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R_1, PAIRWISE COMPACT, AND PAIRWISE COMPLETE SPACES

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1. The R_1 axiom was first introduced by Davis in [1]. It is strictly weaker than the T_2 axiom. Murdeshwar and Naimpally, in [4], have weakened the T_2 hypothesis to R_1 in some well-known theorems. We show that in many topological spaces the R_1 axiom and regularity are equivalent. Also, the definition of local compactness given in [4] can be weakened to the usual definition and still get the same results.

The notion of a bitopological space was first introduced by Kelley in [3]. Fletcher, Hoyle, and Patty discuss pairwise compactness for bitopological spaces in [2]. One of our main results is that a bitopological space (X, P, Q) is pairwise compact if and only if each ultrafilter ν on X, containing a proper P closed set and a proper Q closed set, has a common P and Q limit.

Finally, we discuss biquasi uniform spaces. Some results on these spaces are given by Murdeshwar and Naimpally in [5]. We define pairwise completeness and give some conditions under which pairwise compactness implies pairwise completeness. It is also shown that if (X, Q_1, Q_2) is a pairwise complete, pairwise Hausdorff space, then Q_1 and Q_2 induce the same topology on X.

2. The following definitions and theorems will be needed

DEFINITIONS. The topological space (X, τ) is R_1 if $\bar{x} \neq \bar{y}$ implies that x and y have disjoint neighborhoods [4]. It is *locally compact* if every point has a closed compact neighborhood [4]. The collection of sets Q is a *quasi-uniformity* for X if it is a uniformity less the symmetry axiom [5].

If τ_1, τ_2 are two topologies on X, then (X, τ_1, τ_2) is a *bitopological space* [3]. The space is called *pairwise Hausdorff* if $x \neq y$ implies there is a τ_1 open set and a τ_2 open set containing $\{x\}$ and $\{y\}$ respectively, which are disjoint [3]. It is *pairwise regular* if A is τ_i closed with $x \notin A$ implies that there are τ_j, τ_i disjoint open sets containing A and $\{x\}$ respectively, $i \neq j$ [3]. It is called *pairwise compact* if each cover of X consisting of both nonempty τ_1 and τ_2 open sets has a finite subcover [2]. If Q_1, Q_2 are quasi-uniformities for X, then (X, Q_1, Q_2) is called a *biquasi-uniform space* [5]. A filter $\mathfrak{f}$ on X is *Q_1-Cauchy* if for each $\nu \in Q_1$ implies that there is an $x \in X$ with $\nu(x) \in \mathfrak{f}$ [5]. The space (X, Q_1, Q_2) is called *pairwise complete* if every Q_i-Cauchy filter on X has a Q_j cluster point, $i \neq j$.

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THEOREMS. *The one point compactification of X is R_1 iff X is R_1 and locally compact* [4]. *A locally compact R_1 space is completely regular* [4]. *An R_1 paracompact space is regular and hence normal* [4]. *Thus a compact R_1 space is normal. If X is R_1 and $A \subset X$ is compact with $\overline{\{x\}} \cap A = \varnothing$, then $\{x\}$ and A have disjoint neighborhoods* [4]. *The R_1 axiom is hereditary* [4]. *In an R_1 space (X, τ), $\bar{x} \subset O$ for each $x \in O \in \tau$* [4]. *If A is a τ_1-closed proper set in (X, τ_1, τ_2), then A is τ_2-compact if X is pairwise compact* [2].

3. LEMMA. *A locally compact (i.e. each point has a compact neighborhood) R_1 space (X, τ) has a neighborhood base at each point consisting of closed compact neighborhoods.*

