.18) \quad \begin{aligned} \mathscr{E}(S_\lambda\phi, S_\lambda\phi) &= \mathscr{E}_\lambda^{(1)}(\phi, \phi) + (U_\lambda' S_\lambda\phi - U_\lambda' 1 \cdot S_\lambda\phi, \phi - S_\lambda\phi) \\ &\leqq \mathscr{E}_\lambda^{(1)}(\phi, \phi) + 2\tilde{U}_\lambda\langle S_\lambda\phi, S_\lambda\phi\rangle^{1/2}\{\mathscr{E}_\lambda^{(1)}(\phi, \phi)\}^{1/2} \\ &\leqq \mathscr{E}_\lambda^{(1)}(\phi, \phi) + 2\{\mathscr{E}(S_\lambda\phi, S_\lambda\phi)\}^{1/2}\{\mathscr{E}_\lambda^{(1)}(\phi, \phi)^{1/2}\}. \end{aligned}$$

Hence,

$$(4.19) \quad \mathscr{E}(S_\lambda\phi, S_\lambda\phi) \leqq (1 + \sqrt{2})\mathscr{E}_\lambda^{(1)}(\phi, \phi).$$

In addition

$$(4.20) \quad \begin{aligned} \|S_\lambda\phi\|_2^2 = \tilde{U}_1\langle S_\lambda\phi, S_\lambda\phi\rangle + (S_\lambda\phi, U_1 S_\lambda\phi) &\leqq \mathscr{E}(S_\lambda\phi, S_\lambda\phi) \\ &+ \sqrt{2}\|S_\lambda\phi\|_2 \cdot \left(\int (S_\lambda\phi)^2 d\mu'\right)^{1/2}. \end{aligned}$$

But if $\lambda \geqq 1$

$$(4.21) \quad \begin{aligned} \left(\int (S_\lambda\phi)^2 d\mu'\right)^{1/2} &\leqq \left(\int (\phi - S_\lambda\phi)^2 d\mu'\right)^{1/2} + \left(\int \phi^2 d\mu'\right)^{1/2} \\ &\leqq \sqrt{2}\{\mathscr{E}_\lambda^{(1)}(\phi, \phi)\}^{1/2} + \sqrt{2}\|\phi\|_2. \end{aligned}$$

Combining (4.20) and (4.21) we conclude that if $\lambda \geqq 1$

$$(4.22) \quad \|S_\lambda\phi\|_2 \leqq C'[\mathscr{E}_\lambda^{(1)}(\phi, \phi) + \|\phi\|_2].$$

Thus, from (4.19) and (4.22)

$$(4.23) \quad \sup_{\lambda \geqq 1} \|S_\lambda\phi\|_{\mathscr{F}} < \infty.$$

Since $\mathscr{F}$ is a Hilbert space in the symmetrized form

$$(4.24) \quad \tilde{\mathscr{E}}(\psi, \phi) + (\psi, \phi)_2 = \tfrac{1}{2}\{\mathscr{E}(\psi, \phi) + \mathscr{E}(\phi, \psi)\} + (\psi, \phi)_2$$

there exists a sequence $\lambda_n \to \infty$ and an $\eta \in \mathscr{F}$ such that

$$(4.25) \quad \lim_{n\to\infty} S_{\lambda_n} \to \eta$$

weakly in $\mathscr{F}$. But for each $\psi \in L^2(d\mu')$ the inner product $(\psi, \cdot)'$ defines a continuous linear functional on $\mathscr{F}$, and therefore $\}S_{\lambda_n}\phi\}$ also converges to η weakly in $L^2(d\mu')$. However, for $\psi \in L^2(d\mu')$ we have

$$(4.26) \quad \begin{aligned} |(\psi, \phi - S_\lambda\phi)'| &\leqq \left\{\int (\phi - S_\lambda\phi)^2 d\mu_\lambda'\right\}^{1/2}\left\{\int \psi^2 (U_\lambda' 1)^{-1} d\mu'\right\}^{1/2} \\ &\leqq \sqrt{2}\{\mathscr{E}_\lambda^{(1)}(\phi, \phi)\}^{1/2}\left\{\int \psi^2 (U_\lambda' 1)^{-1} d\mu'\right\}^{1/2}. \end{aligned}$$

Since $\phi \in \mathscr{G}$ and $U_\lambda' 1 \uparrow \infty$, the right side of (4.26) goes to zero as $\lambda \uparrow \infty$. Thus $\phi = \eta \in \mathscr{F}$. This completes the proof.

Using the results of Theorem 4.1, we can now define a second norm on the space $\mathscr{F}$, namely

(4.27) $\|\phi\|_{\mathscr{G}}^2 = \sup_{\lambda \geqq 1} \mathscr{E}_\lambda^{(1)}(\phi, \phi) + \|\phi\|_2^2.$

LEMMA 4.3. *If* $\phi \in \mathscr{F}$ *then*

(4.28) $\mathscr{E}(\psi, \phi) = \lim_{\lambda \to \infty} \mathscr{E}_\lambda^{(1)}(\psi, \phi) = \mathscr{E}^{(1)}(\psi, \phi),$

and therefore

(4.29) $\|\phi\|_{\mathscr{F}} \leqq \|\phi\|_{\mathscr{G}}.$

Proof. As in (2.44) we have

(4.30) $|\mathscr{E}_\lambda(\psi, \phi) - \mathscr{E}_\lambda^{(1)}(\psi, \phi)| \leqq \sqrt{2}\, L_\lambda(\psi)^{1/2}\{\mathscr{E}_\lambda^{(1)}(\phi, \phi)\}^{1/2}.$

Since $\phi \in \mathscr{G}$ and $\psi \in \mathscr{D}$, the right side tends to zero as $\lambda \to \infty$.

