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BOUNDED SPHERICAL FUNCTIONS FOR ABSTRACT OPERATORS

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Abstract

The aim of this paper is to study the spherical functions associated to an operator. These functions can be thought of abstractly as being eigenfunctions of the operator which can be expressed in terms of the operator. The meaning of these properties will be made precise as will a notion of boundedness. The results are obtained by studying a specific shift operator on the algebra of functionals on the complex polynomial ring. For the class studied, we obtain ellipses of eigenvalues for which there exist bounded spherical functions. As an application of the results, we study radial functions on discrete groups.

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Let $A = \mathbf{C}[x]$ and let $B = A^*$ be its dual. What we are searching for are useful expressions for the eigenfuctions of the shift operator $T\colon B \to B$ given by $T(f)[p(x)] = f[xp(x)]$. We will later apply these results to self-adjoint operators on certain spaces. Notice that if ϕ is an eigenfunction with eigenvalue z, then by definition $\phi(xp(x)) = z\phi(p(x))$. Thus by induction on the degree of $p(x), \phi(p(x)) = p(z)\phi(1)$. So defining $\phi_z(p(x)) = p(z)$, ϕ_z generates the eigenspace of z. Thus the solution ϕ_z has a simple expression as a functional on A. What we are looking for are convenient ways of describing ϕ_z in general settings. We need some preliminary facts.

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By "measure", we always mean a positive Radon measure. In particular assume that we are given a linear map $\mu\colon \mathbf{R}[x] \to \mathbf{R}$ such that if $p(x)$ is a real polynomial with nonnegative values then $\mu(p(x)) \geq 0$. Then there is a unique measure $d\mu$ on $\mathbf{R}$ such that $\int q(x)\, d\mu = \mu(q(x))$ for all $q(x) \in \mathbf{R}[x]$. (See [4] Theorems 34.9 and Lemma 34.11.)

Assume that for each $n = 0, 1, 2, \ldots$ there is given an nth degree real polynomial $p_n(x)$ with positive leading coefficient. Since the $p_n(x)$ form a basis for $\mathbf{R}[x]$, there is a unique inner product defined by setting $p_n(x) \cdot p_m(x) = \delta_{n,m}$, the Kronecker delta. For $p(x) \in \mathbf{R}[x]$ define $\mu(p(x)) = 1 \cdot p(x)$.

PROPOSITION 1. *The following are equivalent.*

(a) *x is self-adjoint with respect to the inner product.*

(b) *μ is a real positive measure on $\mathbf{R}$ and $p(x) \cdot q(x) = \mu(p(x)q(x))$.*

(c) *There exist two sequences of real numbers b_n and a_n with $b_n > 0$, such that*

$$xp_n(x) = b_n p_{n+1}(x) + a_n p_n(x) + b_{n-1} p_{n-1}(x). \tag{1}$$

PROOF. Assume (a). First observe that since x is self-adjoint so is any $p(x) \in \mathbf{R}[x]$, so that $p(x) \cdot q(x) = 1 \cdot p(x)q(x) = \mu(p(x)q(x))$. We need to prove that if $p(x)$ is a non-zero polynomial with positive values, then $\mu(p(x)) > 0$. By Lemma 34.10 [4] such a $p(x)$ can be written as a sum of squares of polynomials, so it suffices to prove that if $p(x)$ is any non-zero polynomial, then $\mu(p(x)^2) > 0$. But $\mu(p(x)^2) = p(x) \cdot p(x) > 0$. This proves (b).

Next assume (b). Write $xp_n(x)$ as a linear combination $\sum \alpha_{n,i} p_i(x)$. Since $xp_n(x)$ is of degree $n+1$, $\alpha_{n,i} = 0$ for $i > n+1$. Notice that $\alpha_{n,m} = (xp_n(x)) \cdot p_m(x) = \mu\{(xp_n(x))p_m(x)\} = \mu\{p_n(x)(xp_m(x))\} = p_n(x) \cdot (xp_m(x)) = \alpha_{m,n}$. In particular $\alpha_{n,m} \neq 0$ only if $m = n-1$, n, or $n+1$. Letting $b_n = \alpha_{n,n+1} = \alpha_{n+1,n}$ and $a_n = \alpha_{n,n}$, (1) follows. Note that $b_n > 0$ since all leading coefficients are positive.