Proof. Let $O \in \tau$ and $x \in O$. Then there is a compact neighborhood N of x with $x \in O_N \subset N$ and $O_N \in \tau$. Hence $\bar{x} \subset O \cap O_N$. If $N-(O \cap O_N) \neq \varnothing$, then $\bar{x}$ and $N-(O \cap O_N)$ are nonempty disjoint N-closed sets. Since N is normal, then there exist disjoint N-open sets O_1, O_2 such that $\bar{x} \subset O_1$, and $N-(O \cap O_N) \subset O_2$. Thus $\bar{O}_1^{(N)} \subset N - O_2 \subset O \cap O_N$. But $O_1 \subset O_N$ implies that O_1 is τ-open. We claim that $\bar{O}_1^{(N)} = \bar{O}_1^{(X)}$. Let $y \in X-N$. Then $y \in X - O_N$ and hence $\bar{y} \subset X - O_N$. Thus $\bar{y} \cap \bar{O}_1^{(N)} = \varnothing$. Since $\bar{O}_1^{(N)}$ is compact, then $\{y\}$ and $\bar{O}_1^{(N)}$ have disjoint neighborhoods. Hence our claim follows and thus $x \in \bar{O}_1^{(N)} = \bar{O}_1^{(X)} \subset O$. If $N-(O \cap O_N) = \varnothing$, then $O_N = N \subset O$. If $y \notin N$, then $\bar{y} \cap N = \varnothing$ and hence N is a closed, compact, neighborhood of x.

REMARK 1. From the above lemma, we see that the usual definition of local compactness could have been used in the above theorems of §2.

THEOREM 1. *Let (X, τ) be locally compact (usual definition). Then the following are equivalent*:

(1) *τ is R_1*
(2) *τ is regular*
(3) *the closed compact neighborhoods of each point is a neighborhood base.*

Proof. This follows from the lemma and the fact that regularity implies R_1, which is easily established.

REMARK 2. From the results of §2, we easily see that R_1 and regularity are equivalent in both paracompact and normal spaces.

One might conjecture that a complete R_1 quasi uniform space is regular. The following example shows that even a T_2 complete quasi uniform space is not necessarily regular.

Let X=reals, Q=rationals, I=irrationals, and

$$B = \{O_{x,\epsilon} = [(x-\epsilon, x+\epsilon) \cap Q] \cup \{x\} \mid \epsilon > 0\}.$$

Clearly B is a base for a topology τ on X, which is T_2. Let $x \in Q$. Then $\{x\}$ and I are disjoint closed sets and are not contained in disjoint open sets. Hence τ is not regular. Let T be the quasi uniformity whose base is $\{V_\epsilon = \bigcup_{x \in X} (\{x\} \times O_{x,\epsilon}) \mid \epsilon > 0\}$.

Then T is easily seen to be a quasi uniformity which induces τ. Let $\mathfrak{f}$ be a T-Cauchy filter on X. That is, for each $V \in T$ there exists an $x \in X$ such that $V(x) \in \mathfrak{f}$. We must show that $\mathfrak{f}$ has a cluster point. There is an $x_n \in X$ such that $V_{2^{-n}}(x_n) = O_{x_n, 2^{-n}} \in \mathfrak{f}$ for each natural number n. Now $x_n \to x$ in the usual sense. We claim that x is a cluster point of $\mathfrak{f}$. If not, then there is an $O_{x,\epsilon} \in B$ and $F \in \mathfrak{f}$ such that $O_{x,\epsilon} \cap F = \varnothing$. There is a natural number m such that $O_{x_n, 2^{-n}} \subset O_{x,\epsilon} \cup \{x_n\}$ for each $n \geq m$. Hence $O_{x,\epsilon} \cup \{x_n\} \in \mathfrak{f}$. Since $O_{x,\epsilon} \cap F = \varnothing$, then $F \cap (O_{x,\epsilon} \cup \{x_n\}) = \{x_n\} \in \mathfrak{f}$ for each $n \geq m$, which is a contradiction. Hence (X, T) is complete.

4. The emphasis here will be on pairwise complete spaces

THEOREM 2. *If (X, Q_1, Q_2) is pairwise compact such that for each Q_i-Cauchy filter $\mathfrak{f}$ on X, $\bar{\mathfrak{f}}(Q_i) \neq \{X\} (i=1, 2)$, then it is pairwise complete.*

Proof. Let $\mathfrak{f}$ be a Q_1-Cauchy filter on X. Then there is an $F \in \mathfrak{f}$ with $\bar{F}^{(Q_1)} \neq X$. Hence $\bar{F}^{(Q_1)}$ is Q_2-compact. If there does not exist a Q_2-cluster point x of $\mathfrak{f}$, then for each $x \in \bar{F}^{(Q_1)}$ there is a $V_x \in Q_2$ and $F_x \in \mathfrak{f}$ such that $V_x(x) \cap F_x = \varnothing$. Thus there exists $V_{x_1}, \ldots, V_{x_n}$ such that $\bigcup_{i=1}^{n} V_{x_i}(x_i) \supset \bar{F}^{(Q_1)}$ and we have $F \cap (\bigcap_{i=1}^{n} F_{x_i}) = \varnothing$. This contradicts $\mathfrak{f}$ being a filter. Hence $\mathfrak{f}$ has a Q_2-cluster point and the theorem follows.