It is easily seen that $\mathscr{F}$ is complete in $\| \ \|_{\mathscr{G}}$, and therefore we can find a constant $d > 0$ such that

(4.31) $d\|\phi\|_{\mathscr{G}}^2 \leqq \|\phi\|_{\mathscr{F}}^2 \leqq \|\phi\|_{\mathscr{G}}^2.$

LEMMA 4.4. *The space* $\mathscr{F}$ *is a Hilbert space with the inner product*

(4.32) $\tilde{N}_\lambda(\psi, \phi) = \frac{1}{2}\{N_\lambda(\psi, \phi) + N_\lambda(\phi, \psi)\},$

where N_λ *is defined in* (4.4). *Furthermore, there exist bounded linear mappings* Q_λ *and* $\bar{Q}_\lambda$ *from* $\mathscr{F}$ *to* $\mathscr{F}$ *with continuous linear inverses such that*

(4.33) $N_\lambda(Q_\lambda\psi, \phi) = \tilde{N}_\lambda(\psi, \phi) = N_\lambda(\psi, \bar{Q}_\lambda\phi).$

Proof. We have already shown in Lemma 4.1 that $N_\lambda(\phi, \phi)^{1/2}$ is a norm equivalent to $\|\phi\|_{\mathscr{F}}$. We also have

$$
\begin{aligned}
(4.34) \quad |N_\lambda(\psi, \phi)| &\leqq C\|\psi\|_{\mathscr{F}}\|\phi\|_{\mathscr{F}} + \|\psi\|_2\|\phi\|_2 \\
&\leqq K_\lambda\|\psi\|_{\mathscr{F}}\|\phi\|_{\mathscr{F}} \leqq K_\lambda' N_\lambda(\psi, \psi)^{1/2} N_\lambda(\phi, \phi)^{1/2}.
\end{aligned}
$$

The existence of Q_λ and $\bar{Q}_\lambda$ then follows from the Lax-Milgram Lemma.

THEOREM 4.2. *We have*

(i) $R(M') \subset \mathscr{F}$

and

(ii) $R(M')$ *is dense in* $\mathscr{F}$.

Proof. For each $\psi \in L^2(d\mu')$ the functional $(\psi, \phi)'$ is a continuous linear functional on $\mathscr{F}$ and therefore there exists a continuous linear mapping P_λ from $L^2(d\mu')$ into $(\mathscr{F}, \tilde{N}_\lambda)$ such that

(4.35) $\tilde{N}_\lambda(P_\lambda\psi, \phi) = (\psi, \phi)'.$

Letting

(4.36) $\bar{M}_\lambda\psi = Q_\lambda P_\lambda\psi,$

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where Q_λ is defined as in (4.33), we get

(4.37) $N_\lambda(\bar{M}_\lambda\psi, \phi) = (\psi, \phi)'$

for $\psi \in L^2(d\mu')$ and $\phi \in \mathscr{F}$. In particular,

(4.38) $N_\lambda(\bar{M}_\lambda\psi, M_\lambda\phi) = (\psi, M_\lambda\phi)'$

for $\psi \in L^2(d\mu')$ and $\phi \in L^2(d\mu)$. But we have already shown that

(4.39) $N_\lambda(\eta, M_\lambda\phi) = (\eta, \phi)$

for $\eta \in \mathscr{F}$ and $\phi \in L^2(d\mu)$. Therefore,

(4.40) $N_\lambda(\bar{M}_\lambda\psi, M_\lambda\phi) = (\bar{M}_\lambda\psi, \phi) = (\psi, M_\lambda\phi)'$,

for all $\psi \in L^2(d\mu')$ and $\phi \in L^2(d\mu)$, which proves that

(4.41) $\bar{M}_\lambda\psi = M_\lambda'\psi$.

This establishes part (i).

To prove (ii) suppose $R(M')$ were not dense in $\{\mathscr{F}, \tilde{N}_\lambda\}$. We could then find a non-trivial $\phi_0 \in \mathscr{F}$ such that

$$0 = \tilde{N}_\lambda(M_\lambda'\psi, \phi_0) = N_\lambda(M_\lambda'\psi, \bar{Q}_\lambda\phi_0) = (\psi, \bar{Q}_\lambda\phi_0)'$$

for all $\psi \in L^2(d\mu')$. But this is impossible unless $\phi_0 = 0$, which is a contradiction. This completes the proof of part (ii).

Later on we shall find it convenient to have the following lemma:

LEMMA 4.5. *If ϕ and ψ belong to $\mathscr{F}$, then*

(4.42) $\mathscr{E}(\psi, \phi) = \lim_{\lambda\to\infty} \bar{\mathscr{E}}_\lambda^{(1)}(\psi, \phi)$.