Next assume (c). Then (1) shows that $(xp_n(x)) \cdot p_m(x) = p_n(x) \cdot (xp_m(x))$. (a) follows since $\{p_n(x)\}$ is a basis for $\mathbf{R}[x]$.

Let μ be a measure on the real line. We will always assume that the support of μ consists of more than a finite number of points and that all polynomials are μ-integrable. We can always find a unique sequence $\{p_n(x)\}$ of *orthonormal polynomials* for μ with $p_n(x) \in \mathbf{R}[x]$ of degree n having a positive leading coefficient. Setting $p(x) \cdot q(x) = \mu\{p(x)\overline{q(x)}\}$ yields an inner product on $A = \mathbf{C}[x]$ with respect to which $\{p_n(x)\}$ is an orthonormal basis.

Since we have an inner product on A for which the $\{p_n(x)\}$ are orthonormal, we can represent any element of B, its dual, as an infinite sum $\sum \alpha_n p_n(x)$ where the action on A is given formally by the inner product. By Proposition 1, then, the shift operator $T\colon B \to B$ satisfies $T(p_n(x))[p_m(x)] = (xp_n(x))[p_m(x)]$. Thus the operator T can be thought of as multiplication by x *in* B.

Now since $\phi_z(p_n(x)) = p_n(z)$, we can write $\phi_z = \sum p_n(z)p_n(x)$. Thus

THEOREM 1. *For every* $z \in \mathbb{C}$, $T\colon B \to B$ *has a one-dimensional eigenspace which is generated by*

$$\phi_z = \sum p_n(z)p_n(x).$$

Note. Using (1) one can also calculate directly that $x\phi_z = z\phi_z$.

Since T is multiplication by x, ϕ_z can be expressed as $\sum p_n(z)p_n(T)$. This is what was meant in the introduction by describing the eigenfunction in terms of T.

Let us now look at applications of this in terms of operators. For the moment we will not specify the type of space on which they are to operate.

If V is a von Neumann algebra, then by a *trace* on V we mean a linear functional $\eta\colon V \to \mathbb{C}$ such that if P is a non-zero projection then $0 < \eta(P) \leq 1$. It follows that if S is a positive element of V then $\eta(S) > 0$: The sub-von Neumann algebra W generated by the identity 1 and by S is isomorphic to the bounded functions $L^\infty(X,\nu)$ of the spectrum X of S with respect to some Borel measure ν (see for example, Theorem 2.2.4 of [1]). X is a compact subset of $[0,\infty)$, and making the identification of W with $L^\infty(X,\nu), S$ corresponds to the inclusion map $X \subset \mathbb{C}$. We consider η a trace on $L^\infty(X,\nu)$. But now η induces a measure σ on X by $\sigma(U) = \eta(\chi_U)$ where χ_U is the characteristic function on the ν-measurable set U. Note that if $\nu(U) > 0$, then χ_U is a non-zero projection so $\sigma(U) > 0$ by the hypothesis on η. Now $\eta(S) = \int t\, d\sigma(t)$. Since σ is a non-trivial measure with compact support in $[0,\infty)$ this integral must be positive.

Suppose that we have the following situation: There exists a sequence of operators T_n and polynomials $q_n(x) \in \mathbf{R}[x]$ of degree n with positive leading coefficient, such that $T_n = q_n(T)$, where T is some given self-adjoint operator, and T_0 is the identity. Assume further that there is a trace $\eta\colon V \to \mathbb{C}$, where V is some von Neumann algebra containing the T_n such that the T_n are orthogonal to one another with repsect to η. This is, $\eta(T_nT_m) = r_n\delta_{n,m}$, $r_n > 0$. In particular $\eta(q_n(T)q_m(T)) = r_n\delta_{n,m}$. So η defines an inner product on V, with respect to which $p_n(T) = q_n(T)/\sqrt{r_n}$ yields an orthonormal sequence.