THEOREM 3. *Let Q_1, Q_2 be uniformities on X with either $Q_1 \neq \{X \times X\}$ and $Q_2 \neq \{X \times X\}$, or $Q_1 = Q_2 = \{X \times X\}$. If (X, Q_1, Q_2) is pairwise compact, then it is pairwise complete.*

Proof. Clearly the latter part follows. Let $Q_1 \neq \{X \times X\}$, $Q_2 \neq \{X \times X\}$ and $\mathfrak{f}$ a Q_1-Cauchy filter. Hence $\mathfrak{f} \times \mathfrak{f} \geq Q_1$. Let $V \in Q_1$ with $V \neq X \times X$ and such that V is Q_1-closed. There is an $F \in \mathfrak{f}$ with $F \times F \subset V$. Hence $\bar{F}^{(Q_1)} \times \bar{F}^{(Q_1)} \subset V$. That is, $\bar{F}^{(Q_1)} \neq X$. Hence by Theorem 2, the result follows.

If (X, Q_1, Q_2) is pairwise compact and $Q_1 = \{X \times X\}$, $Q_2 \neq \{X \times X\}$ or vice-versa, then (X, Q_1, Q_2) may not be pairwise complete. Let X be a nonfinite set, $Q_1 = \{X \times X\}$, $Q_2 = \{V \subset X \times X \mid V \supset \Delta\}$. Clearly (X, Q_1, Q_2) is pairwise compact. Let $\mathfrak{f}$ be a filter on X with $\bigcap_{F \in \mathfrak{f}} F = \varnothing$. Hence $\bigcap_{F \in \mathfrak{f}} \bar{F}^{(Q_2)} = \varnothing$. Since every filter on X is Q_1-Cauchy, then (X, Q_1, Q_2) is not pairwise complete.

THEOREM 4. *If (X, Q_1, Q_2) is pairwise Hausdorff and pairwise complete, then $\tau_{Q_1} = \tau_{Q_2}$. That is, the induced topologies are equal.*

Proof. Let $O_1 \in \tau_{Q_1}$, $x \in O_1$. If for each $O_2 \in \tau_{Q_2}$ with $x \in O_2$ implies that $O_2 \cap (X - O_1) \neq \varnothing$, then $B = \{O_2 \cap (X - O_1) \mid x \in O_2 \in \tau_{Q_2}\}$ is a base for a filter $\mathfrak{f}$ on X. Clearly $\mathfrak{f}$ is Q_2-Cauchy. Hence $\mathfrak{f}$ has a Q_1-cluster point, say x_1. Also $x_1 \neq x$. Hence there is a $G_1 \in \tau_{Q_1}$, $G_2 \in \tau_{Q_2}$ such that $x_1 \in G_1$, $x \in G_2$ and $G_1 \cap G_2 = \varnothing$. This contradicts $x_1 \in \overline{G_2 \cap (X - O_1)}^{(Q_1)}$ and the theorem follows.

Pairwise Hausdorff cannot be dropped from the hypothesis of the above theorem. One can easily show that the following would be a counter-example. Let X be the closed interval $[0, 1]$, $Q_1 = \{X \times X\}$ and let Q_2 be the usual uniformity on X.

5. THEOREM 5. *The bitopological space (X, P, Q) is pairwise compact iff each ultrafilter ν on X, with at least one proper P-closed subset and at least one proper Q-closed subset, P and Q converges to a common point.*

Proof. Let (X, P, Q) be pairwise compact. If there is an ultrafilter ν on X such that ν has no P and Q common limit, then for each $x \in X$ there is an O_x, either in P or Q, such that $O_x \notin \nu$. Hence $X - O_x \in \nu$. Let $C = \{O_x \mid x \in X\}$. If there are elements of C in both P and Q, then $\bigcup_{i=1}^n O_{x_i} = X$ for some finite set. Hence $\bigcap_{i=1}^n (X - O_{x_i}) \in \nu$, which implies that $\phi \in \nu$ and we have a contradiction. Next, assume all elements of C are in one topology, say P. Let $A \in \nu$ such that A is a proper Q-closed subset of X. Hence A is P-compact and we have that $\bigcup_{i=1}^n O_{x_i} \supset A$ for some finite set. This implies that

$$X - \bigcup_{i=1}^{n} O_{x_i} \subset X - A, \quad \text{with} \quad X - \bigcup_{i=1}^{n} O_{x_i} = \bigcap_{i=1}^{n} (X - O_{x_i}) \in \nu$$

which is a contradiction since $A \in \nu$. Hence the necessity follows.

