BEST DIFFERENCE EQUATION APPROXIMATION TO DUFFING’S EQUATION

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Abstract

Duffing’s differential equation in its simplest form can be approximated by a variety of difference equations. It is shown how to choose a ‘best’ difference equation in the sense that the solutions of this difference equation are successive discrete exact values of the solution of the differential equation.

1. Introduction

A differential equation, such as the simple Duffing equation [2]

\[ \ddot{x}(t) + ax(t) + bx^3(t) = 0, \quad (1.1) \]

can be approximated by difference equations in a variety of ways. In numerical analysis, the aim is to choose a difference equation (\(\Delta E\)) which gives a high-order approximation to the differential equation (\(DE\)) and which can be solved computationally in a stable way without trouble from round-off errors.

The simplest \(\Delta E\) approximation to (1.1) is

\[ h^{-2}(x_{n+1} - 2x_n + x_{n-1}) + ax_n + bx_n^3 = 0, \quad (1.2) \]

where \(h\) is a chosen constant time interval, but for theoretical purposes this is not suitable because it cannot be solved in closed form, nor can the general behaviour of its solutions be analysed.

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In a recent paper [3] it has been shown that an alternative $\Delta E$ approximation, namely

$$h^{-2}(x_{n+1} - 2x_n + x_{n-1}) + ax_n + \frac{1}{2}bx_n^2(x_{n+1} + x_{n-1}) = 0, \quad (1.3)$$

can be solved in closed form and for appropriate choice of the value of $h$ periodic solutions are obtained with the same qualitative features as the solutions of the DE (1.1).

The problem solved in this paper is: find a best $\Delta E$ approximation to the DE (1.1).

By "best" we shall mean the following: if $x(t)$ is the solution of a DE, then a best $\Delta E$ approximation to the DE using a time interval $h$ is one for which the solution $x_n$ of the $\Delta E$ is given by

$$x_n = x(nh). \quad (1.4)$$

Geometrically, the $x_n$ satisfying the $\Delta E$ are required to be points on the solution curve $x(t)$ of the DE at successive values $t = nh$, where $n$ is an integer.

In previous papers [4], [5] this question has been answered for linear ODE's with constant coefficients and for some non-linear ODE's describing ecological problems. To answer the question for the Duffing equation (1.1) we consider the three cases which arise giving the Jacobian elliptic functions $cn, dn$ and $sn$ as periodic solutions and a further case of a non-periodic solution.

In general, there will be more than one $\Delta E$ if with the property (1.4), but the particular difference equations we derive are simple in form and directly recognizable as approximations to the DE.

2. Case I

Consider first the equation (1.1) with initial conditions

$$x(0) = A \quad \text{and} \quad \dot{x}(0) = 0, \quad (2.1)$$

and for the range of parameter values

$$a > -\frac{1}{2}bA^2 \quad \text{where} \ b > 0. \quad (2.2)$$

Then the well-known [2] solution of (1.1) is

$$x(t) = A \ cn\left[\left(a + bA^2\right)^{1/2}t \frac{1}{2}bA^2 / (a + bA^2)\right], \quad (2.3)$$

using the notation of [1].

To find a $\Delta E$ satisfied by this function we use the following.
Best approximation to Duffing's equation

**Lemma. If**

\[ x(t_2) = f[t_2 - t_1, x(t_1)] \quad \text{for all } t_1, t_2 \]  \hspace{1cm} (2.4)

and

\[ x_n = x(nh) \quad \text{where } n \text{ is an integer} \]  \hspace{1cm} (2.5)

then

\[ x_{n+1} = f[h, x_n], \]  \hspace{1cm} (2.6)

and

\[ x_{n-1} = f[-h, x_n]. \]  \hspace{1cm} (2.7)

The proof of the lemma is immediate.

For the function

\[ x(t) = A \cos[l(t_0 + t_1) \mid m] = A \cos[l(t_0 + t)] \]  \hspace{1cm} (2.8)

the addition formula for \( \cos \) can be used to derive from

\[ x(t_2) = A \cos[l(t_2 - t_1) + l(t_0 + t_1)] \]  \hspace{1cm} (2.9)

the relation

\[ x(t_2) \{1 - m \sin^2[l(t_2 - t_1)]\sin^2[l(t_0 + t_1)]\} = A \cos[l(t_2 - t_1)]\cos[l(t_0 + t_1)] - A \sin[l(t_2 - t_1)]\sin[l(t_0 + t_1)] \sin[l(t_2 - t_1)] \sin[l(t_0 + t_1)]. \]  \hspace{1cm} (2.10)

Use of the lemma gives the \( \Delta E \)

\[ (x_{n+1} + x_{n-1}) \{1 - m \sin^2(lh)\sin^2[A^{-2}x_n^2]\} = 2x_n \cos(lh). \]  \hspace{1cm} (2.11)

Specific application to (2.3) accordingly gives

\[ (x_{n+1} + x_{n-1})\{\cos^2 + x_n^2[\sin^2(lh)\sin^2(A^{-2}x_n^2)]\} = 2x_n \cos(lh), \]  \hspace{1cm} (2.12)

where, in this \( \Delta E \), we use the abbreviated notation

\[ \sin = \sin[(a + bA^2)^{1/2}h \mid l/2bA^2 / (a + bA^2)] \]  \hspace{1cm} (2.13)

and similarly for \( \cos \) and \( \sin \).