Assume further that there is a space containing V on which multiplication by T is defined and in which $\phi_z = \sum p_n(z)p_n(T) = \sum q_n(z)T_n/r_n$ converges. Since Theorem 1 is proved in a purely formal algebraic context and since the T_n are linearly independent by their orthogonality, we may treat this ϕ_z just like that of Theorem 1. This says that ϕ_z generates the eigenspace of T with eigenvalue z. ϕ_z has as constant term 1_V with respect to the T_n and is by definition a

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spherical function. Thus we get the following, which is our main use of Theorem 1:

COROLLARY. *Assume that the T_n, q_n, and r_n are as above. If $\phi = \sum \alpha_n T_n$ converges in some space and $T\phi = z\phi$, then ϕ is necessarily some constant multiple of $\phi_z = \sum q_n(z)T_n/r_n$.*

Examples of this phenomenon are the spherical functions on free groups which are found in [8] and [11]. We will discuss these in detail in the examples.

There is another way of interpreting these results. In [6] it is shown that given T_n, q_n, r_n, p_n and μ as above, the spectral measure for T can be described as

$$dQ = \sum q_n(x)\, d\mu(x) T_n/r_n.$$

Heuristically, a spherical function ought to be the evaluation of the spectral measure at a point; and indeed if we take neighborhoods N of a point z, then $Q(N)/\mu(N)$ does converge to $\sum q_n(z)T_n/r_n$ as $\mu(N)$ goes to zero.

Since the ϕ_z are necessarily the only spherical functions which can be described as infinite linear combinatons of T_n, one can study directly the question of the existence of spherical functions with given properties simply by studying the ϕ_z.

One interesting problem is that of finding spherical functions $\sum \alpha_n p_n(x)$ such that the coefficients of the $p_n(x)$ are bounded. That is, giving A the norm $|\sum s_i p_i(s)| = \sum |s_i|$, we seek spherical functions that are continuous on A. So we need to find out when the sequence $p_n(z)$ is bounded. We say that ϕ_z is a *bounded spherical function* (w.r.t. $\{p_n(x)\}$) in this case. Let us take the special case of the constant recursive formula (1) with $b_n = 1$ and $a_n = 0$ for all n. We prove later a general result, Theorem 3, which subsumes Proposition 2, but it seems worthwhile to look at this special case, because it contains the general ideas of the proof of Theorem 3, but with much simpler calculations.

PROPOSITION 2. *If the polynomials $p_n(x)$ satisfy*

$$xp_n(x) = p_{n+1}(x) + p_{n-1}(x),$$

with $p_0(x) = 1$ and $p_1(x) = x$, then the spherical function ϕ_z is bounded if and only if $z \in (-2, 2)$.

PROOF. Let $\theta(z, w) = \sum p_n(z)w^n$. Then $\theta = 1 + \sum p_{n+1}(z)w^{n+1} = 1 + \sum(zp_n(z) - p_{n-1}(z))w^{n+1} = 1 + zw\theta - w^2\theta$. Thus $(1 - zw + w^2)\theta = 1$. Then if $z \neq \pm 2$, we can find A, B, r, and s such that $\theta = A/(1 - rw) + B/(1 - sw)$. Thus $p_n(z)$, the coefficient of w^n, is bounded if and only if $|r| \leq 1$ and $|s| \leq 1$.

But $rs = 1$ and $r + s = z$. Thus the condition becomes $|r| = 1$, whence $z = r + \bar{r} \in (-2, 2)$. For $z = \pm 2$, $\theta = (1 \pm w)^{-2}$ and so $p_n(z) = (\pm 1)^n(n+1)$ is unbounded.

It follows from the proof that $p_n(x)$ can never converge to zero. In particular, then, in this case ϕ_z can never be in a Hilbert space in which the $p_n(x)$ form an orthonormal system.

Since in many applications, it is more natural to study orthogonal *non-normalized* polynomials, we give here a translation and generalization of our results in terms of the recursive formula for orthogonal polynomials which are not necessarily orthonormal. Assume the sequence $\{q_n(x)\}$ of orthogonal polynomials satisfies

$$xq_n(x) = \nu_n q_{n+1}(x) + \alpha_n q_n(x) + \beta_{n-1} q_{n-1}(x), \tag{2}$$

for real numbers $\nu_n, \beta_n > 0$ and α_n. This holds for all $n \geq 0$ if we set $q_{-1}(x) = 0$ and let β_{-1} be arbitrary.