Conversely, if (X, P, Q) is not pairwise compact then there is an open cover of X, say C, containing both nonempty P-open and Q-open sets that has no finite subcover. Let $\delta = \{X - O \mid O \in C\}$. Then $(X - O_1) \cap (X - O_2) = X - (O_1 \cup O_2) \neq \varnothing$. Thus δ is a subbase for a filter $\mathfrak{f}$. Let ν be an ultrafilter with $\mathfrak{f} \subset \nu$. Clearly ν has proper P and Q closed subsets. Hence by hypothesis νP and Q converges to some $x \in X$. Let $O \in C$ with $x \in O$. Hence $O \in \nu$; however, $X - O \in \nu$ and we have a contradiction. Hence (X, P, Q) is pairwise compact.

THEOREM 6. *If the bitopological space (X, P, Q) is pairwise compact, then for each filter $\mathfrak{f}$ on X there is an $x \in X$ with x being a P-cluster point of $\bar{\mathfrak{f}}^{(Q)}$ and a Q-cluster point of $\bar{\mathfrak{f}}^{(P)}$.*

Proof. Let $\mathfrak{f}$ be a filter on X. If $\bar{\mathfrak{f}}^{(Q)}$ does not have a P-cluster point, then there is an $F \in \mathfrak{f}$ with $\bar{F}^{(Q)} \neq X$. Hence $\bar{F}^{(Q)}$ is P-compact. For each $x \in \bar{F}^{(Q)}$ there is an $O_x \in P$, $F_x \in \mathfrak{f}$ with $O_x \cap F_x = \varnothing$. Let $\bigcup_{i=1}^n O_{x_i} \supset \bar{F}^{(Q)}$. Hence $F \cap (\bigcap_{i=1}^n F_{x_i}) = \varnothing$ which is a contradiction. Consequently, $\bar{\mathfrak{f}}^{(Q)}$ has a P-cluster point. Similarly $\bar{\mathfrak{f}}^{(P)}$ has a Q-cluster point. If $\mathfrak{f}$ has a proper P-closed subset and a proper Q-closed subset, then we easily see that the desired result follows from Theorem 5. If $\mathfrak{f}$ does not have a proper P-closed subset, then $\bar{F}^{(P)} = X$ for each $F \in \mathfrak{f}$. Hence every $x \in X$ is Q-cluster point of $\bar{\mathfrak{f}}^{(P)}$. Since $\bar{\mathfrak{f}}^{(Q)}$ has a P-cluster point, then the theorem follows.

The conclusion of Theorem 6 seems to be a reasonable, weaker definition of pairwise compactness. The converse of Theorem 6 is false as one sees from the following example. Let $X = [0, 1]$ with P being the usual topology, Q the discrete topology. It is easily shown that (X, P, Q) is not pairwise compact but that the conclusion of Theorem 6 holds.

THEOREM 7. *If (X, P) is compact, disconnected and (X, P, Q) is pairwise regular, then (X, Q) is disconnected.*

Proof. Let $X=O_1 \cup O_2$ with both O_1, $O_2 \neq \varnothing$ and disjoint P-open subsets. Since (X, P, Q) is pairwise regular, then for each $y \in O_1$ there is an $O_y \in P$, $G_y \in Q$ with $y \in O_y$ and $O_2 \subset G_y$, where $O_y \cap G_y = \varnothing$. Now O_1 is P-compact. Hence $\bigcup_{i=1}^{n} O_{y_i} = O_1$ for some finite set and we have that $\bigcap_{i=1}^{n} G_{y_i} = O_2 \in Q$. Similarly $O_1 \in Q$ and the result follows. Hence we have the following theorem.

THEOREM 8. *If (X, P), (X, Q) are compact, (X, P, Q) is pairwise regular, then (X, P) is connected iff (X, Q) is connected.*

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