Proof. For ϕ and ψ in $\mathscr{F}$ we have the identity

(4.43) $$\mathscr{E}_\lambda^{(1)}(\psi, \phi) - \bar{\mathscr{E}}_\lambda^{(1)}(\psi, \phi) = \int (\bar{T}_\lambda\psi - \psi)(\phi \cdot U_\lambda 1 - U_\lambda\phi)d\mu + \int (\psi \cdot U_\lambda'1 - U_\lambda\psi)(T_\lambda\phi - \phi)d\mu'.$$

Thus

(4.44) $$|\mathscr{E}_\lambda^{(1)}(\psi, \phi) - \bar{\mathscr{E}}_\lambda^{(1)}(\psi, \phi)| \leqq \sqrt{2}\{V_\lambda(\psi, \psi)^{1/2}L_\lambda(\phi)^{1/2} + V_\lambda(\phi, \phi)^{1/2}\cdot\bar{L}_\lambda(\psi)^{1/2}\}.$$

Using Lemma 4.3 and Corollary 2.2, we see that $V(\phi, \phi)$ and $V(\psi, \psi)$ exist. Furthermore both $L_\lambda\phi$ and $\bar{L}_\lambda\psi$ approach zero as $\lambda \to \infty$ since ϕ and ψ are in $\mathscr{D}$, so the right side of (4.44) approaches zero as $\lambda \to \infty$.

5. The Dirichlet conditions. Suppose we have an L^2 space relative to a finite measure ν on X. Following [**3**] we make the definition:

Definition 5.1. Suppose B is a real bilinear form defined on $D(B) \times D(B)$, where $D(B)$ is a linear subspace of $L^2(d\nu)$, satisfying the following conditions:

B1: B is bounded below, i.e., there exists a non-negative constant β_0 for which

$$(5.1) \qquad B(\phi, \phi) + \beta_0(\phi, \phi)_2 \geqq 0$$

holds for every $\phi \in D(B)$.

Let

$$B_\alpha(\psi, \phi) = B(\psi, \phi) + \alpha(\psi, \phi)_2.$$

B2: B satisfies the continuity condition: for some $\alpha > \beta_0$ there exists a constant $C > 0$ such that

$$(5.2) \qquad |B(\psi, \phi)| \leqq CB_\alpha(\phi, \phi)^{1/2} \cdot B_\alpha(\psi, \psi)^{1/2}$$

for all ϕ and ψ in $D(B)$.

B3: $D(B)$ is complete relative to the norms

$$(5.3) \qquad |||\phi|||_\alpha{}^2 = B_\alpha(\phi, \phi) \qquad (\alpha > \beta_0).$$

B4: $D(B)$ is a sublattice of $L^2(d\nu)$ such that for each constant $c \geqq 0$

$$\phi \in D(B) \Rightarrow (\phi \wedge c) \in D(B).$$

Furthermore

$$(5.4) \qquad B((\phi - c)^+, \phi \wedge c) \geqq 0$$

for each constant $c \geqq 0$ and $\phi \in D(B)$. (Note: since $\phi = (\phi - c)^+ + (\phi \wedge c)$ the fact that ϕ and $\phi \wedge c$ both belong to $D(B)$ assures that $(\phi - c)^+ \in D(B)$.)

Then B is called a *Dirichlet bilinear form*.

Definition 5.2. If B is an unsymmetric Dirichlet bilinear form, then the pair $\{D(B), B\}$ is called an *unsymmetric Dirichlet space* relative to $L^2(d\nu)$. If B is symmetric, then $\{D(B), B\}$ is a Dirichlet space relative to L^2 in the sense of [**2**]. The term "Dirichlet space relative to L^2" will include both cases.

The following theorem is basic in the study of these Dirichlet spaces. For convenience, we give the proof here since it is not overly long.

THEOREM 5.1. *Suppose B is a Dirichlet bilinear form, either symmetric or unsymmetric. There exists, for each $\lambda > \beta_0$, a continuous linear mapping $G_\lambda : L^2(d\nu) \to D(B)$ such that*

(i) $B(\psi, G_\lambda\phi) + \lambda(\psi, G_\lambda\phi) = (\psi, \phi)$ *for* $\psi \in D(B)$, $\phi \in L^2(d\nu)$;

(ii) $G_\lambda - G_\mu = (\mu - \lambda)G_\lambda G_\mu$;

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(iii) $0 \leqq \phi \leqq 1 \Rightarrow 0 \leqq \lambda G_\lambda \phi \leqq 1$;
(iv) $(\lambda - \beta_0)\|G_\lambda \phi\|_2 \leqq \|\phi\|_2$;
(v) Rng G_λ *is independent of* λ *and is dense in* $D(B)$ *relative to the norms* $|||\cdot|||_\alpha$ *for* $\alpha > \beta_0$.

Proof. Since all the norms $|||\cdot|||_\lambda$ for $\lambda > \beta_0$ are equivalent, $D(B)$ is a Banach space relative to each of these norms. Now let

$$(5.5) \quad \tilde{B}_\lambda(\psi, \phi) = \tfrac{1}{2}\{B(\psi, \phi) + B(\phi, \psi)\} + \lambda(\psi, \phi)_2.$$

Then $D(B)$ is a Hilbert space with inner product $B_\lambda(\psi, \phi)$.