Let $k_n^2 = q_n(x) \cdot q_n(x)$. Then, since $(xq_n(x)) \cdot q_{n-1}(x) = q_n(x) \cdot (xq_{n-1}(x))$, we get $k_n^2 = k_{n-1}^2 \beta_{n-1}/\nu_{n-1}$, and hence

$$k_{n+1}^2 = (\beta_n \beta_{n-1} \cdots \beta_2 \beta_1 \beta_0 k_0^2)/(\nu_n \cdots \nu_1 \nu_0).$$

Putting $p_n(x) = k_n^{-1} q_n(x)$, $\{p_n(x)\}$ satisfies (1) with $b_n = \sqrt{\beta_n \nu_n}$ and $a_n = \alpha_n$. In particular, the spherical functions have the form $\phi_z = \sum q_n(z) q_n(x)/k_n^2$. We now ask a different bundedness question: when are the coefficients of $q_n(x)$ bounded? In this case the coefficient is $q_n(z)/k_n^2 = p_n(z)/k_n$. We now say that $\phi = \sum \alpha_n q_n(x)$ is a *bounded spherical function* w.r.t. $\{q_n(x)\}$ if the α_n are bounded even though the $\{q_n(x)\}$ may not be orthonormal.

NOTATION. (a) We write $r_n \twoheadrightarrow r$, r_n goes quickly to r, if there exist constants $\varepsilon < 1$ and M such that $|r_n - r| < M\varepsilon^n$.

(b) If $f(z)$ is the sum of a power series, ρ_f represents its radius of convergence.

THEOREM 2. *Assume that $\{q_n(x)\}$ satisfy $xq_n(x) = \nu_n q_{n+1}(x) + \alpha_n q_n(x) + \beta_{n-1} q_{n-1}(x)$ where $\alpha_n \twoheadrightarrow \alpha$, $\beta_n \twoheadrightarrow \beta$, and $\nu_n \twoheadrightarrow \nu$. Consider the elliptical region E of all points $z = u + iv$ satisfying*

$$\left[\frac{u - \alpha}{\beta + \nu}\right]^2 + \left[\frac{v}{\beta - \nu}\right]^2 \leq 1 \quad \text{if } \nu \neq \beta,$$

or

$$z \in (\alpha - 2\beta, (\alpha + 2\beta)b) \quad \text{if } \nu = \beta.$$

Then the spherical function ϕ_z is bounded if $\nu \leq \beta$ and $z \in E$ and is unbounded if $\nu > \beta$ and $z \in E$. There exists an analytic function $h(w)$ calculable from the $\{q_n(x)\}$ such that for all other z, ϕ_z is bounded if and only if $h(\sigma) = 0$, where $\sigma = \{(z - \alpha) \pm \sqrt{(z-\alpha)^2 - 4\beta\nu}\}/2\nu$ and $|\sigma| < 1$, except if $\beta = \nu$ and $z = \alpha \pm 2\beta$ in which case $\sigma = \pm 1$.

PROOF. Fix z and let $\theta(w) = \sum (q_n(z)/k_n^2) w^n$. Then

$$\begin{aligned} z\theta &= \sum \frac{\gamma_n q_{n+1}(z)}{k_n^2} w^n + \alpha_n \frac{q_n(z)}{k_n^2} w^n + \beta_{n-1} \frac{q_n(z)}{k_n^2} w^n \\ &= \sum w^{-1} \beta_n \frac{q_{n+1}(z)}{k_n^2} w^{n+1} + \alpha_n \frac{q_n(z)}{k_n^2} w^n + \gamma_{n-1} w \frac{q_{n-1}(z)}{k_{n-1}^2} w^{n-1} \\ &= w^{-1}\beta \sum \frac{q_{n+1}(z)}{k_{n+1}^2} w^{n+1} + \alpha \sum \frac{q_n(z)}{k_n^2} w^n + \gamma w \sum \frac{q_{n-1}(z)}{k_{n-1}^2} w^{n-1} \\ &\quad + \sum \left[w^{-1}(\beta_n - \beta) \frac{q_{n+1}(z)}{k_n^2} w^n + (\alpha_n - \alpha) \frac{q_n(z)}{k_n^2} w^n \right. \\ &\qquad\qquad \left. + w(\gamma_{n-1} - \gamma) \frac{q_{n-1}(z)}{k_{n-1}^2} w^{n-1} \right] \\ &= w^{-1}\beta(\theta - 1) + \alpha\theta + \nu w \theta + k(w), \end{aligned}$$