For \( t_0 = 0 \) and \( x_0 = A, x_1 = x_{-1} \) and the solution of (2.12) is

\[ x_n = A \cos[(a + bA^2)^{1/2}nh \mid l/2bA^2 / (a + bA^2)]. \]  \hspace{1cm} (2.14)
The $\Delta E$ (2.12) is thus a best approximation to the $DE$ (1.1) in the sense that, comparing (2.3) with (2.14),

$$x_n = x(nh). \quad (2.15)$$

It should be noted that that the constant time interval $h$ does not need to be 'small' in any sense, nor is it restricted to being positive. As $h$ is made smaller, the points $x_n$ on the solution curve $x(t)$ simply come closer together.

The fact that in the limit $h \to 0$ the $\Delta E$ tends to the $DE$ can be verified as follows. The $\Delta E$ (2.12) can be written

$$(x_{n+1} - 2x_n + x_{n-1})(a + bA^2)ns^2dn^2 + 2x_n(a + bA^2)ns^2(dn^2 - cn) + \frac{1}{2}b x_n^2(x_{n+1} + x_{n-1}) = 0. \quad (2.16)$$

And since

$$(a + bA^2)ns^2dn^2 = h^{-2} + O(h^{-1}) \quad (2.17)$$

and

$$2(a + bA^2)ns^2(dn^2 - cn) = a + O(h) \quad (2.18)$$

the $\Delta E$ (2.16) tends to the $DE$ (1.1) as $h \to 0$.

It is interesting to note from (2.16) that this $\Delta E$ is obtained from the $DE$ by replacing $x$ by $(x_{n+1} - 2x_n + x_{n-1})(a + \frac{1}{2}bA^2)ns^2dn^2$ instead of the usual $h^{-2}(x_{n+1} - 2x_n + x_{n-1})$. And as has been observed previously [3], $x^3$ has been replaced not by $x^3$ but by $\frac{1}{2}x_n(x_{n+1} + x_{n-1})$.

3. Case II

For the range of parameters

$$-bA^2 < a < -\frac{1}{2}bA^2 \quad \text{where } b > 0, \quad (3.1)$$

we transform the results for case I using the reciprocal parameter [1]

$$\mu = m^{-1} = 2(a + bA^2)/(bA^2) \quad \text{where } 0 < \mu < 1, \quad (3.2)$$

and

$$v = \left(\frac{1}{2}bA^2\right)^{1/2}h, \quad (3.3)$$

so that

$$\text{sn}\left[(a + bA^2)^{1/2}h|\frac{1}{2}bA^2/(a + bA^2)\right] = \left[2(a + bA^2)/(bA^2)\right]^{1/2}\text{sn}(v|\mu), \quad (3.4)$$
and similarly

\[ \begin{align*}
    cn &= dn(\nu | \mu), \\
    dn &= cn(\nu | \mu).
\end{align*} \tag{3.5} \tag{3.6} \]

Hence (2.12) becomes

\[ (x_{n+1} + x_{n-1})\{cn^2 + x_n^2A^{-2}sn^2\} = 2x_n dn, \tag{3.7} \]

where

\[ sn = sn\left[ \left( \frac{1}{2}bA^2 \right)^{1/2} h \bigg| 2(a + bA^2) / (bA^2) \right], \tag{3.8} \]

and similarly for cn and dn.

For the range of parameters (3.1) the solution of the DE (1.1) is [2]

\[ x(t) = A \, dn\left[ \left( \frac{1}{2}bA^2 \right)^{1/2} t \bigg| 2(a + bA^2) / (bA^2) \right], \tag{3.9} \]

while the solution of (3.7) is

\[ x_n = A \, dn\left[ \left( \frac{1}{2}bA^2 \right)^{1/2} nh \bigg| 2(a + bA^2) / (bA^2) \right]. \tag{3.10} \]

Again we have \( x_n = x(nh) \) so that in the sense defined, (3.7) is a best \( \Delta E \) approximation to the DE (1.1) for the range of parameters (3.1).