Since B2 holds, the Lax-Milgram lemma gives the existence of bounded linear operators Q_λ and $\bar{Q}_\lambda$ from $\{D(B), \tilde{B}_\lambda\}$ to itself such that

$$(5.6) \quad B_\lambda(\psi, \bar{Q}_\lambda \phi) = \tilde{B}_\lambda(\psi, \phi) = B_\lambda(Q_\lambda \psi, \phi).$$

Furthermore, Q_λ and $\bar{Q}_\lambda$ have continuous linear inverses.

Now, given $\phi \in L^2$, we can define a continuous linear functional on the Hilbert space $\{D(B), \tilde{B}_\lambda\}$ by

$$(5.7) \quad F_\phi(\psi) = (\psi, \phi)_2.$$

Thus, there exists a unique element $J_\lambda \phi \in D(B)$ such that

$$(5.8) \quad \tilde{B}_\lambda(\psi; J_\lambda \phi) = (\psi, \phi)_2.$$

But by (5.6)

$$(5.9) \quad B_\lambda(\psi, \bar{Q}_\lambda J_\lambda \phi) = (\psi, \phi)_2.$$

Now define

$$(5.10) \quad G_\lambda \phi = \bar{Q}_\lambda J_\lambda \phi.$$

Then for each ϕ, the unique element of $D(B)$ satisfying (i) is $G_\lambda \phi$. The resolvent equation (ii) follows from the uniqueness.

We next establish the submarkov property (iii). Suppose $\lambda > \beta_0$ and $\phi \leqq C$ where $C \geqq 0$ is a constant. We shall show that $\lambda G_\lambda \phi \leqq C$. By taking $C = 0$ and $C = 1$ in turn, we get (iii). Let

$$\psi_\lambda = (\lambda G_\lambda \phi - c)^+.$$

Conditions (5.1), (5.4) and (i) imply that

$$(5.11) \quad 0 \leqq B(\psi_\lambda, \lambda G_\lambda) + \beta_0\|\psi_\lambda\|^2 = (\psi_\lambda, \lambda(\phi - \lambda G_\lambda \phi) + \beta_0 \psi_\lambda)_2.$$

If $\|\psi_\lambda\|_2 \neq 0$, then from (5.11),

$$(5.12) \quad \lambda(\psi_\lambda, \phi - (\lambda G_\lambda \phi \wedge c))_2 \geqq (\lambda - \beta_0)\|\psi_\lambda\|^2 > 0,$$

which is a contradiction, since

$$(5.13) \quad (\phi \leqq c) \Rightarrow ((v - c)^+, \phi - (v \wedge c)) \leqq 0$$

for all $v \in L^2$. Thus,

(5.14) $\psi_\lambda = (\lambda G_\lambda \phi - c)^+ = 0,$

i.e., $\lambda G_\lambda \phi \leqq c$. This completes the proof of (iii).

Condition (iv) is an immediate consequence of (5.1) and (i).

In (v) the independence on λ of range G_λ follows from (ii). If range $G_\lambda = R(G)$ were not dense in $\{D(B), |||\cdot|||_\lambda\}$, then range $\bar{Q}_\lambda^{-1}R(G)$ would not be dense in $\{D(B), \tilde{B}_\lambda\}$. But that contradicts (5.8). This completes the proof of the theorem.

If, in addition to (5.4), we also have

(5.15) $B(\phi \wedge c, (\phi - c)^+) \geqq 0$

then the symmetrized form $\tilde{B}$ also satisfies (5.4), and the space $\{D(B), \tilde{B}\}$ is a Dirichlet space relative to $L^2(d\nu)$ in the sense of [**2**].

We now apply these concepts to the spaces constructed in the previous sections.

THEOREM 5.2. *The pair $\{\mathscr{F}, \mathscr{E}\}$ defined at the beginning of Section 4 is a Dirichlet space relative to $L^2(d\tilde{\mu})$ in the sense of Definition 5.2. The form $\mathscr{E}$ satisfies the additional conditions:*

(5.16) $\mathscr{E}(\phi, \phi) \geqq \tilde{U}\langle \phi, \phi \rangle$

(5.17) $\mathscr{E}(\phi \wedge c, (\phi - c)^+)$

$$\geqq \int (\phi - c)^+ \cdot \left\{ \int U(\xi, \eta)(\phi - c)^- \mu(d\xi) \right\} \mu'(d\eta),$$

and

(5.18) $\mathscr{E}((\phi - c)^+, \phi \wedge c) \geqq \int (\phi - c)^+ \{U(\xi, \eta)(\phi - c)^- \mu'(d\eta)\} \mu(d\xi).$

Proof. The only part we need to prove is the pair of inequalities (5.17) and (5.18). The rest of the proof has been established throughout the previous sections.