where $k(w)$ is the part under the final summation. Thus $(w^{-1}\beta + \alpha + \nu w - z)\theta = w^{-1}\beta - k(w)$. Hence if we let

$$h(w) = 1 - w\beta^{-1}k(w), \quad \text{and} \quad g(w) = (1 - \beta^{-1}(z - \alpha)w + \beta^{-1}\nu w^2)^{-1},$$

we see that $\theta(w) = g(w)h(w)$. Notice that $\rho_h > \rho_\theta$, since $\alpha_n \twoheadrightarrow \alpha, \beta_n \twoheadrightarrow \beta$, and $\nu_n \twoheadrightarrow \nu$. The result now follows by applying the following lemma with $b = \beta^{-1}(z - \alpha)/2$ and $c = \beta^{-1}\nu$.

LEMMA. *Let $f(w) = g(w)h(w)$ where $g(w) = (1 - 2bw + cw^2)^{-1}, \rho_h > \rho_f$ and $c > 0$. Consider the elliptical disk E:*

$$E = \left\{ x + iy \mid \left(\frac{x}{1+c}\right)^2 + \left(\frac{y}{1-c}\right)^2 \le \frac{1}{4} \right\}$$

if $c \neq 1$ and $E = (-1, 1)$ for $c = 1$. Then the coefficients of f are bounded if $b \in E$ and $c \le 1$, and are unbounded if $b \in E$ and $c > 1$. In all other cases, they are bounded if and only if $h(\sigma) = 0$ where $\sigma = \{b \pm \sqrt{(b^2 - c)}\}/c$, with the sign chosen so that $|\sigma| < 1$ except in the case $b = \pm 1,\ c = 1$ where $\sigma = b$.

PROOF. We are going to use a special case of Darboux's Theorem, which follows immediately from the version given in Theorem 4 of [2]: Assume that $F(z) = \sum a_n z^n$ is analytic near 0 with radius of convergence r. Assume further that on the circle of radius r, F has a finite number of singularities $\alpha_1, \dots, \alpha_t$ and that for each k, there exist an integer $\varepsilon_k \ge 0$ and a function F_k analytic near α_k such that $F_k(\alpha_k) \neq 0$ and $F(z) = (1 - z/\alpha_k)^{1+\varepsilon_k} F_k(z)$ near α_k. Let $\varepsilon = \max\{\varepsilon_k\}$. Then Darboux's Theorem says that

$$a_n = \sum \frac{F_k(\alpha_k) n^{\varepsilon_k}}{\varepsilon_k! \alpha_k^n} + o(r^{-n} n^{\varepsilon}).$$

Note that this formula remains the same even if $|\alpha_k| > r$ for some k.

Write $f(w) = \sum a_n w^n$. Note that $\rho_f \geq \min(\rho_g, \rho_h)$. Since $\rho_f < \rho_h$, this minimum must be $\rho = \rho_g$.