**4. Case III**

For the range of parameters

\[ a > -bA^2 \quad \text{where} \quad b < 0, \tag{4.1} \]

we use the ‘negative parameter’ transformation [1]

\[ \mu = \frac{-m}{1 - m} = \frac{-bA^2}{2a + bA^2} \quad \text{where} \quad 0 < \mu < 1, \tag{4.2} \]

and

\[ \nu = \left( a + \frac{1}{2}bA^2 \right)^{1/2} h, \tag{4.3} \]

so that

\[ sn\left[ \left( a + bA^2 \right)^{1/2} h \bigg| \frac{1}{2}bA^2 / (a + bA^2) \right] = (a + bA^2)^{1/2}(a + \frac{1}{2}bA^2)^{-1/2} sd(\nu | \mu) \tag{4.4} \]

and similarly

\[ \begin{align*}
    cn &= cd(\nu | \mu), \\
    dn &= nd(\nu | \mu).
\end{align*} \tag{4.5} \tag{4.6} \]
Hence (2.12) becomes
\[
(x_{n+1} + x_{n-1})\left\{1 + x_{n}^{2}\left[b/(2a + bA^2)\right]sn^{2}\right\} = 2x_{n}cn\ dn, \tag{4.7}
\]
where
\[
\text{sn} = \text{sn}\left[(a + \frac{1}{2}bA^2)^{1/2}t | -\frac{1}{2}bA^2/(a + \frac{1}{2}bA^2)\right], \tag{4.8}
\]
and similarly for \(cn\) and \(dn\).

For the range of parameters (4.1) the solution of the DE (1.1) satisfying the conditions
\[
x(0) = 0 \text{ and } |x| = A \text{ when } \dot{x} = 0, \tag{4.9}
\]
is
\[
x(t) = A \text{ sn}\left[(a + \frac{1}{2}bA^2)^{1/2}t | -\frac{1}{2}bA^2/(a + \frac{1}{2}bA^2)\right]. \tag{4.10}
\]
A solution of the DE (4.7) is
\[
x_n = A \text{ cn}\left[(a + \frac{1}{2}bA^2)^{1/2}(t_0 + nh) | -\frac{1}{2}bA^2/(a + \frac{1}{2}bA^2)\right]. \tag{4.11}
\]
If we choose \(t_0\) so that
\[
(a + \frac{1}{2}bA^2)^{1/2}t_0 = -K, \tag{4.12}
\]
where \(K\) is the quarter-period of \(sn\), then
\[
x_n = A \text{ sn}\left[(a + \frac{1}{2}bA^2)^{1/2}nh | -\frac{1}{2}bA^2/(a + \frac{1}{2}bA^2)\right] \tag{4.13}
\]
satisfying \(x_0 = 0\) and \(\text{max } x = A\). Comparing (4.13) with (4.10) we again see that \(x_n = x(nh)\).

5. A case of a non-periodic solution

The Duffing equation
\[
\ddot{x} + \omega^2 x - 2\omega^2 A^{-2}x^3 = 0, \tag{5.1}
\]
with
\[
x(0) = A \text{ and } \dot{x}(0) = 0,
\]
has values for the parameters outside the ranges considered in the above three cases since \(a = -\frac{1}{2}bA^2 < -bA^2\) and \(b < 0\). The DE (5.1) has the non-periodic solution
\[
x(t) = A \sec \omega t. \tag{5.2}
\]
A best approximating $\Delta E$ to the DE (5.1) in this case is obtained using the addition formula for sec, namely

$$A \sec(\omega t_2) = A \sec[\omega(t_2 - t_1) + \omega t_1]$$

$$= A/\{\cos \omega(t_2 - t_1) \cos \omega t_1 - \sin \omega(t_2 - t_1) \sin \omega t_1\},$$

so that

$$(x_{n+1} + x_{n-1})\left(1 - A^{-2}x_n^2 \sin^2(\omega h)\right) = 2x_n \cos(\omega h) \quad (5.3)$$

or

$$(x_{n+1} - 2x_n + x_{n-1})/(4\omega^{-2} \sin^2\frac{1}{2}\omega h) + \omega^2 x_n$$

$$- \omega^2 A^{-2}x_n^2(x_{n+1} + x_{n-1}) \cos^2\frac{1}{2}\omega h = 0. \quad (5.4)$$

Again the $\Delta E$ (5.3) or (5.4) is a best approximation to the DE (5.1) in the sense that the solution of the $\Delta E$ is

$$x_n = A \sec(\omega nh) = x(nh). \quad (5.5)$$

The form (5.4) is revealing in showing that $\bar{x}$ in (5.1) is replaced by $(x_{n+1} - 2x_n + x_{n-1})/(4\omega^{-2} \sin^2\frac{1}{2}\omega h)$ and $x^3$ by $\frac{1}{2}x_n^2(x_{n+1} + x_{n-1}) \cos^2\frac{1}{2}\omega h$.

6. Discussion

This paper has shown that it is possible to find from the infinite choices available a difference equation which in an obvious sense is a best approximation to Duffing's differential equation in its simplest form. The result extends to an important classical non-linear differential equation a general theory applicable to linear differential equations with constant coefficients. Because the method is based on the knowledge of the solution of the differential equation it cannot be expected to apply to equations for which closed form solutions are not known. For these, recourse is often made to perturbation methods, as for example in Duffing's equation (1.1) when there is an additional small friction term $\varepsilon \dot{x}$. Then it is an advantage to use for the unperturbed equation the difference equation derived above.

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References


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