We have,

(5.19) $$\mathscr{E}_\lambda^{(1)}(\psi, \phi) + (\psi, U_\lambda \phi) - \int \psi \phi d\mu_\lambda' = (\psi, U_\lambda \phi) - \int \psi S_\lambda \phi d\mu_\lambda'$$

$$= (\psi - \bar{T}_\lambda \psi, U_\lambda \phi) = (\psi - \bar{T}_\lambda \psi, U_\lambda \phi - \phi \cdot U_\lambda 1)$$

$$+ (\psi - \bar{T}_\lambda \psi, \phi \cdot U_\lambda 1).$$

The first term on the right hand side of (5.19) is bounded by

(5.20) $\sqrt{2} L_\lambda(\phi)^{1/2} \cdot V_\lambda(\psi, \psi)^{1/2},$

where V_λ is defined in (2.27). Since $\phi \in \mathscr{F}$, Theorem 2.1 implies that

$\phi \in \mathscr{D}$ and that $V(\phi, \phi)$ exists. Therefore, the expression in (5.20) approaches zero as $\lambda \to \infty$.

Next, we observe that

$$(5.21) \quad (U_\lambda'(\phi \wedge c) - (\phi \wedge c)U_\lambda'1, (\phi - c)^+)' = (U_\lambda'(\phi \wedge c - c), (\phi - c)^+) = (U_\lambda'(\phi - c)^-, (\phi - c)^+)$$

and

$$(5.22) \quad (\phi \wedge c - \bar{T}_\lambda(\phi \wedge c), (\phi - c)^+ U_\lambda 1) \geqq 0.$$

Therefore,

$$(5.23) \quad \mathscr{E}_\lambda^{(1)}(\phi \wedge c, (\phi - c)^+) \geqq (U_\lambda'(\phi - c)^-, (\phi - c)^+)' + (\phi \wedge c - \bar{T}_\lambda(\phi \wedge c), U_\lambda(\phi - c)^+ - (\phi - c)^+ U_\lambda 1).$$

If we now let $\lambda \to \infty$, and use Lemma 4.3 we get (5.17). The proof of (5.18) is similar, using $\bar{\mathscr{E}}^{(1)}$ and Lemma 4.5.

COROLLARY 5.1. *If $\tilde{\mathscr{E}}$ is the symmetrized form corresponding to $\mathscr{E}$, then $\{\mathscr{F}, \tilde{\mathscr{E}}\}$ is a symmetric Dirichlet space relative to $L^2(d\tilde{\mu})$.*

6. The converse construction.

THEOREM 6.1. *Suppose $\{\mathscr{F}, \mathscr{E}\}$ is a Dirichlet space relative to $L^2(d\tilde{\mu})$ in the sense of Definition 5.2, where $\mathscr{E}$ is a Dirichlet form (with $\beta_0 \equiv 0$ in Definition 5.1) satisfying (5.16)-(5.18). Then there exists a family of continuous linear mappings $\{M_\lambda : \lambda > 0\}$ from $L^2(d\mu)$ into $\mathscr{F}$ satisfying (1.8) and*

$$(6.1) \quad N_\lambda(\psi, M_\lambda\phi) = \mathscr{E}(\psi, M_\lambda\phi) + (\psi, U_\lambda\phi) = (\psi, \phi)$$

for each $\lambda > 0$, $\psi \in \mathscr{F}$ and $\phi \in L^2(d\mu)$. If M_λ' denotes the adjoint of M_λ when the latter is regarded as a mapping from $L^2(d\mu)$ into $L^2(d\mu')$, then M_λ' is also a continuous linear mapping from $L^2(d\mu')$ into $\mathscr{F}$ satisfying (1.9) and

$$(6.2) \quad N_\lambda(M_\lambda'\psi, \phi) = \mathscr{E}(M_\lambda'\psi, \phi) + (U_\lambda' M_\lambda'\psi, \phi)' = (\psi, \phi)$$

for $\lambda > 0$, $\psi \in L^2(d\mu')$, and $\phi \in \mathscr{F}$.

Furthermore, the mappings M_λ and M_λ' satisfy conditions M1-M3 *(Section* 1*) and the continuity condition* M4 *(formula* (3.2)*) for each $\nu > 0$.*

Proof. As in the previous sections, we have defined the form N_λ by (4.4), and $\|\phi\|_{\mathscr{F}}^2$ by (4.1). Since (5.16) holds, the arguments in the proof of Lemma 4.1 apply and (4.5) is valid. Also, (4.34) is also applicable and shows that N_λ satisfies the continuity condition.

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We next define a new form

$$(6.3)\qquad B_\lambda(\psi, \phi) = N_\lambda(\psi, \phi) - \int \psi\phi d\mu_\lambda$$

and show it to satisfy conditions B1-B4 of Theorem 5.1 with $d\nu = d\mu_\lambda$.

Using (5.16), (see (4.6) and (4.7)),

$$(6.4)\qquad B_\lambda(\phi, \phi) + \frac{1}{2}\int \phi^2 d\mu_\lambda = N_\lambda(\phi, \phi) - \frac{1}{2}\int \phi^2 d\mu_\lambda \geqq \frac{1}{2}\int \phi^2 d\mu_\lambda' \geqq 0.$$

Thus (5.1) holds for B_λ with $\beta_0 = 1/2$ and $d\nu = d\mu_\lambda$.

The continuity condition (5.2) for B follows easily from (4.34), the continuity condition for N_λ.