Let σ_1 and σ_2 be the roots of the polynomial $g(w)^{-1} = (1 - 2bw + cw^2)$. We shall assume that $|\sigma_1| \leq |\sigma_2|$, so that $\rho = |\sigma_1|$. Let $\beta_i = 1/\sigma_i$ so that $(1 - \beta_1 w)(1 - \beta_2 w) = (1 - 2bw + cw^2)$. Then $\beta_1 + \beta_2 = 2b$ and $\beta_1\beta_2 = c$. Let $\lambda = re^{i\theta}$ be one of the β_i, whence c/λ is the other. Now $2b = \lambda + c/\lambda = re^{i\theta} + (c/r)e^{-i\theta} = (r + cr^{-1})\cos\theta + i(r - cr^{-1})\sin\theta$. Here we shall note two special cases: if $r = \sqrt{c}$ this gives $b = \sqrt{c}\cos\theta$, hence b is in the interval $[-\sqrt{c}, +\sqrt{c}]$; and if $r = c$ we get points in the ellipse $2b = (1+c)\cos\theta + i(1-c)\sin\theta$, so that setting $b = x + iy$ gives

$$\left(\frac{x}{1+c}\right)^2 + \left(\frac{y}{1-c}\right)^2 = \frac{1}{4}$$

if $c < 1$, and a point of the interval $[-1, 1]$ for $c = 1$.

Applying Darboux's Theorem now, we have that a_n is $\{\sigma_1/(\sigma_2 - \sigma_1)\}[\sigma_1^{-n}h(\sigma_1) - \sigma_2^{-n}h(\sigma_2)] + o(\rho^{-n})$, if $\sigma_1 \neq \sigma_2$ and $h(\sigma_1) \neq 0$; $\{\sigma_1/(\sigma_1 - \sigma_2)\}[\sigma_2^{-n}h(\sigma_2)] + o(|\sigma_2|^{-n})$, if $\sigma_1 \neq \sigma_2$ and $h(\sigma_1) = 0$; $n\sigma^{-n}h(\sigma) + o(\rho^{-n}n)$, if $\sigma_1 = \sigma_2 = \sigma$ and $h(\sigma) \neq 0$; and finally $\sigma^{-n}h'(\sigma) + o(\rho^{-n})$, if $\sigma_1 = \sigma_2 = \sigma$ and $h(\sigma) = 0$.

Thus we see that $\{a_n\}$ is bounded in the cases $\rho > 1$; or $\rho = 1$ and $\sigma_1 \neq \sigma_2$; or $\rho = 1, \sigma_1 = \sigma_2 = \sigma$ and $h(\sigma) = 0$; it is unbounded in the case $\rho = 1, \sigma_1 = \sigma_2 = \sigma$ and $h(\sigma) \neq 0$.

Now for the case $\rho < 1$: If $\sigma_1 = \sigma_2$, then $\{a_n\}$ is unbounded, so assume that $\sigma_1 \neq \sigma_2$. If $h(\sigma_1) \neq 0$, then $\{a_n\}$ is unbounded; and finally if $h(\sigma_1) = 0$, then $\{a_n\}$ is bounded if and only if $|\sigma_2| \geq 1$.

We shall now examine what these conditions mean in terms of b and c.

First consider the condition that $\rho \geq 1$. Thus $|\lambda| \leq 1$ and $|c/\lambda| \leq 1$; that is, $c \leq r \leq 1$. In particular we have $c \leq 1$. Now $2b = \lambda + c/\lambda$ remains invariant under the transformation $\lambda \to c/\lambda$, so the set of b for $c \leq r \leq 1$ is the same as the set for $c \leq r \leq \sqrt{c}$. We discussed above the two end points and all other values are between the two. So $b = x + iy$ is in the elliptical disk

$$\left(\frac{x}{1+c}\right)^2 + \left(\frac{y}{1-c}\right)^2 \leq \frac{1}{4}$$

if $c < 1$, and in the interval $(-1, 1)$ for $c = 1$ since the case $c = 1$ and $b = \pm 1$ is the exceptional case $\rho = 1$ and $\sigma_1 = \sigma_2 = \sigma$. In this latter case $\sigma = b = \pm 1$ and $\sigma = b$, and as pointed out above $\{a_n\}$ is bounded if and only if $h(\sigma) = 0$. This completes the case $\rho \geq 1$.

Now assume that $\rho < 1$. Let us consider first the case $|\beta_i| > 1$ for $i = 1, 2$. Thus the conditions are $|\lambda| > 1, |c/\lambda| > 1$, that is $c > r > 1$. In particular, $c > 1$. Since $2b = \lambda + c/\lambda$ remains invariant under the transformation $\lambda \to c/\lambda$, the set of b for $c > r > 1$ is the same as the set for $c > r > \sqrt{c}$, and once again b is in the elliptical disk

$$\left(\frac{x}{1+c}\right)^2 + \left(\frac{y}{1-c}\right)^2 \leq \frac{1}{4}.$$

Finally we examine the condition $|\beta_2| \leq 1 < |\beta_1|$. Thus the conditions are $|\lambda| > 1, |c/\lambda| \leq 1$, that is, $r > 1, r \geq c$. Now since $2b = \lambda + c/\lambda$ remains invariant under the transformation $\lambda \to c/\lambda$, the set of b for which $r > 1, r \geq c$ is the same as the set for which $r < 1, r \leq c$. Thus the condition on $b = x + iy$ is

$$\left(\frac{x}{1+c}\right)^2 + \left(\frac{y}{1-c}\right)^2 > \frac{1}{4}$$

if $c \neq 1$, and $b \notin (-1, 1)$ for $c = 1$. That is $b \notin E$.

This completes the proof of the Lemma.

Applications

(1) Let F be a free group on a fixed set of t generators, and let χ_n be the characteristic function on the words of F of length n. Then there is a sequence of polynomials $q_n(x)$ (see [5]) such that $q_n(\chi_1) = \chi_n$. The polynomials satisfy (2) with $\nu_n = 1$, $\alpha_n = 0$, and $\beta_n = 2t - 1$, for $n > 0$, and $\beta_0 = 2t$. Thus applying Theorem 2, there are spherical functions (eigenfunctions of the operator "convolution with χ_1") for all $z \in \mathbb{C}$, which are bounded exactly for $z = x + iy$ inside the ellipse $[x/2t]^2 + [y/(2t-2)]^2 = 1$. This result, first proved in [10], can be found in [8]. In this case we do not have any "interference" from the analtyic function $h(w)$ of Theorem 2. The only value of z for which $h(\sigma) = 0$ is $z = 2t - 1$, which is already in the ellipse. $h(w)$ is simple to calculate because $\alpha_n = \alpha$, $\beta_n = \beta$, and $\nu_n = \nu$ for all n, with the single exception of $\beta_0 = \beta + 1$.

(2) Let G be a free product of k finite groups, each of order r. Then defining the elements of the original factor groups to have length 1, we can define χ_n as above, and find a recursive relation (see [7], [9], and [5]) (2) with $\alpha_n = r - 2$, and $\beta_n = (k-1)(r-1)$, and $\nu_n = 1$ always except for $\beta_0 = k(r-1)$. Thus applying Theorem 2, there are spherical functions for all $z \in \mathbb{C}$, which are bounded exactly for $z = x + iy$ inside the ellipse $[(x - r + 2)/(kr - k - r + 2)]^2 + [y/(k^2 - k - r)]^2 = 1$. Just as in the first example $h(w)$ does not influence the boundedness area because the only value of z for which $h(z) = 0$ is already inside the ellipse.

The above examples are the important special cases of a more general sequence of polynomials studied in [7] where $\alpha_n = \alpha$, and $\beta_n = \beta$, and $\nu_n = 1$ for all n except for $\beta_0 = \beta + r$. In this case $h(w) = 1 - \beta^{-1} r(z - \alpha)$, a constant with respect to w. Thus the only value of z for which $h(\sigma) = 0$ is $z = \alpha + \beta/r$. This is inside the ellipse if and only if $r \geq \beta/(\beta+1)$, which is always true in the group case since $r \geq 1$.

Other examples can be found whenever the Plancherel measure of a discrete group is known. In addition, Theorem 2 may be generalized, following the same lines, to the case in which the α_n, β_n, and ν_n do not converge, but do converge "cyclically"; that is for some fixed d, $\alpha_{dn+k}, \beta_{dn+k}$, and ν_{dn+k} go quickly to some limits that depend on k. Here we expect the region of bounded coefficients to be a higher degree curve. An example of this occurs in [3] where $d = 2$, and the form of the region is a quartic, interestingly of the type that a certain square lies inside an ellipse.

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