To verify B3, suppose $\alpha > 1/2$, and $\gamma = 1 \wedge (2\alpha - 1)$. Then using (4.5) and (4.7),

$$(6.5)\qquad (\lambda \vee 1)\|\phi\|_{\mathscr{F}}^2 \geqq N_\lambda(\phi, \phi) \geqq (1 \wedge \alpha^{-1})\left[B_\lambda(\phi, \phi) + \alpha\int \phi^2 d\mu_\lambda\right]$$

$$= (1 \wedge \alpha^{-1})\left[N_\lambda(\phi, \phi) - (1 - \alpha)\int \varphi^2 d\mu_\lambda\right]$$

$$\geqq (1 \wedge \alpha^{-1})$$

$$\cdot\left[\gamma N_\lambda(\phi, \phi) + (1 - \gamma)\int \phi^2 d\tilde{\mu}_\lambda - (1 - \alpha)\int \phi^2 d\mu_\lambda\right]$$

$$\geqq (1 \wedge \alpha^{-1})\left[\gamma N_\lambda(\phi, \phi) + \left\{\frac{1 - \gamma}{2} - (1 - \alpha)\right\}\int \phi^2 d\mu_\lambda\right]$$

$$\geqq (1 \wedge \alpha^{-1})[\gamma N_\lambda(\phi, \phi)] \geqq \tfrac{1}{4}(1 \wedge \alpha^{-1})(\gamma)(\lambda \wedge 1)\|\phi\|_{\mathscr{F}}^2.$$

Since by hypothesis, $\mathscr{F}$ is complete in the norm $\|\cdot\|_{\mathscr{F}}$, it must also be complete in all the norms

$$(6.6)\qquad B_\lambda(\phi, \phi) + \alpha\int \phi^2 d\mu_\lambda$$

for $\lambda > 0$, $\alpha > \frac{1}{2}$.

Finally, to verify (5.4), we note that

$$(6.7)\qquad B_\lambda((\phi - c)^+, \phi \wedge c) = \mathscr{E}((\phi - c)^+, \phi \wedge c)$$

$$+ ((\phi - c)^+, U_\lambda(\phi \wedge c)) - \int (\phi - c)^+ \cdot (\phi \wedge c) d\mu_\lambda$$

$$= \mathscr{E}((\phi - c)^+, \phi \wedge c) - ((\phi - c)^+, U_\lambda(\phi - c)^-) \geqq 0$$

by (5.18).

We have now verified all the conditions of Definition 5.1 with $B = B_\lambda$. Thus, Theorem 5.1 (with $\beta_0 = 1/2$ and $d\nu = d\mu_\lambda$) assures the existence of a family $\{G_\alpha^\lambda : \alpha > 1/2\}$ of continuous linear mappings from $L^2(d\mu)$

into $\mathscr{F}$ satisfying the conclusions (i)-(v) of that theorem, with the parameter λ there replaced by the parameter α in the present proof. In particular, for $\alpha = 1$, we have

$$(6.8) \quad B_\lambda(\psi, G_1^\lambda\phi) + \int \psi G_1^\lambda \phi d\mu_\lambda = N_\lambda(\psi, G_1^\lambda\phi) = \int \psi\phi d\mu_\lambda = \int \psi\phi U_\lambda 1\, d\mu.$$

If we set

$$(6.9) \quad M_\lambda\phi = G_1^\lambda(\phi\cdot\{U_\lambda 1\}^{-1})$$

we get (6.1). Since G_1^λ is a positive mapping, so is M_λ, and $G_1^\lambda 1 \leqq 1$ implies $M_\lambda U_\lambda 1 \leqq 1$.

We can carry through an exactly analogous treatment for

$$(6.10) \quad \bar{B}_\lambda(\psi, \phi) = N_\lambda(\phi, \psi) - \int \phi\psi d\mu_\lambda',$$

using (5.17) instead of (5.18). In applying Definition 5.1 and Theorem 5.1 we use $\beta_0 = 1/2$ and $d\nu = d\mu_\lambda'$, to obtain a family $\{\bar{M}_\lambda\}$ of continuous, positive linear mappings that satisfy

$$N_\lambda(\bar{M}_\lambda\psi, \phi) = (\psi, \phi)',$$

plus the submarkov condition $\bar{M}_\lambda U_\lambda' 1 \leqq 1$. We then show that $\bar{M}_\lambda = M_\lambda'$ exactly as in the proof of (4.41). The continuity condition (3.2) is satisfied, because of the continuity condition for $\mathscr{E}$ (by Theorem 3.2).

Finally, we must prove M3 at the end of Section 1. Since we know that

$$R(M) \subset \mathscr{F} \subset L^2(d\tilde{\mu}),$$

it is sufficient to prove that $\bar{S}_\lambda\psi \to \psi$ weakly in $L^2(d\mu)$ as $\lambda \to \infty$ for $\psi \in \mathscr{F}$. Actually, we prove more, namely that

$$(6.11) \quad \lim_{\lambda\to\infty} \|\psi - \bar{S}_\lambda\psi\|_{\mathscr{F}} = 0$$

for all $\psi \in \mathscr{F}$.

Equation (6.2) implies that $R(M')$ is dense in $\mathscr{F}$, using the Lax-Milgram lemma as in the proof of Theorem 4.2 (ii). We know that $\bar{S}_\lambda\psi \to \psi$ strongly in $L^2(d\mu)$ if $\psi \in R(M')$. Using an argument parallel to (4.10) we have

$$(6.12) \quad \|\psi - \bar{S}_\lambda\psi\|_{\mathscr{F}}^2 \leqq 4N_\lambda(\psi - \bar{S}_\lambda\psi, \psi - \bar{S}_\lambda\psi) = 4\mathscr{E}(\psi, \psi - \bar{S}_\lambda\psi) \leqq 4C\|\psi\|_{\mathscr{F}}\|\psi - \bar{S}_\lambda\psi\|_{\mathscr{F}},$$

so

$$(6.13) \quad \|\psi - \bar{S}_\lambda\psi\|_{\mathscr{F}} \leqq 4C\|\phi\|_{\mathscr{F}}.$$

From this we conclude that (6.11) holds.

7. Examples. As we have already pointed out, the most familiar example occurs when $U_\lambda = \lambda I = U_\lambda$, and $d\mu = d\mu' = dm = dm'$. In this case the form $\tilde{U}_\lambda$ is identically 0. The mappings $M_\lambda = G_\lambda$ are submarkov pseudo-resolvents whose adjoints G_λ' are also submarkov. This implies that

$$\|\lambda G_\lambda\|_{L^2} = \|\lambda G_\lambda'\|_{L^2} \leqq 1.$$

The form $\mathscr{E}$ in this case is given by

$$(7.1) \qquad \mathscr{E}(\psi, \phi) = \lim_{\lambda\to\infty} \lambda(\psi, \phi - \lambda G_\lambda \phi)_2 = \lim_{\lambda\to\infty} \mathscr{E}_\lambda(\psi, \phi).$$

In this paper we are interested in the case where $\mathscr{E}$ satisfies the continuity condition (3.1). Such resolvent families have been studied by Kunita [**3**].

For our second example we return to the unsymmetric L^2-diffusion resolvents treated by Kunita in [**4**]. Let E be a bounded domain in $\mathbf{R}^n$ with dx the Lebesgue measure on E, and define the elliptic operator L as in (0.13). A submarkov family of pseudo-resolvents $\{G_\lambda : \lambda > 0\}$ of operators on $L^2(E)$ to itself is called an *L-diffusion resolvent* if it maps L^∞ into $C(E)$ and satisfies

$$(7.2) \qquad (\lambda - L)G_\lambda f = f$$

for all $f \in L^\infty$.

In [**4**] Kunita considers conservative L-diffusion resolvents $\{G_\lambda : \lambda > 0\}$ satisfying the two conditions:

R1. The adjoint G_λ^* in $L^2(E, dx)$ satisfies

$$(7.3) \qquad (\lambda - L^*)G_\lambda^* f = f$$

where L^* is the formal adjoint

$$(7.4) \qquad L^* = \sum \partial/\partial x_i(a_{ij}\partial/\partial x_j) - \sum \partial/\partial x_i(b_i \cdot);$$

R2. There exists a strictly positive h with $\partial h/\partial x_i \in L^2(E, dx)$ such that

$$(7.5) \qquad \lambda G_\lambda^* h = h.$$

By replacing dx with the measure $d\bar{x} = hdx$ and G_λ^* with the new adjoint

$$(7.6) \qquad G_\lambda^{*,h} f = h^{-1} G_\lambda^*(hf)$$

relative to the inner product in $L^2(E, hdx)$, we can reduce this to the case where $\lambda G_\lambda^* 1 = 1$. The operator L^* would be replaced by

$$(7.7) \qquad L^{*,h} = \sum \partial/\partial x_i(a_{ij}\partial/\partial x_j) - \sum_i (b_i - 2\sum_j a_{ij}\partial/\partial x_j(\log h))\partial/\partial x_i.$$

From now on we assume that our measure on E has been so chosen, and we shall simply write G_λ^* instead of $G_\lambda^{*,h}$. We are primarily interested here in the mappings $\{M_\lambda\}$ arising from the Feller representation for G_λ.

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There is a minimal "absorbing barrier" resolvent G_λ^0 and a generalized boundary ∂E (see [**4**], Theorem 2), along with the Poisson-Martin kernels for representing solutions of $Lu = 0$ and $L^*u = 0$ $(= L^{*,h}u = 0$ if we are using the measure $hdx)$. If these kernels are denoted by $K_\lambda(\xi, x)$ and $K_\lambda^*(\xi, x)$, resp., then the Feller representation of $G_\lambda(x, y)$ is (0.4) with

$$(7.8) \qquad H_\lambda(x, y) = \int_{\partial E} \int_{\partial E} K_\lambda(\xi, x) M_\lambda(\xi, \eta) K_\lambda^*(\eta, y) \mu_0(d\xi) \bar{h}(\eta) \mu_0^*(d\eta),$$

where μ_0 and μ_0^* are the canonical representing measures of 1 and h, respectively, and $\bar{h}$ is the boundary function for h. (Note that h satisfies $L^{*,h}(h) = 0$.)

In this example

$$\mathscr{U}_\lambda \phi(\xi) = \lambda \int_E K_\lambda^*(\xi, x) H\phi(x) d\bar{x}$$

where H_ϕ satisfies $LH_\phi = 0$ with boundary function ϕ. If $\phi \in L^\infty$, and $dm = \bar{h} d\mu_0^*$, then

$$\int_{\partial E} \mathscr{U}_\lambda |\phi|(\xi) dm = \lambda \int_E H|\phi| d\bar{x} < \infty.$$

Now define U_λ, U_λ', $d\mu$ and $d\mu'$ as in (1.1)-(1.3). The mappings

$$M_\lambda \phi = \int_{\partial E} M_\lambda(\eta, \xi) \phi(\xi) \mu(d\xi)$$

and

$$M_\lambda' \psi = \int_{\partial E} \psi(\eta) M_\lambda(\eta, \xi) \mu'(d\eta)$$

satisfy (1.8) and (1.9). The boundary functions of G_λ and G_λ^* are equal to $M_\lambda U_\lambda$ and $M_\lambda' U_\lambda'$, respectively. Conditions U1-U4 of Section 1 are satisfied, and the representation (2.5) holds with $f_\lambda = 0$. There is a one to one correspondence between the conservative families $\{M_\lambda\}$ given by the Feller representation and the conservative diffusion resolvents satisfying R1 and R2. We are interested here in those families $\{M_\lambda\}$ that give rise to Dirichlet spaces on the boundary in the sense of Definitions 5.1 and 5.2. This requires that the Dirichlet forms determined by them satisfy the continuity condition (3.1).

A discrete example afforded by the Kolmogorov differential equations

$$\frac{dP}{dt} = \Omega P, \qquad \frac{dP}{dt} = P\Omega$$

where P and Ω are infinite matrices, i.e.,

$$P = P(t, i, k) \qquad (i, k = 1, 2, \ldots)$$
$$\Omega(i, k) = -q_i + q_i \Pi(i, k)$$

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with $q_i \geqq 0$, $\Pi(i, k) \geqq 0$ and $\sum_j \Pi(i, j) = 1$. Boundary value problems for these equations were treated by Feller [**1**], under the assumption that the equation

$$\lambda x_\lambda - \Omega x_\lambda = 0, \qquad \lambda u_\lambda - u_\lambda \Omega = 0$$

passes only finitely many linearly independent solutions, say M and N, respectively. The analogue of (0.4) and (7.8) for the associated resolvent matrix $\{P_\lambda(i, k)\}$ is (see [**1**], (13.1)),

$$(7.9) \qquad P_\lambda(i, k) = P_\lambda^{\min}(i, k) + \sum_{\xi=1}^{M} \sum_{\eta=1}^{N} x_\lambda^{(\xi)}(i) M_\lambda(\xi, \eta) u_\lambda^{(\eta)}(k)$$

$i, k = 1, 2, \ldots$ We have

$$(7.10) \qquad \mathscr{U}_\lambda(\xi, \eta) = \lambda u_\lambda^{(\xi)} x^{(\eta)}$$

(where the $x^{(\eta)}$ are solutions of $\Omega x^{(\eta)} = 0$), for $\xi = 1, \ldots, N$ and $\eta = 1, \ldots, M$.

If $M < N$, we define the measure m' so that the integers $1, \ldots, M$ get measure 1, and the integers from $M + 1$ to N get measure 0. The measure m gives measure one to all the integers $1, \ldots, N$. If $M > N$, then m' and m give measure one to the integers $1, \ldots, M$ and m gives measure zero to $N + 1, \ldots, M$. We are therefore taking

$$X = \{1, 2, \ldots, K = M \vee N\}.$$

The kernels $\mathscr{U}_\lambda(\xi, \eta)$ and $M_\lambda(\xi, \eta)$ are defined almost everywhere on $X \times X$ with the product measure.

In this paper we are interested in the case where as $\lambda \to \infty$,

$$(7.11) \qquad \sum_{\eta=1}^{k} \mathscr{U}_\lambda(\xi, \eta) m'(\eta) \uparrow \infty$$

and

$$(7.12) \qquad \sum_{\xi=1}^{k} \mathscr{U}_\lambda(\xi, \eta) m(\xi) \uparrow \infty,$$

although the case of a finite limiting matrix $\{\mathscr{U}(\xi, \eta)\}$ is also considered in [**1**]. It is shown in [**1**] that

$$\mathscr{U}_\lambda(\xi, \eta) \uparrow \mathscr{U}(\xi, \eta) < \infty \qquad (\xi \neq \eta).$$

Thus, if (7.11) and (7.12) hold we must have

$$\mathscr{U}_\lambda(\xi, \xi) \uparrow \infty \qquad (1 \leqq \xi \leqq M \vee N).$$

We then introduce the norming of Section 1, given by (1.1)-(1.3). The operator M_λ will be given by

$$M_\lambda \phi(\xi) = \sum_{\eta=1}^{N} M_\lambda(\xi, \eta) \phi(\eta) \mathscr{U}_1 1(\eta) m(\eta),$$

and

$$M_\lambda'\psi(\eta) = \sum_{\xi=1}^{M} M_\lambda(\xi, \eta)\psi(\xi)\mathscr{U}_1'1(\xi)m'(\xi).$$

The condition that $\lambda P_\lambda 1 \leqq 1$ is equivalent to $M_\lambda U_\lambda 1 \leqq 1$. If we have a strictly positive solution of $\Omega h = 0$ then, as in the previous example, we can reduce considerations to the case where $M_\lambda' U_\lambda' 1 \leqq 1$. The general structure of this discrete example is exactly analogous to the diffusion example.

In a subsequent paper we shall investigate some other examples, but the technical details are too lengthy to present here.

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